Course of "Industrial Automation"
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Discrete equivalents – Digital control design

Prof. Francesco Montefusco

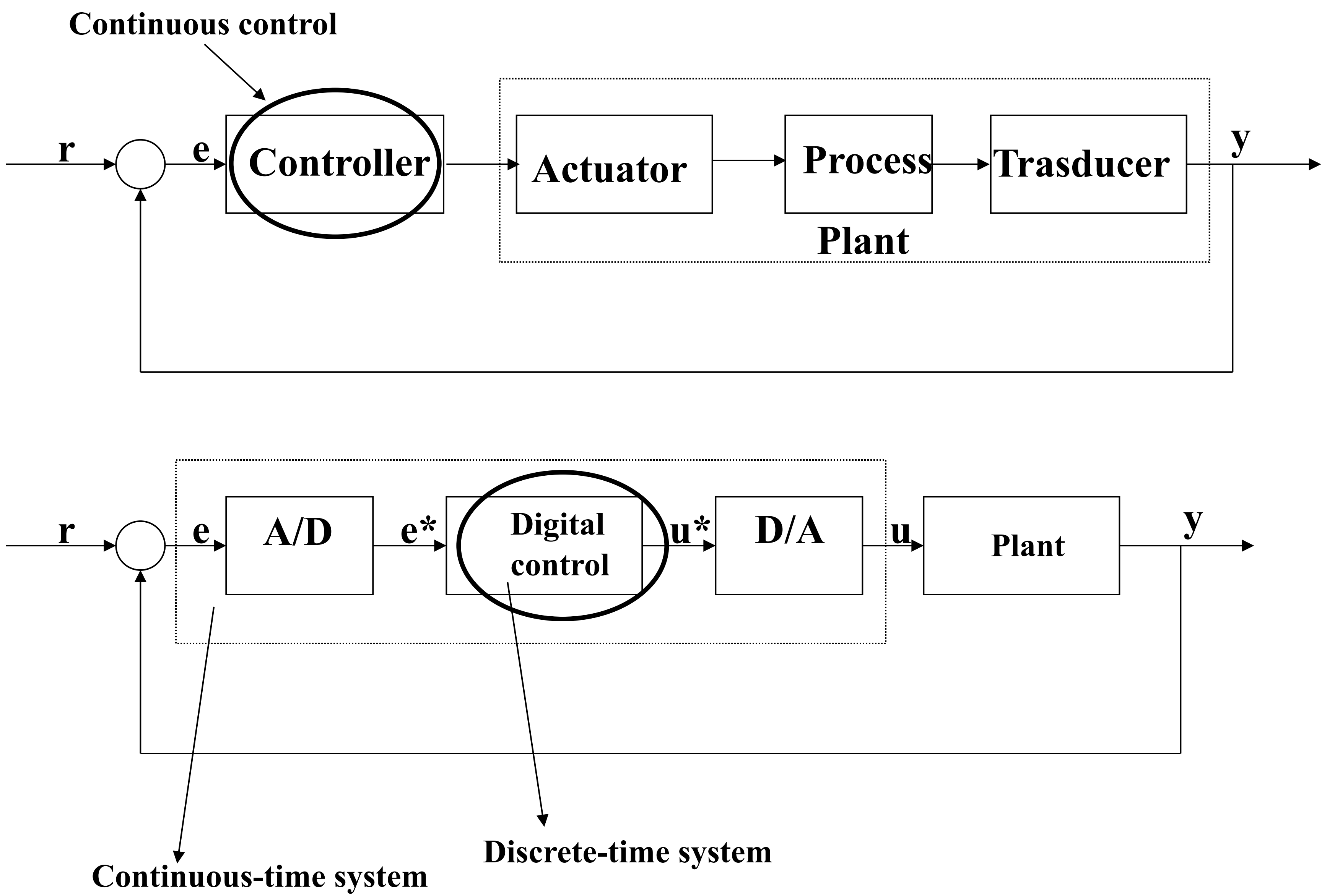
Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli Studi di Napoli Parthenope

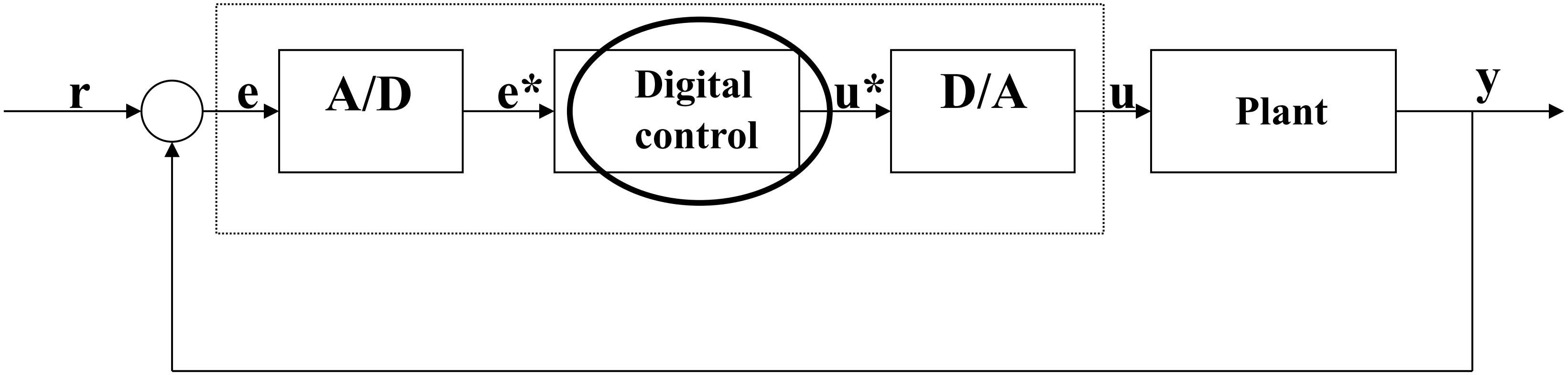
francesco.montefusco@uniparthenope.it

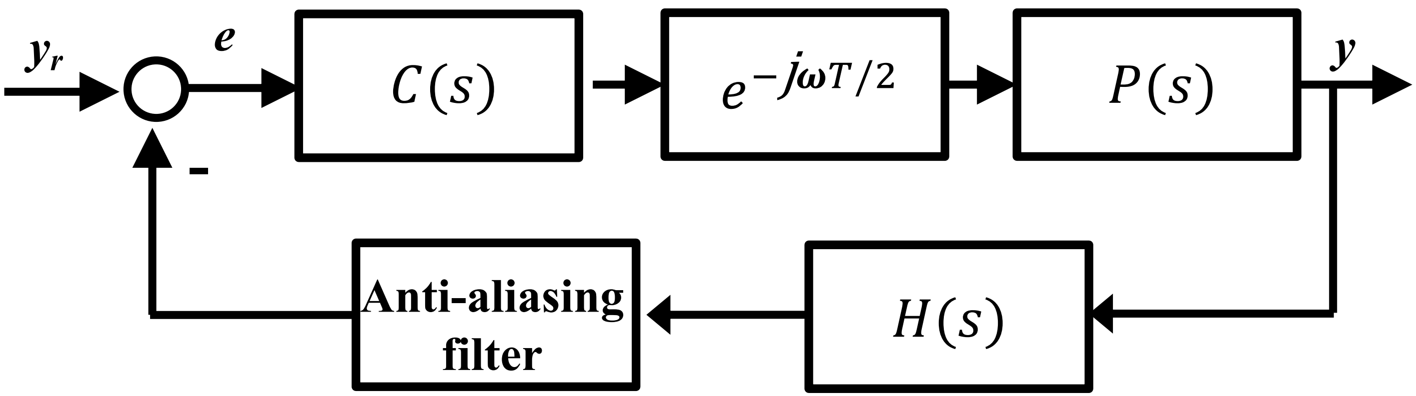
Team code: **5qf4mll**

Continuous vs. digital



Scheme of the digital control system in continuous-time





Introduction - 1

- From $C(s)$ we want to find a discrete equivalent $D(z)$:
- A transformation ($\boldsymbol{s} \longrightarrow \boldsymbol{z}$) allows the transition from continuous time to discrete time such that

same static and dynamic perfomance

- Same static perfomance:

$$\boldsymbol{D}(z)|_{\boldsymbol{z=1}} \cong \boldsymbol{C}(\boldsymbol{s})|_{\boldsymbol{s=0}}$$

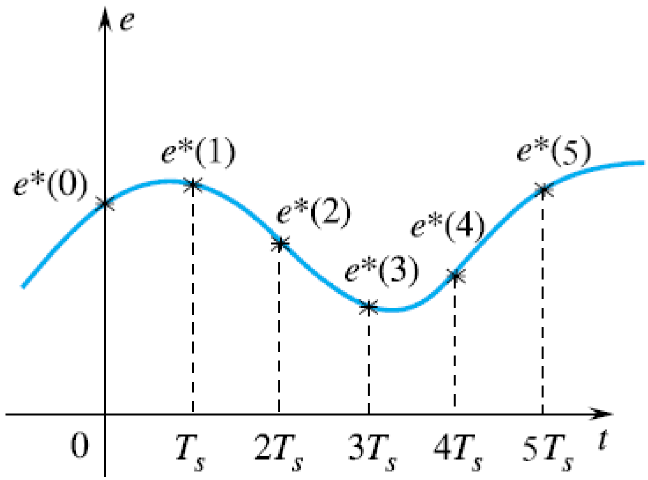
- Same dynamic performance, i.e., $\boldsymbol{D}(\boldsymbol{z})$ with same frequency behavior of $\boldsymbol{C}(\boldsymbol{s})$ in a given range of $\boldsymbol{\omega}$:

$$\boldsymbol{D}(\boldsymbol{z})|_{\boldsymbol{z}=e^{\boldsymbol{j\omega T}}} \cong \boldsymbol{C}(\boldsymbol{s})|_{\boldsymbol{s=j\omega}}$$

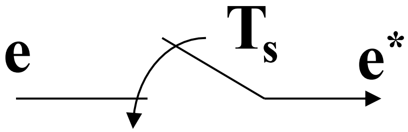
The transformation from contoinouos to discret time domain is given by

$$z = e^{sT}$$

Indeed, we use the impulse modulation as the mathematical representation of the sampling operation as it follows:



$$\omega_S = 2\pi f_S = \frac{2\pi}{T_S}$$



$$e_s(t) = e(t)\sum_{k=0}^{\infty}\delta(t - kT_s) = \sum_{k=0}^{\infty} e(kT_s)\delta(t - kT_s)$$

Then, assuming $x_s(t)$ the sampled representation of a continuous-time signal $x(t)$:

$$x_s(t) = x(t)\sum_{k=0}^{\infty}\delta(t-kT_s) = \sum_{k=0}^{\infty}x(kT_s)\delta(t-kT_s)$$

$\mathcal{L}$

$$\mathcal{L}(x_s(t)) = \int_0^{\infty}\sum_{k=0}^{\infty}x(kT_s)\delta(\tau-kT_s)e^{-s\tau}\,d\tau$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty}x(kT_s)\delta(\tau-kT_s)e^{-s\tau}d\tau =$$

$$\sum_{n=0}^{\infty}x(kT_s)e^{-skT_s} = X(z)|_{z=e^{sT_s}}$$

$$X_s(s) = X(z)|_{z=e^{sT_s}}$$

Introduction - 4

Therefore, we could assume the inverse transformation

$$\boldsymbol{s} = \frac{\mathbf{1}}{\boldsymbol{T}} \ln \boldsymbol{z},$$

But we get:

- a function $\boldsymbol{D}(z)$ which is not rational, and cannot be associated with a finite-dimensional discrete-time system

Basically, design of discrete equivalents via **numerical integration**

Let us consider a continuous Linear Time Invariant (LTI) system in the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

By integrating the state equation between kT and $(k+1)T$ and denoting with $\boldsymbol{x}^*(\boldsymbol{k}) = \boldsymbol{x}(\boldsymbol{kT})$ the state vector at the instant time kT:

$$\boldsymbol{x}^*(\boldsymbol{k}+\mathbf{1}) - \boldsymbol{x}^*(\boldsymbol{k}) = \boldsymbol{A}\int_{\boldsymbol{kT}}^{(\boldsymbol{k}+\mathbf{1})\boldsymbol{T}} \boldsymbol{x}(\boldsymbol{t})\boldsymbol{dt} + \boldsymbol{B}\int_{\boldsymbol{kT}}^{(\boldsymbol{k}+\mathbf{1})\boldsymbol{T}} \boldsymbol{u}(\boldsymbol{t})\boldsymbol{dt}\,.$$

By exploiting the following formula for the numerical integration $\boldsymbol{f}(t)$:

$$\int_{\boldsymbol{kT}}^{(\boldsymbol{k}+\mathbf{1})\boldsymbol{T}} \boldsymbol{f}(\boldsymbol{t})\boldsymbol{dt} \cong [(\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{f}(\boldsymbol{kT}) + \boldsymbol{\alpha}\boldsymbol{f}((\boldsymbol{k}+\mathbf{1})\boldsymbol{T})]\boldsymbol{T}$$

with $\mathbf{0} \leq \boldsymbol{\alpha} \leq \mathbf{1}$

We obtain:

$$\boldsymbol{x}^*(\boldsymbol{k}+\mathbf{1})-\boldsymbol{x}^*(\boldsymbol{k})=\boldsymbol{A}[(\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{x}^*(\boldsymbol{k})+\boldsymbol{\alpha}\boldsymbol{x}^*(\boldsymbol{k}+\mathbf{1})]\boldsymbol{T}$$
$$+\boldsymbol{B}[(\mathbf{1}-\boldsymbol{\alpha})\boldsymbol{u}^*(\boldsymbol{k})+\boldsymbol{\alpha}\boldsymbol{u}^*(\boldsymbol{k}+\mathbf{1})]\boldsymbol{T}$$

$$\boldsymbol{y}^*(\boldsymbol{k})=\boldsymbol{C}\boldsymbol{x}^*(\boldsymbol{k})+\boldsymbol{D}\boldsymbol{u}^*(\boldsymbol{k}).$$

where $\boldsymbol{u}^*(\boldsymbol{k})=\boldsymbol{u}(\boldsymbol{k}\boldsymbol{T})$

By applying zeta-Transform:

$$\left[\frac{\mathbf{1}}{\boldsymbol{T}}\frac{\boldsymbol{z}-\mathbf{1}}{\boldsymbol{\alpha}\boldsymbol{z}+\mathbf{1}-\boldsymbol{\alpha}}\boldsymbol{I}-\boldsymbol{A}\right]\boldsymbol{X}^*(\boldsymbol{z})=\boldsymbol{B}\boldsymbol{U}^*(\boldsymbol{z})$$

and

$$\frac{\boldsymbol{Y}^*(\boldsymbol{z})}{\boldsymbol{U}^*(\boldsymbol{z})}=\boldsymbol{G}^*(\boldsymbol{z})=\boldsymbol{C}\left[\frac{\mathbf{1}}{\boldsymbol{T}}\frac{\boldsymbol{z}-\mathbf{1}}{\boldsymbol{\alpha}\boldsymbol{z}+\mathbf{1}-\boldsymbol{\alpha}}\boldsymbol{I}-\boldsymbol{A}\right]^{-\mathbf{1}}\boldsymbol{B}+\boldsymbol{D}$$

Recall the tf of a continuous LTI a tempo continuo,

$$G(s) = C(sI - A)^{-1}B + D,$$

the discrete equivalent tf $G^*(z)$,

$$G^*(z) = C\left[\frac{1}{T}\frac{z-1}{\alpha z + 1 - \alpha}I - A\right]^{-1}B + D$$

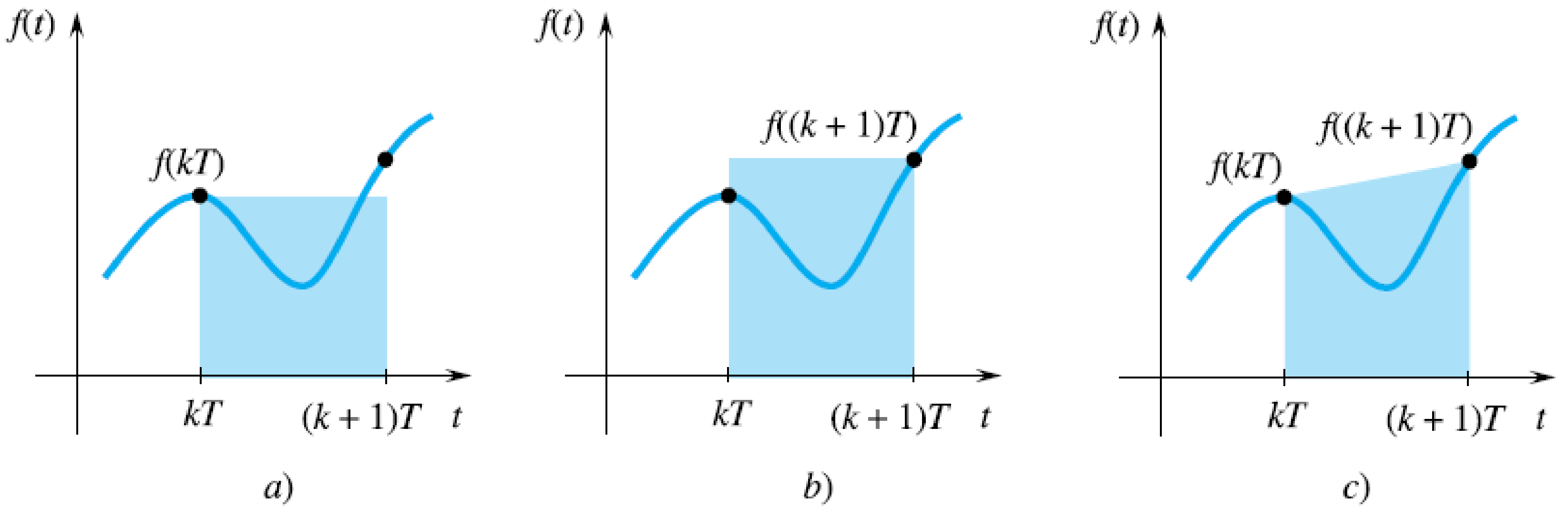
is given by

$$G^*(z) = G\left(\frac{1}{T}\frac{z-1}{\alpha z + 1 - \alpha}\right)$$

by exploiting the following transformation

$$\boldsymbol{s = \frac{1}{T}\frac{z-1}{\alpha z + 1 - \alpha}}$$

Geometric interpretation



Forward rule (Euler's method, α=0)

$$s = \frac{z-1}{T}$$

Backward rule (α=1)

$$s = \frac{z-1}{zT}$$

Trapezoid rule (Tustin, α=0.5)

$$s = \frac{T}{2}\frac{z+1}{z-1}$$

By approximating the differential equation via difference equation – Euler's method

From the definition of a derivative

$$\dot{y} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$$

Even if δt is not quite equal to zero

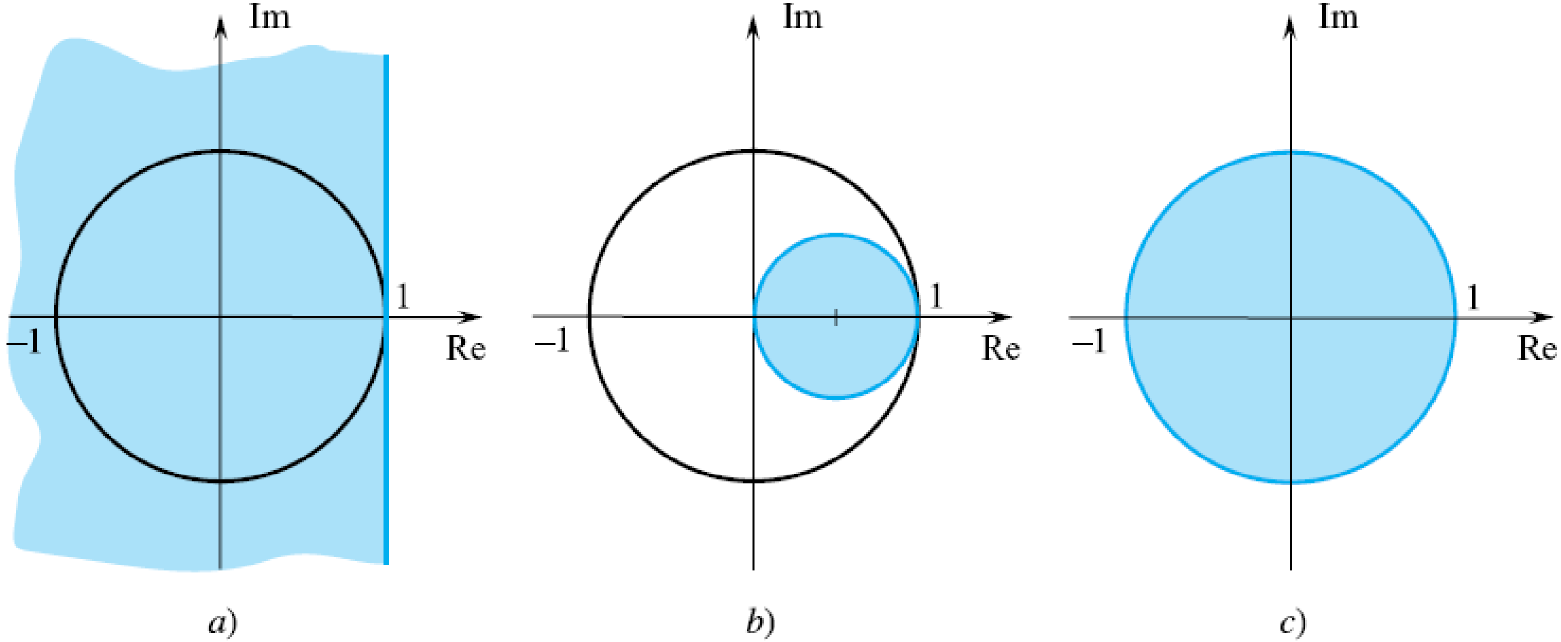
$$\dot{y}(k) = \dot{y}(kT) \cong \frac{y\big((k+1)T\big) - y(kT)}{(k+1)T - kT} = \frac{y(k+1) - y(k)}{T}$$

$\boldsymbol{\mathcal{L}}, \mathbf{Z}$

$$sY(s) \cong \frac{z-1}{T} Y(z)$$

$$s = \frac{z-1}{T}$$

A map from the left-half of the s-plane (s<0) to the z-plane



Forward rule (Euler's method, α=0)

Backward rule (α=1)

Trapezoid rule (Tustin, α=0.5)

Inverse transformation:

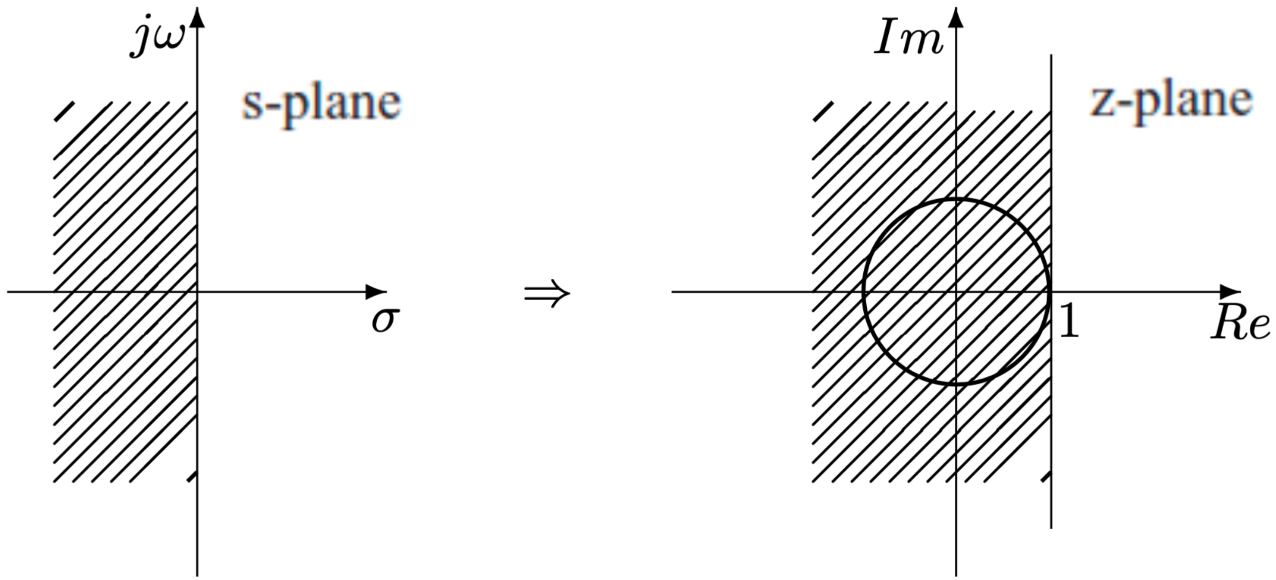
$$z = \frac{1 + (1 - \alpha)\, Ts}{1 - \alpha Ts}$$

Map from s to z: s<0 for forward rule

Forward rule: $z = 1 + sT \leftrightarrow s = \frac{z-1}{T}$

By approximating $z = e^{sT}$

$$z = e^{sT} \cong 1 + \mathrm{sT}$$



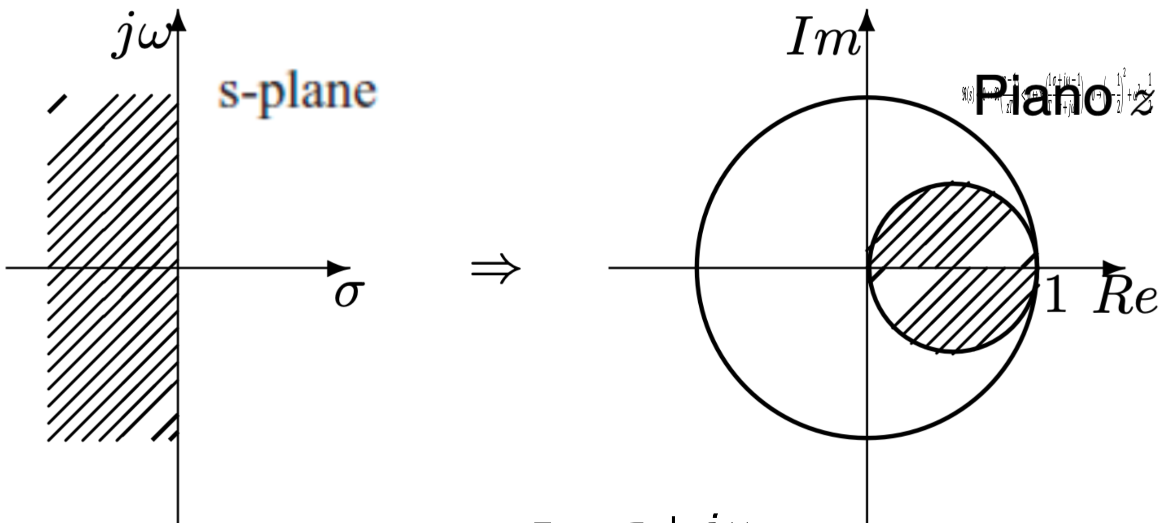
$$\Re(s) < 0 \leftrightarrow \Re\left(\frac{z-1}{T}\right) < 0 \leftrightarrow \Re(z) < 1$$

Then it is possible to achieve unstable $G^*(z)$ from stable $G(s)$.

Map from s to z: s<0 for backward rule

Backward rule: $z = \frac{1}{1-sT} \leftrightarrow s = \frac{z-1}{zT}$

By approximating: $z = e^{sT} = \frac{1}{e^{-sT}} \cong \frac{1}{1-sT}$



$$\Re(s) < 0 \leftrightarrow \Re\left(\frac{z-1}{zT}\right) < 0 \overset{z=\sigma+j\omega}{\leftrightarrow} \Re\left(\frac{1}{T}\frac{\sigma + j\omega - 1}{\sigma + j\omega}\right) < 0 \rightarrow \left(\sigma - \frac{1}{2}\right)^2 + \omega^2 < \left(\frac{1}{2}\right)^2$$

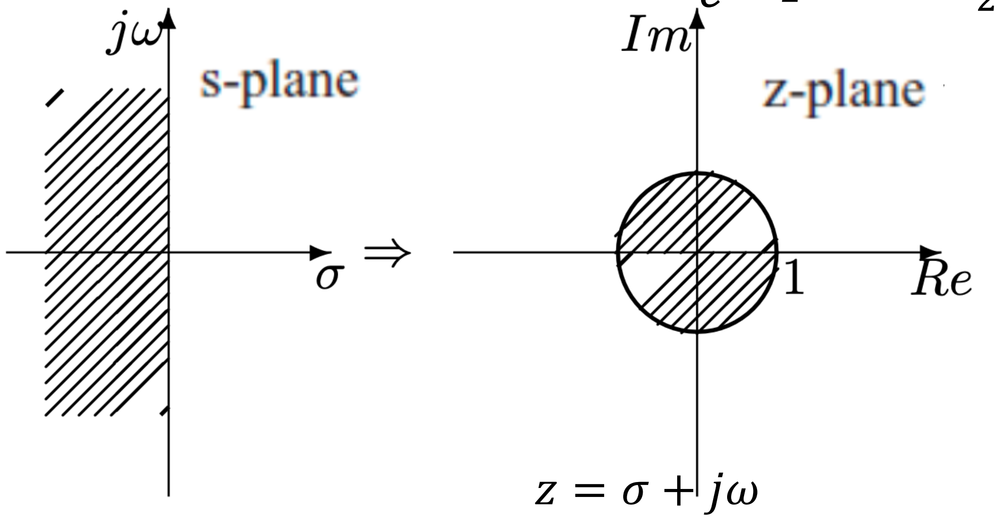
All z points inside the radius circle with r=½ and center (½, 0).

Stable $G(s)$ systems correspond to stable $G^*(z)$ system. However, there is the chance to achieve stable $G^*(z)$ systems from unstable $G(s)$.

Map from s to z: s<0 for Tustin

Tustin: $z = \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}} \leftrightarrow s = \frac{2}{T}\frac{z-1}{z+1}$

By approximating: $z = e^{sT} = \frac{e^{s\frac{T}{2}}}{e^{-s\frac{T}{2}}} \cong \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}$



$$z = e^{j\omega T}\Big|_{\omega=\frac{\omega_s}{2}} = -1$$

$$z \cong \frac{2j\omega - 1}{2j\omega + 1}\Big|_{\omega=\frac{\omega_s}{2}} = \begin{cases} |z| = 1 \\ \arg(z) = 115^{o} \end{cases}$$

i.e. frequency compression

$$\Re(s) < 0 \leftrightarrow \Re\left(\frac{2}{T}\frac{z-1}{z+1}\right) < 0 \leftrightarrow \Re\left(\frac{2}{T}\frac{\sigma + j\omega - 1}{\sigma + j\omega + 1}\right) < 0 \rightarrow \sigma^2 + \omega^2 < 1$$

➡ All z points inside the radius circle with r=1 and center (0, 0).

The stability of the system is preserved: stable systems $G(s)$ in continuous time are transformed into stable systems $G^*(z)$ in discrete time (and vice versa)

Frequency behavior– Tustin

$$G^*(z)|_{z=e^{j\omega T}} \cong G(s)|_{s=j\omega}$$

By using **Tustin:**

$$G^*(z) \cong G(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

In terms of frequency:

$$G^*\left(z = e^{j\omega T}\right) \cong G\left(\frac{2}{T}\frac{e^{j\omega T}-1}{e^{j\omega T}+1}\right) = G\left(j\frac{2}{T}\tan\frac{\omega T}{2}\right)$$

Then

$$G^*\left(e^{j\omega T}\right) \cong G(j\omega) \leftrightarrow j\frac{2}{T}\tan\frac{\omega T}{2} = j\omega \text{ iff } \frac{\omega T}{2} \ll 1 \leftrightarrow \omega \ll \frac{\omega_s}{8}$$

$$\text{with } \omega_s = \frac{2\pi}{T}$$

Tustin/ bilinear transformation with prewarping

If we want that at a given frequency ω_1

$$\boldsymbol{G}^*\left(e^{j\omega_1 T}\right) = G(j\omega_1),$$

then it is sufficient to employ this transformation

$$\boldsymbol{G}^*\left(e^{j\omega_1 T}\right) = G\left(\frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\frac{z-1}{z+1}\right)$$

where

$$s = \frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\frac{z-1}{z+1}$$

represents the bilinear transformation with prewarping.

zeta-Transform method: hold equivalent – impulse invariant discretization

This discretization method allows maintaining unchanged the impulse response of the discrete equivalent $\boldsymbol{G}^*(\boldsymbol{z})$ of the continuous-time system $\boldsymbol{G}(\boldsymbol{s})$.

By definition $\boldsymbol{G}^*(\boldsymbol{z})$ is the zeta-Transform of output sequence y_δ^* in response to the unit pulse $\delta(k)$.

$$\boldsymbol{G}^*(\boldsymbol{z}) = Z(y_\delta^*) = \mathrm{Z}\left(\mathcal{L}^{-1}\left(G(s)\right)\Big|_{t=KT}\right)$$

$\boldsymbol{G}^*(\boldsymbol{z})$ is given by the the zeta-Transform of the response to the ideal pulse $\boldsymbol{G}(\boldsymbol{s})$ ($y_\delta(t) = \mathcal{L}^{-1}(G(s))$) sampled at multiple instants of the sampling period T, $y_\delta^*(k) = y_\delta(KT)$.

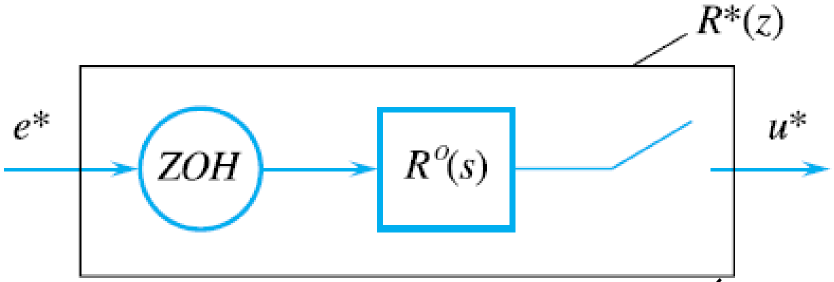
Given the continuous LTI system defined by the transfer function $\boldsymbol{G(s)}$,

$$\boldsymbol{G(s) = \frac{1}{s(s+1)}}$$

determine the discrete equivalent $\boldsymbol{G^*(z)}$ by using zeta-Transform method

Solution:

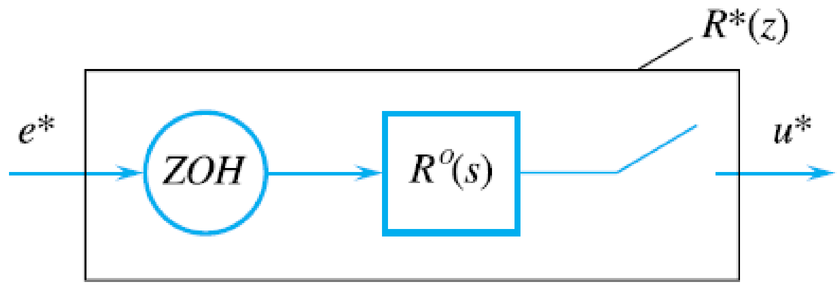
1. $y_\delta(t) = \mathcal{L}^{-1}(G(s)) = 1(t) - e^{-t}1(t) = (1 - \ e^{-t})1(t)$
2. $y_\delta^*(kT) = \left(1 - \ e^{-kT}\right)1(kT)$
3. $G^*(z) = Z(y_\delta^*) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z\left(1-e^{-T}\right)}{(z-1)(z-e^{-T})}$



$$R^*(z) = U_\delta^*(z) = \frac{z-1}{z}U(z) = \frac{z-1}{z}\mathrm{Z}\left(\mathcal{L}^{-1}\left(\frac{R^o(s)}{s}\right)\Bigg|_{t=KT}\right)$$

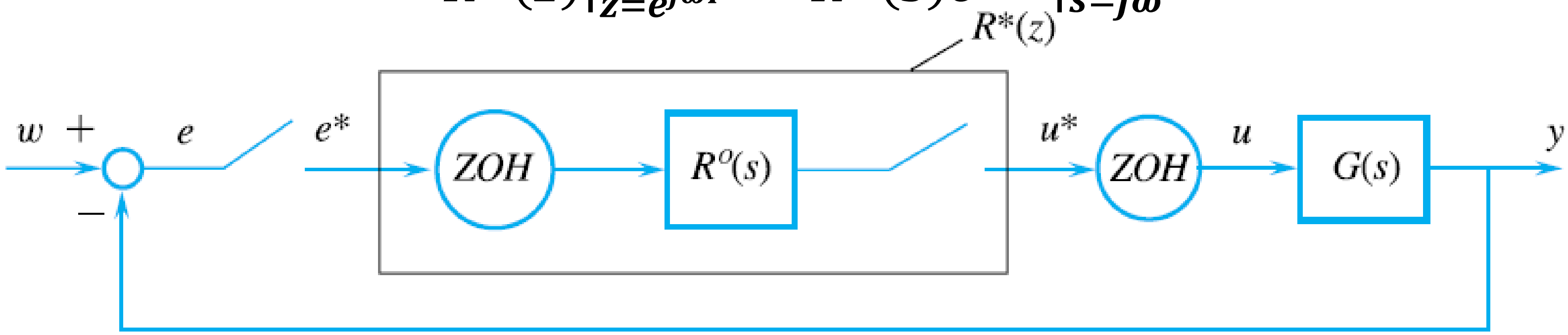
Summarizing, by this procedure it is possible to obtain the discrete tf of the digital controller, $R^*(z)$:

1. **Determine the step response of the continuous controller in the Laplace domain, $U_s(s)= \frac{R^o(s)}{s}$.**
2. **Antitransform $U_s(s)$ thereby determining the samples of the controller output $u_s(kT)$.**
3. **Compute the z-transform of the output samples $u_s(kT)$: $\boldsymbol{Z}\big(u_s(kT)\big)$**
4. **Determine the tf of $R^*(z)= \frac{z-1}{z}\boldsymbol{Z}\big(u_s(kT)\big)$**



$R^o(s)$, analog controller - **$R^*(z)$**, digital controller by sampled-data model (ZOH method)

$$R^*\,(z)|_{z=e^{j\omega T}} \cong R^*\,(s)e^{-\frac{sT}{2}}|_{s=j\omega}$$



i.e., double pair of sampler and hold devices

Zero-pole matching equivalents

The technique consists of a set of heuristic rules for locating the zeros and poles according to the sampling transformation

The discrete equivalent $G^*(z)$ can be obtained as it follows:

1. the transformation of the individual poles and zeros is carried out using the sampling transformation $\boldsymbol{z = e^{sT}}$;

2. introduce as many zeros into $\boldsymbol{z = -1}$ as there are poles of $G(s)$ in excess of the finite zeros;

3. the static gain is compensated.

Zero-pole matching equivalents - Example

Example: $G(s) = \dfrac{\mathbf{10(s+5)}}{\mathbf{(1+10s)\,(s+1)}}$

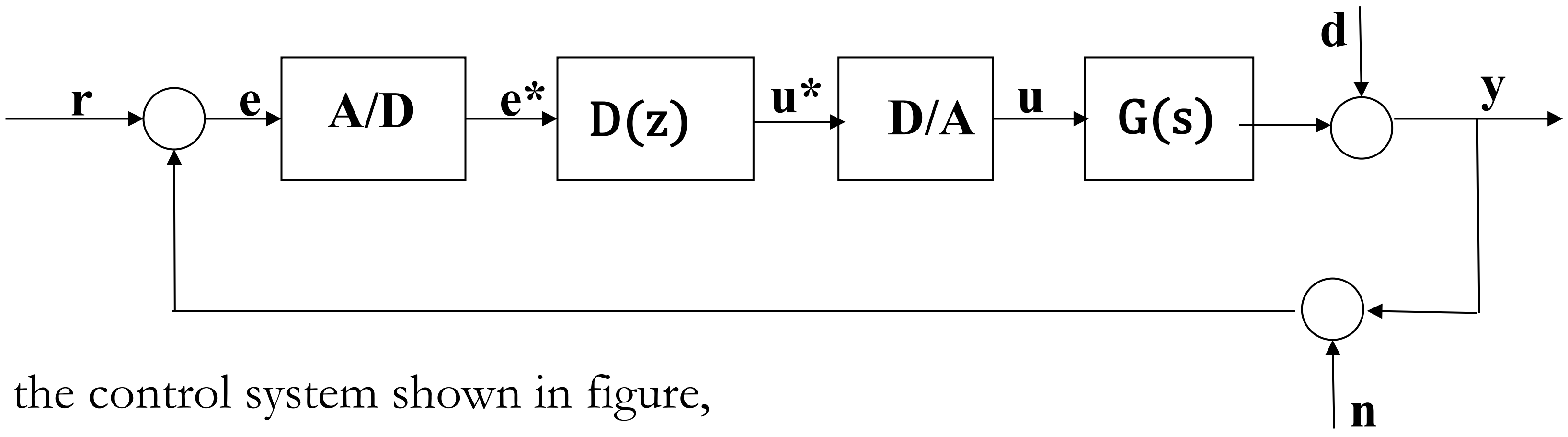
For the zero in s $= -5$, $(s+5)$ ⇒ $\left(1 - \dfrac{e^{-5T}}{z}\right)$

$n - m = 1$ ⇒ **One zero in** $z = -1$

⇒ $$G^*(z) = k\,\frac{(z+1)(z-e^{-5T})}{(z-e^{-T})(z-e^{-0.1T})}$$

$$G^*(1) = 2k\,\frac{(1-e^{-5T})}{(1-e^{-T})(1-e^{-0.1T})} = G(0) = 50$$

$$k = \frac{50}{2}\frac{(1-e^{-T})(1-e^{-0.1T})}{(1-e^{-5T})}$$



For the control system shown in figure, where

$$G(s) = \frac{1}{s(s+1)},$$

design a digital control D(z) by emulation of a continuous design (i.e. by computing the discrete equivalent using Tustin) in order to satisfy the following requirements

- $e_\infty = 0$ wrt to a step disturbance d
- $s \leq 15\%$
- $t_{a5\%} < 300$ ms
- $T = 30$ ms

Discuss the action to be implemented for reducing the effect of high-frequency noise n (i.e., $n_1(t) = 0.1\sin(400t), n_2(t) = 0.1\sin(500t)$)

- $e_\infty = 0$ wrt to a step disturbance d
  $\Longrightarrow$ one integrator in the open loop function $F(s)$ (i.e., one pole in zero)
- $s \leq 15\%$
  $\Longrightarrow$ $\zeta \geq 0.5$, where ζ is the damping factor of the closed-loop system ($\varphi_m > 50°$)
- $t_{a5\%} < 300$ ms
  $\Longrightarrow \frac{3}{\zeta\omega_n} < \frac{3}{10} \Longrightarrow \begin{cases} \zeta\omega_n > 10 \\ \omega_c \cong \omega_n \end{cases} \Longrightarrow \omega_c > 20\,\text{rad/s}$, where ω_c is the crossing frequency of the open-loop function, $F(s)$, and ω_n is the natural frequency of the second order approximation of the closed loop system.
- $T = 30$ ms ($\omega_s = \frac{2\pi}{T}$)
  $\Longrightarrow$ delay at $\omega_c \Longrightarrow$ in terms of phase, $-\frac{\omega_c T}{2} \Longrightarrow \varphi_m > 50° + \left(\frac{\omega_c T}{2}\right)°$

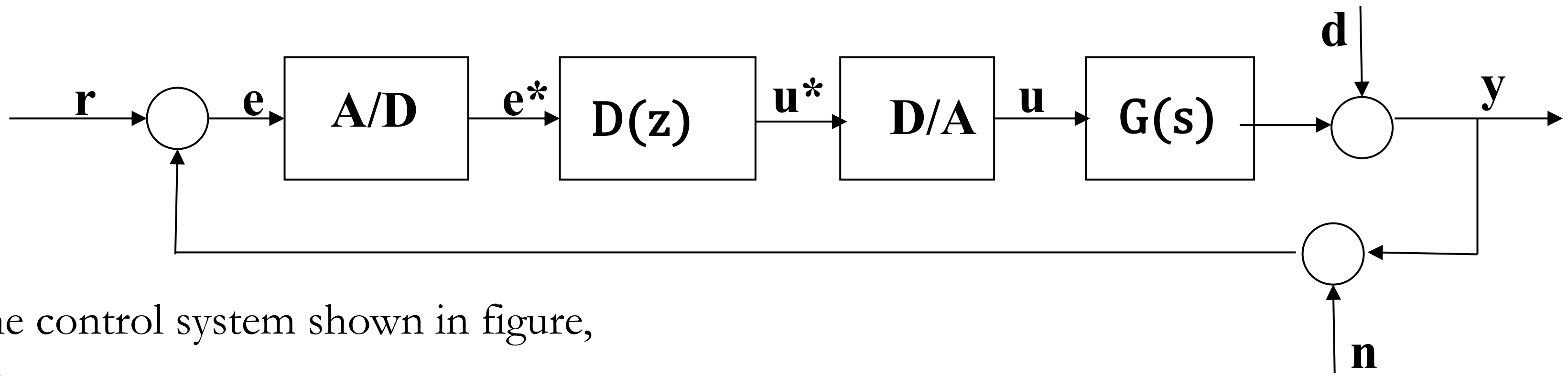
- See the relative Matlab code and the schemes implemented in Simulink (included in *Matlab_Simulink* folder):
  - design_control_system_analog_vs_digital.m
  - feedback_control_scheme_analog_vs_digital.slx

Problem - 1

Discuss the action to be implemented for reducing the effect of high-frequency noise n (i.e., $n_1(t) = 0.1\sin(400t), n_2(t) = 0.1\sin(500t)$)

$\Longrightarrow$ anti-aliasing filter with $\boldsymbol{\omega_f} < \frac{\omega_s}{2}$ and $\boldsymbol{\omega_f} \gg \omega_c$

Problem - 2



For the control system shown in figure, where

$$G(s) = \frac{1}{(s+10)},$$

the following continuous controller has been designed:

$$C(s) = \frac{10(s+10)}{s(s/50+1)}$$

Then, a digital controller has been implemented by discretization using ZOH method with $T = 50$ ms.

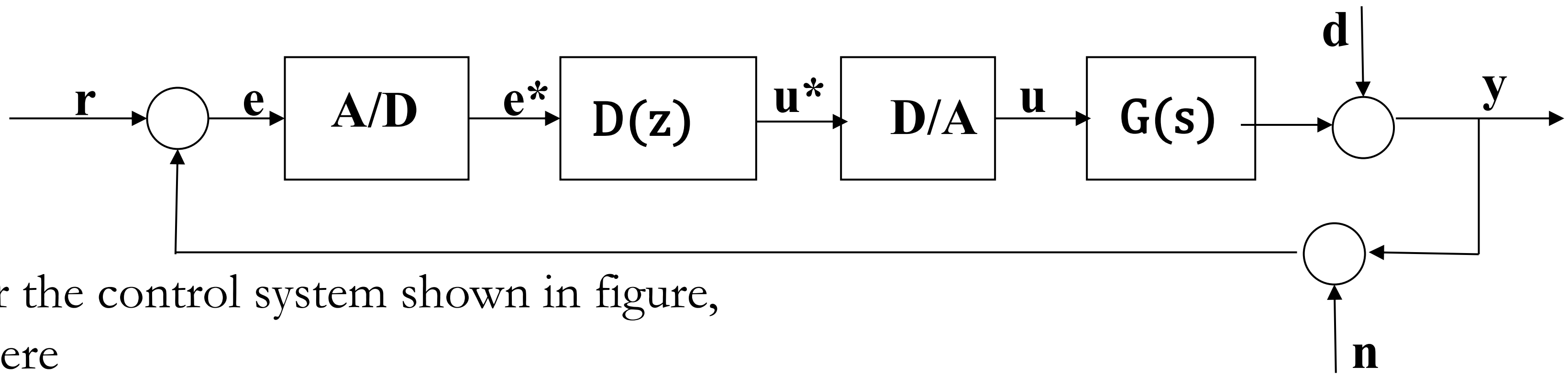
- Evaluate the performance achieved by the continuous controller
- Evaluate the performance achieved by the digital controller
- By assuming a high-frequency noise n (i.e., $n_1(t) = 0.5\sin(400t)$), compare the performance obtained by the analaog and digital controllers
- For the digital controller discuss the action to be implemented for reducing the effect of n.
- Evaluate the performance of the digital controller by using $T = 25$ ms.

Problem - 2

- Implement the controller: give the difference equation that corresponds to D(z) for both the values of T

- See the relative Matlab code and the schemes implemented in Simulink (included in *Matlab_Simulink* folder):
  - design_control_system_analog_vs_digital_pr2.m
  - feedback_control_scheme_analog_vs_digital.slx

Problem - 3



For the control system shown in figure, where

$$G(s) = \frac{1}{(s+2)},$$

a. Design a digital control D(z) by emulation of a continuous design (i.e. by computing the discrete equivalent using ZOH and/or Tustin)

- by setting opportunely the sampling time T,
- in order to satisfy the following requirements:
  - $e_\infty = 0$ to a step disturbance d
  - $e_\infty \leq 0.1$ wrt a reference ramp signal $r(t)$ of slope 0.5.
  - $s \leq 20\%$ to a step input r
  - $t_{a5\%} < 1s$
  - attenuation factor ≥ 20 dB for multi-frequency noise in the range $[50 + \infty]$ rad/s

Problem - 3

b. Discuss the action to be implemented for reducing the effect of high-frequency noise n (i.e., $n_1(t) = 0.2\sin(50t), n_2(t) = 0.2\sin(100t)$)

c. Implement the controller: give the difference equation that corresponds to D(z) for both cases (Tustin and ZOH)

For the continuous design:

- $e_\infty = 0$ wrt to a step disturbance d
  $\Longrightarrow$ one integrator in the open loop function $F(s) = C(s)G(s)$ (i.e., one pole in zero)
  Then $C(s) = \frac{k_0}{s}$
- $e_\infty \leq 0.1$ to a ramp signal of slope R_0 equal to 0.5, $r(t) = 0.5t\,1(t)$ (R_0=0.5)
  $\Longrightarrow e_\infty = \frac{R_0}{F_0}$, with $F_0 = k_0 G(0) = \frac{k_0}{2} \Longrightarrow e_\infty = \frac{R_0}{F_0} = \frac{\frac{1}{2}}{\frac{k_0}{2}} \leq \frac{1}{10} \Longrightarrow k_0 \geq 10$
- $s \leq 20\%$
  $\Longrightarrow$ $\zeta \geq 0.45$, where ζ is the damping factor of the closed-loop system ($\varphi_m > 45°$)
- $t_{a5\%} < 1$ s
  $\Longrightarrow \frac{3}{\zeta\omega_n} < 1 \Longrightarrow \begin{cases} \zeta\omega_n > 3 \\ \omega_c \cong \omega_n \end{cases} \Longrightarrow \omega_c > 6.6$ rad/s, where ω_c is the crossing frequency of the open-loop function, $F(s)$, and ω_n is the natural frequency of the second order approximation of the closed loop system $\Longrightarrow C(s) = \frac{k_0}{s}\frac{(s+1)}{(\frac{s}{30}+1)}$
  - attenuation factor ≥ 20dB for noise in the range $[50 + \infty]$ rad/s

Problem - 3

See the relative Matlab code and the schemes implemented in Simulink (included in *Matlab_Simulink* folder):

- design_control_system_analog_vs_digital_pr3.m
- feedback_control_scheme_analog_vs_digital_pr3.slx