

Course of "Industrial Automation" 2024/25

Analysis of the Sample and Hold -Sampled-Data Systems

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Continuous vs. digital







Scheme of the digital control system in continuous-time







- The digital controller is a discrete-time system and the plant to be controlled is a continuous-time system.
- It is needed a device that transforms a continuous signal into a discrete one.



• Such device is the analog-to-digital converter (A/D).



Ideal sampler







- Periodic sampling: the sampling instants are equally spaced, or k, i.e. t_k=kT_s (k=0,1,2,..), with T_s representing the sampling time.
- The hold circuit holds the value of the sampled signal over a specified period of time.



• The common problem when sampling a signal is the loss of information.

Indeed, it is obvious that the same signal e*(k) can be generated by infinite continuous-time functions e(t).

Hence, given a signal e^{*}(k) it is impossible to go back to the original signal e(t).



Sampling operation - example

Given a set of signals, $e_h(t) = \sin((\alpha + 2h)\pi t), \alpha \in \mathbb{R}, h \in \mathbb{Z}$



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Ideal sampler – Spectrum of a sampled signal



Assume $e_s(t)$ as the mathematical representation of the sampling operation:

$$\mathcal{F}_{(\text{Fourier transform})} = e(t) \sum_{k=0}^{\infty} \delta(t - kT_s) = \sum_{k=0}^{\infty} e(kT_s) \delta(t - kT_s)$$
(Fourier transform)

Ideal sampler – Spectrum of a sampled signal



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Ideal sampler – Spectrum of a sampled signal



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In order for an analog signal (e(t)) to be reconstructed from its sampled version $(e^*(k))$, by Shannon's theorem, it must have a strictly limited bandwidth and $\omega_s > 2\omega_{max}$



 $e_0(t) = \sin(0.6\pi t)$ $e_{-1}(t) = \sin(-1.4\pi t)$ $|E_0(j\omega)|$ Aliasing π -0.6π 0.6π ω a) $|E_0^*(j\omega)| = |E_{-1}^*(j\omega)|$ π $-5.4\pi - 4.6\pi - 3.4\pi - 2.6\pi - 1.4\pi - 0.6\pi$ 0.6\epsilon 1.4\epsilon 2.6\epsilon 3.4\epsilon 4.6π 5.4π ω b)





In this way the spectrum of the signal to be sampled doesn't have significant components for $\omega > \omega_f$

$$\omega_f < \omega_N = \frac{\omega_s}{2}$$
$$\omega_s > 2\omega_f$$



Taking into accout the bandwidth of the control system, approximated by ω_c , then for Shannon's Theorem

$$\omega_s > 2\omega_c$$





The antialiasing filter can reduce the phase margin by destabilizing the entire control system, then

 $\omega_f \gg \omega_C$

By taking into account all the constraints

 $\omega_s > 2\omega_f \gg \omega_c$

In general $\boldsymbol{\omega}_{\boldsymbol{s}}$ is given by

$$\alpha\omega_{\mathcal{C}} < \omega_{s} < 10\alpha\omega_{\mathcal{C}}$$

with $5 \leq \alpha \leq 10$





- It is a device that transforms a digital input (binary numbers) to an analog output.
- The most commonly used D/A converter is the zero order hold (ZOH), which operates as follows:

$$u(t) = u^*(k) \quad t \in \left[kT_m, (k+1)T_m\right]$$

• T_m is the sample time



ZOH circuit





ZOH - 1



$$\omega_m = 2\pi f_m = \frac{2\pi}{T_m}$$
$$u(t) = u^*(k) \quad t \in [kT_m, (k+1)T_m]$$

Relationship in terms of transform between the input samples $u^*(k)$ and the output signal u(t) ($T = T_m$)





ZOH - 2

In general, for an input sequence $u^*(k)$ with zeta-Transform $U^*(z)$, the corresponding output signal

$$u(t) = \sum_{k=0}^{\infty} u^{*}(k) h_{0}(t-kT) \quad \square \qquad U(s) = \sum_{k=0}^{\infty} u^{*}(k) e^{-skT} H_{0}(s) = H_{0}(s) U^{*}(e^{sT}).$$

 $H_0(s) = \frac{U(s)}{U^*(e^{sT})}, \text{ it can be assumed as the tf of ZOH}$

Moreover, for $s = j\omega$, the ZOH frequency response can be evalueted

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = e^{-j\omega T/2} \frac{e^{-j\omega T/2} - e^{-j\omega T/2}}{2j} 2j = T e^{-j\omega T/2} \frac{\sin \omega T/2}{\omega T/2}$$



ZOH - 3

Frequency response of **ZOH**: Bode Diagrams of $H_0(j\omega)/T$ in linear and log. scale ($\omega_N = \pi/T$)





Sampler – ZOH series

• By working in the frequency range $\omega < \frac{\omega_s}{8}$, it is possible to approximate the sampler-zoh series (hp $T_s = T_m = T$) with a delay element



- where this term introduces a maximum delay equal to
- The presence of a numerical control tends to destabilize the entire system.



Scheme of the digital control system in continuous-time







From analog to digital

Implementation of the digital control:



- From C(s) we want to find an equivalent D(z):
- A transformation allows the transition from continuous time to discrete time such that $C(s) \cong D(z)$

same static and dynamic perfomance $s \rightarrow z$

• Several transformations:

The Laplace trasform of an ideally sampled signal corresponds to the Zeta of the sampled sequence with the sostitution

$$z = e^{sT}$$

By a first-order Padé approximant of the natural logarithm function, 2z-1

thers: By Euler's method,
$$s = \frac{z}{T} \frac{z}{z+1}$$

$$s = \frac{z-1}{T}$$
 (forward rectangular rule) and $s = \frac{z-1}{zT}$ (backward)

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Sampled -data system - 1

Here, we develop the analysis needed to compute the discrete transfer function between the samples that come from the digital computer to the D/A converter and the samples picked up by A/D converter. $G^*(z)$



This scheme represents a discrete time system describing the continuous time systems at the sampling times (it describes the relationship between u^* and y^*)

The discrete tf, $G^*(z)$, corresponds to the z-Transform of the output samples $y^*_{\delta}(Y^*_{\delta}(z))$ when the input samples are the unit pulse $\delta(k)$ at k = 0:

$$u_{\delta}^*(k) = \delta(k) \to U_{\delta}^*(z) = Z(\delta(k)) = 1 \to \mathbf{G}^*(\mathbf{z}) = \frac{Y_{\delta}^*(z)}{U_{\delta}^*(z)} = Y_{\delta}^*(z)$$



Sampled -data system - 2

 $\xrightarrow{u^*} \underbrace{ZOH}_{u} \qquad \text{For } u^*(k) = \delta(k) \rightarrow u(t) = h_0(t) = 1(t) - 1(t - T)$ $\stackrel{\mathcal{L}}{\longleftarrow} \qquad H_0(s) = \frac{1 - e^{-sT}}{s}$

Then $y_{\delta}(t)$ is the difference between the step response of G(s) (i.e., response to 1(t), denoted with $y_s(t)$) and the delayed (T) step response, denoted with $y_s(t-T)$. Thus, by applying Laplace,

$$Y_{\delta}(s) = H_0(s)G(s) = \frac{1 - e^{-sT}}{s}G(s) = (1 - e^{-sT})\frac{G(s)}{s} = (1 - e^{-sT})Y_s(s)$$

$$\mathcal{L}^{-1} \longrightarrow y_{\delta}(t) = y_s(t) - y_s(t - T)$$

By sampling: $y_{\delta}^*(kT) = y_s(kT) - y_s((k-1)T)$ And by applying z-transform $(Y(z) = Z(y_s(KT)))$: $Y_{\delta}^*(z) = Y(z) - \frac{Y(z)}{z} = \frac{z-1}{z}Y(z)$



Summarizing, by this procedure it is possible to obtain the discrete tf of the sampled-data system, $G^*(z)$:

- 1. Determine the step response of the continuous LTI system in the Laplace domain, $Y_s(s) = \frac{G(s)}{s}$.
- 2. Antitransform $Y_s(s)$ thereby determining the samples of the output $y_s(kT)$.
- 3. Compute the z-transform of the output samples $y_s(kT)$: $Z(y_s(kT))$
- 4. Determine the tf of $G^*(z) = \frac{z-1}{z} Z(y_s(kT))$



What is the discrete transfer function of

$$G(s)=\frac{1}{s(s+1)},$$

preceded by a ZOH?

Solution:

1.
$$Y(s) = \frac{G(s)}{s} = \frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

2. $y_s(kT) = \mathcal{L}^{-1}(Y(s))|_{t=KT} = (t-1+e^{-t})1(t)|_{t=KT} = (kT-1+e^{-kT})1(kT)$
3. $Z(y_s(kT)) = \frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}$
4. $G^*(z) = \frac{z-1}{z} \left(\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right) = \frac{T}{z-1} - 1 + \frac{z-1}{z-e^{-T}}$