Solution - Automatic Control Systems – June 11th, 2024

Exercise 1

a. ***Compute the analytic expression of the step response of the LTI system described by the following transfer function:***

$$G(s) = \frac{2(5s+1)}{(s^2+8s+12)}$$

Solution:

$$Y(s) = G(s)U(s) = \frac{2(5s+1)}{(s^2+8s+12)}\frac{1}{s}$$

By partial fraction decomposition,

$$Y(s) = G(s)U(s) = \frac{2(5s+1)}{(s^2+8s+12)}\frac{1}{s} = \frac{2(5s+1)}{(s+6)(s+2)}\frac{1}{s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

and by applying residual method,

$$A = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,\frac{2(5s+1)}{(s+6)(s+2)}\frac{1}{s} = \frac{1}{6}$$

$$B = \lim_{s\to -2}(s+2)\,Y(s) = \lim_{s\to -2}(s+2)\,\frac{2(5s+1)}{(s+6)(s+2)}\frac{1}{s} = \frac{9}{4}$$

$$C = \lim_{s\to -6}(s+6)\,Y(s) = \lim_{s\to -6}(s+6)\,\frac{2(5s+1)}{(s+6)(s+2)}\frac{1}{s} = -\frac{29}{12}$$

By Laplace anti-transformation, we achieve the analytic expression of *y(t)*:

$$\boldsymbol{y(t) = \left(\frac{1}{6} + \frac{9}{4}e^{-2t} - \frac{29}{12}e^{-6t}\right)1(t).}$$

MATLAB code to verify the computation:

```
% use the residue function to verify the fractional decomposition
% define the coefficients of the numerator of Y(s), Y(s)=N(s)/D(s), i.e.
% N(s)= 10*s+2;
num=[10 2];
% define the coefficients of the denominator of Y(s), i.e.,
% D(s)=s^3+8*s^2+12*s
den=[1 8 12 0];
[r,p,k]=residue(num,den)
r =
  -2.4167 %  equal to C, r(1) corresponds to the residual of p(1), i.e. -6
   2.2500 %  equal to B, r(2) corresponds to the residual of p(2), i.e. -2
   0.1667 %  equal to A, r(3) corresponds to the residual of p(3), i.e. 0
p =
   -6
   -2
    0
k =
    []

% define a symbolic variable p

syms p

% define Y(p)

Y_p=2*(5*p+1)/(p^2+8*p+12)/p;

% compute y(t) by using ilaplace function (inverse Laplace transform)

y_t=ilaplace(Y_p)

y_t =

(9*exp(-2*t))/4 - (29*exp(-6*t))/12 + 1/6
```

b. **draw the qualitative step response**

Solution:

Compute the parameters for drawing the step response:

$$y(0) = \lim_{s\to\infty} sY(s) = 0$$

$$\dot{y}(0) = \lim_{s\to+\infty} s^2\, Y(s) = 10$$

$$y_\infty = \lim_{t\to+\infty} y(t) = \lim_{s\to 0} s\, Y(s) = A = 1/6$$

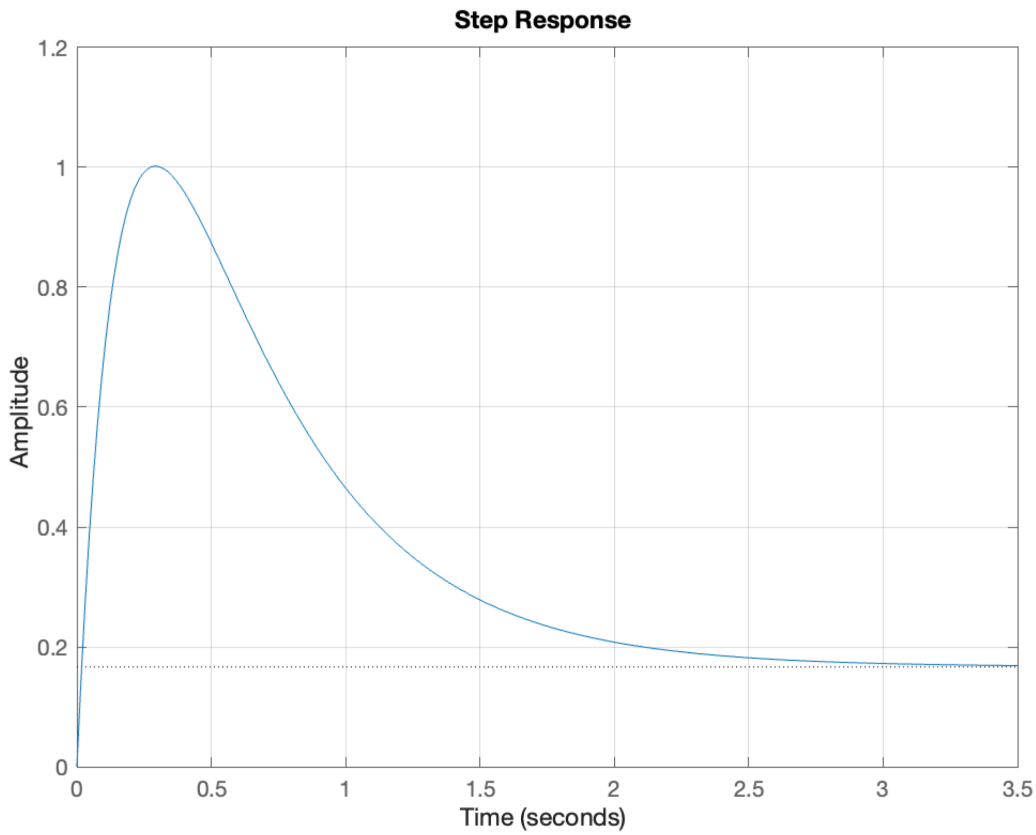
Compute the maximum time constant, τ_{max}, and the settling time:

$$\begin{cases} \tau_1 = -\dfrac{1}{p_1} = -\dfrac{1}{-2} = 0.5 \\ \tau_2 = -\dfrac{1}{p_2} = -\dfrac{1}{-6} \cong 0.17 \end{cases} \Rightarrow \tau_{max} = 0.5 \Rightarrow t_{s5\%} = 3\tau_{max} = 1.5\ \text{sec}$$

Note the presence of the zero ($s = z_1 = -1/5$) at low frequency that determines an overshoot during the transient (but without oscillations).

By using MATLAB, we can achieve the behavior of *y* (define the system transfer function by the function ***tf*** and then use the command ***step*** for achieving the plot – see figure below) and verify how the estimated parameters (computed above) are close to the real ones (from the MATLAB plot).

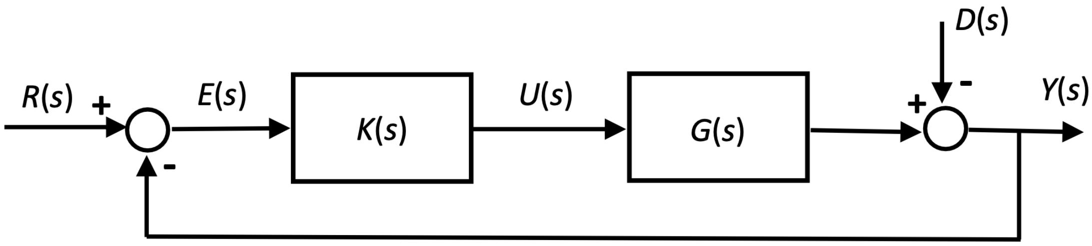
```
% define the tf s variable
s=tf('s');
% define the tf of the LTI system
Gs=2*(5*s+1)/(s^2+8*s+12);
% plot the step response
figure
step(Gs)
grid on
```



Step Response
Amplitude
Time (seconds)
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5

Exercise 2

For the closed loop system shown in figure,



where

$$G(s) = \frac{10}{s(s+10)},$$

a. **devise *K*(*s*) in order to satisfy the following requirements:**

   i. **e_∞=0 w.r.t. a step disturbance $d(t) = d_0 1(t - t_0)$;**

   ii. ***y*(*t*) without overshoot to a step reference input *r*(*t*);**

   iii. **settling time $t_{s5\%} \leq 0.3$ sec.**

b. **draw the qualitative response *y*(*t*) of the devised closed loop system to the following inputs:**

   $r(t) = 1(t)$;

   $d(t) = d_0 \cdot 1(t - t_0)$ **with** $d_0 = 0.2$ **and** $t_0 = 1$ **sec.**

Solution:

Assume the controller is in the form

$$K(s) = K_1(s)K_2(s),$$

where $K_1(s)$ takes into account the static requirement (**i.**) and K_2**(s)** the transient requirements (**ii.** and **iii.**).

i. In order to satisfy the **first requirement**, **i. e_∞=0** w.r.t. a step disturbance $d(t) = d_0 1(t - t_0)$, i.e the complete rejection of a disturbance signal of order *k*=0 ($d(t)$ is a step signal, then *k*=0), it is necessary an open loop function, ***F*(*s*)=*K*(*s*)*G*(*s*)**, of type 1 (i.e., *k+1*) (***F*(*s*) should have one integrator**, i.e. one pole in the origin).

Note that ***G*(*s*) contains one integrator (i.e. one pole in the origin)**, then, it is not necessary to include an additional integrator in ***K*(*s*)**, $\boldsymbol{K_1}$ can be in the form

$$\boldsymbol{K_1(s) = k_0},$$

and the open loop function as

$$\boldsymbol{F_1(s) = K_1(s)G(s) = k_0 \frac{10}{s(s+10)} = k_0 \frac{10}{10\left(\frac{s}{10}+1\right)s} = k_0 \frac{1}{s\left(\frac{s}{10}+1\right)}}.$$

ii. Regarding the second requirement (**ii. no overshoot**), the phase margin $\boldsymbol{\varphi_m}$ of the open loop function ***F*(*s*)** must be $\boldsymbol{\varphi_m > 60^\circ}$.

Then, the closed loop system can be approximated by a first order system in the form

$$\boldsymbol{T_a(s) = \frac{1}{1+s\tau}},$$

with the time constant $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau = \frac{1}{\omega_c}},$$

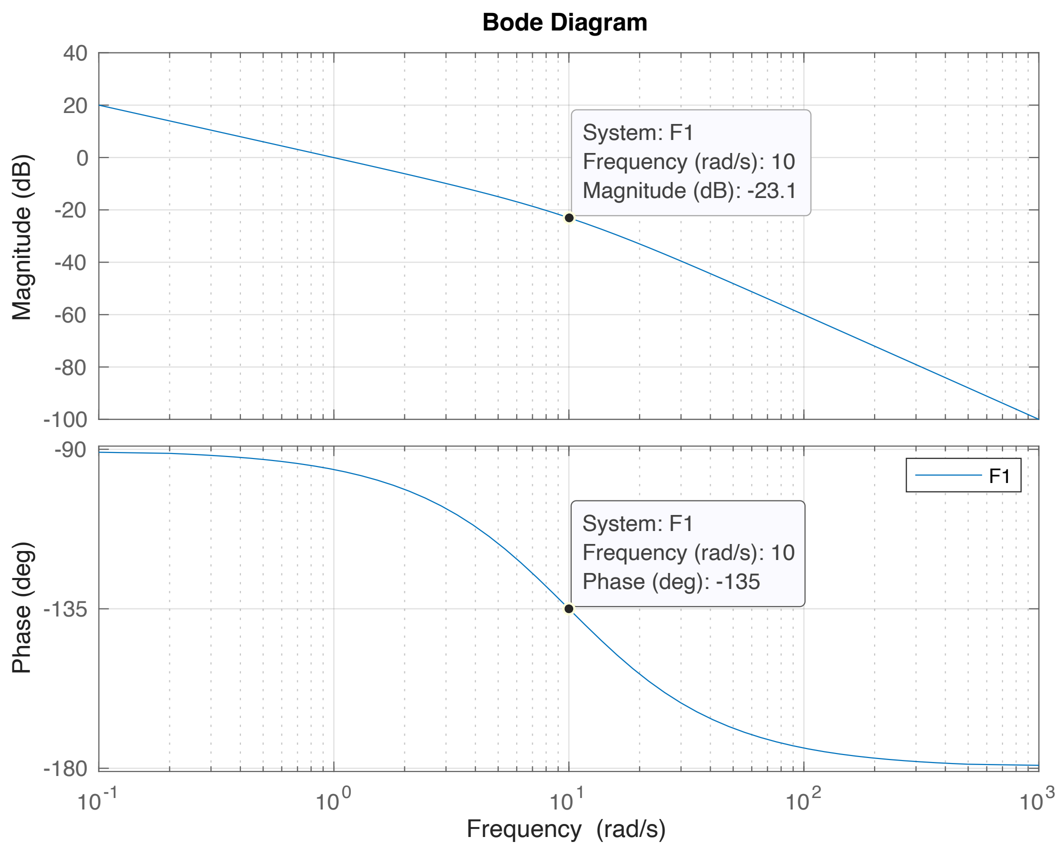
where $\boldsymbol{\omega_c}$ **is the crossing frequency of the open loop system.**

iii. The third requirement (**iii. settling time $t_{s5\%} \leq 0.3$ sec**) determines the constraint on $\boldsymbol{\omega_c}$:

$$\boldsymbol{t_{s5\%} = 3\tau = \frac{3}{\omega_c} \leq \frac{3}{10} \Rightarrow \omega_c \geq 10 \text{ rad/sec}}.$$

Therefore, the transfer function $\boldsymbol{F(s)}$ should have a crossing frequency $\boldsymbol{\omega_c \geq 10 \text{ rad/sec}}$ and a phase margin $\boldsymbol{\varphi_m}$>**60°**.

By checking the Bode diagrams of $\boldsymbol{F_1(s) = k_0 \frac{1}{s\left(\frac{s}{10}+1\right)}}$, with $\boldsymbol{k_0 = 1}$, we get that $\boldsymbol{|F(j10)|_{dB} = -23\text{dB}}$ **(-20 dB** for the asymptotic Bode diagrams) and $\boldsymbol{\arg(F(j10)) = -135^\circ}$, as shown in the figure below.



Therefore, **we require a magnitude amplification of about 20 dB for F_1 to have a crossing frequency $\omega_c \cong 10$ rad/sec and a phase increase** $> 15°$ to have $\varphi_m > 60°$ ($\varphi_m = \pi - |\arg(F(j10))|$).

A possible solution could be obtained

- **by setting $k_0 = 10$** (in order to get the desired magnitude amplification and have $\omega_c \cong$ **10 rad/sec**)
- **by adding a zero in $s = -10$** (resulting in a **phase increase** of $45°$ **in $\omega = 10$ rad/sec** $\Longrightarrow \varphi_m \cong 90$; note that this zero ($s = -10$) cancels the pole of $G(s)$).

Then, the controller is in the form

$$K(s) = K_1(s)K_2(s) = 10\left(\frac{s}{10} + 1\right).$$

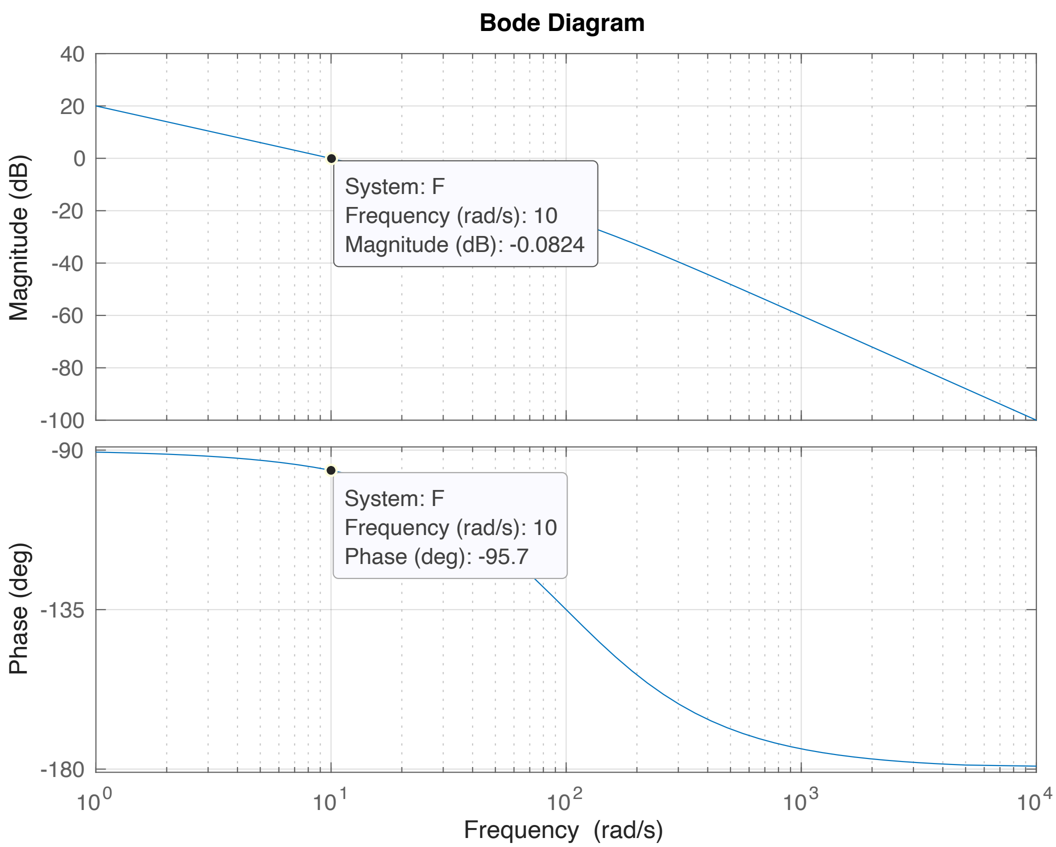
Note that such a controller is not feasible, therefore **we add a pole at high frequency**, for example in $s = -100$, leading to the following controller:

$$K(s) = K_1(s)K_2(s) = 10\frac{\left(\frac{s}{10}+1\right)}{\left(\frac{s}{100}+1\right)}.$$

Then, the resulting open loop function is given by

$$F(s) = K(s)G(s) = 10\frac{\left(\frac{s}{10}+1\right)}{\left(\frac{s}{100}+1\right)}\frac{1}{s\left(\frac{s}{10}+1\right)}$$

and the below figure shows the relative Bode diagrams.



By using MATLAB, it is possible to verify the performance of the devised control system and simulate the response to the following inputs (as required at point .b) (see the below figure):

$$\boldsymbol{r(t) = 1(t)};$$

$$\boldsymbol{d(t) = d_0 \cdot 1(t - t_0)} \textbf{ with } \boldsymbol{d_0 = 0.2} \textbf{ and } \boldsymbol{t_0 = 1} \textbf{ sec.}$$