



Course of "Industrial Automation"

Discrete-Time LTI systems analysis – part 1

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli Studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **5qf4mll**

✧ Let's assume the LTI time discrete system:

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k), \quad x(k_0) = x_0 \\y(k) &= Cx(k) + Du(k)\end{aligned}\tag{1}$$

✧ The solution of the linear difference equation (1) defines the *time evolution of the state variables* and it is given

$$x(k) = A^{k-k_0}x_0 + \sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h), \quad k \geq k_0\tag{2}$$

✧ The time evolution of the outputs is

$$y(k) = CA^{k-k_0}x_0 + C \sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1}Bu(h) + D u(k), \quad k \geq k_0\tag{3}$$

- ✧ The *time evolution of the state and output variables* can be conceptually divided in two parts:

$$\begin{array}{ccc}
 x(k) = & A^{k-k_0} x_0 & + \sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1} B u(h), \quad k \geq k_0 \\
 & \underbrace{\hspace{10em}} & \underbrace{\hspace{10em}} \\
 & \textit{Free evolution} & \textit{Forced evolution} \\
 & \underbrace{\hspace{10em}} & \underbrace{\hspace{10em}} \\
 y(k) = & C A^{k-k_0} x_0 & + C \sum_{h=k_0}^{k-k_0-1} A^{k-k_0-h-1} B u(h) + D u(k), \quad k \geq k_0
 \end{array}$$

- ✧ The *free evolution* indicate the evolution of state and output vectors that would be obtained in the absence of input ($u(t) = 0$).
- ✧ The *forced evolution* indicate the evolution of state and output vectors that would be obtained in the presence of input and null initial conditions ($x_0 = 0$).



Free evolution

- ✧ The free evolution can be achieved by setting in (1) $u(k) = 0$

$$\begin{aligned}x(k+1) &= Ax(k), \quad x(k_0) = x_0 \\y(k) &= Cx(k)\end{aligned}$$

- ✧ By iterations, $k = k_0, k = k_0 + 1, \dots, k = k_0 + n$, it follows

$$\begin{aligned}x(k_0 + 1) &= Ax(k_0) = Ax_0 \\x(k_0 + 2) &= Ax(k_0 + 1) = A^2x_0 \\&\dots \\x(k_0 + n) &= A^n x_0\end{aligned}$$

- ✧ The solution for a general k is given by

$$x(k) = A^{k-k_0} x_0$$



Free evolution: matrix 'A' diagonalizable

- ✧ The free evolution of a discrete-time LTI system is determined by A^k
- ✧ In case the matrix A has real and distinct eigenvalues, it is diagonalizable and A^k

$$A^k = U \Lambda^k U^{-1} = U \operatorname{diag}\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\} V$$

where $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigenvalues of the A matrix, U is eigenvector matrix and $V = U^{-1}$ is the left eigenvector matrix.

- ✧ The free evolution of a discrete LTI system when the matrix A is diagonalizable turns out to be

$$\begin{aligned} A^k x_0 &= U \operatorname{diag}\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\} V x_0 \\ &= (u_1 \quad \dots \quad u_n) \begin{pmatrix} \lambda_1^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n^k \end{pmatrix} \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} x_0 \\ &= \sum_{i=1}^n \lambda_i^k u_i v_i^T x_0 \\ &= \sum_{i=1}^n \lambda_i^k u_i c_i \end{aligned}$$

An aperiodic modes

where the coefficient $c_i \in \mathbb{R}$ are the projection of the initial state x_0 on the eigenvector u_i .

Aperiodic evolution modes (1/4)

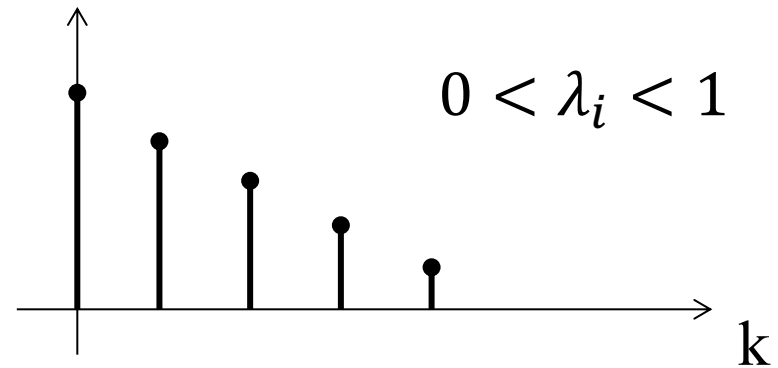
- ✧ *An aperiodic mode* is an evolution mode of a linear system related to a real eigenvalue of the matrix A of multiplicity 1. It can be written in the form

$$c_i \lambda_i^k u_i$$

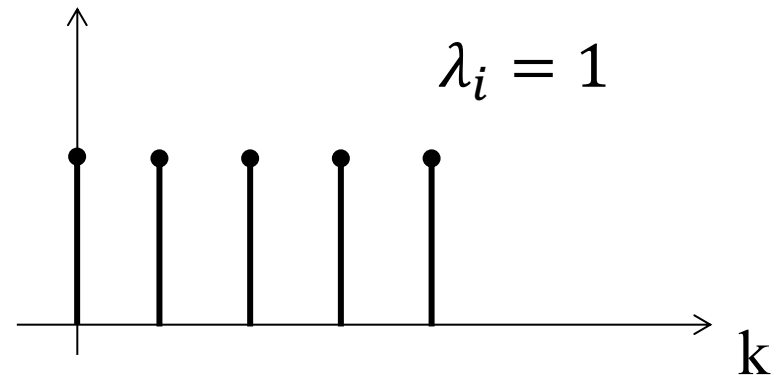
- ✧ It gives us the evolution of the state along the direction defined by the eigenvector u_i starting from an initial value c_i (projection of the initial state x_0 on the eigenvalue u_i).
- ✧ Depending on the value of the eigenvalue λ_i , an aperiod evolution modes can be
- ✧ convergent for $|\lambda_i| < 1$;
 - ✧ constant for $|\lambda_i| = 1$;
 - ✧ divergent for $|\lambda_i| > 1$.

Aperiodic evolution modes (2/4)

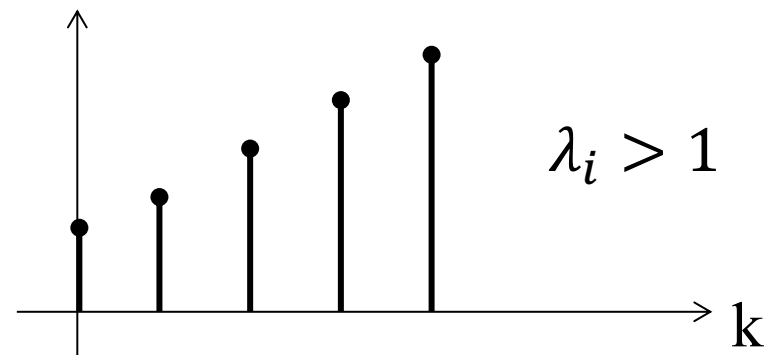
✧ *Convergent aperiodic mode*



✧ *Constant aperiodic mode*

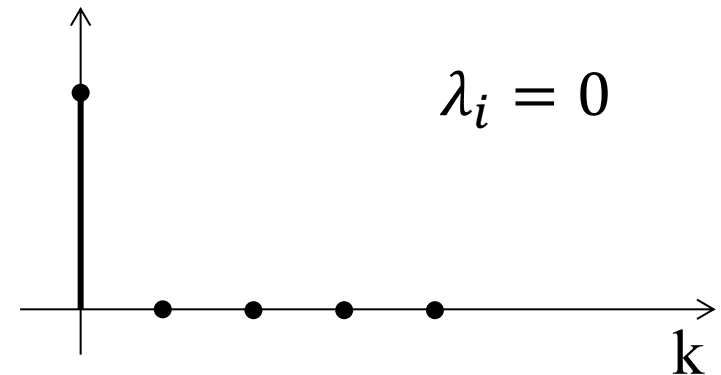


✧ *Divergent aperiodic mode*



Aperiodic evolution modes (3/4)

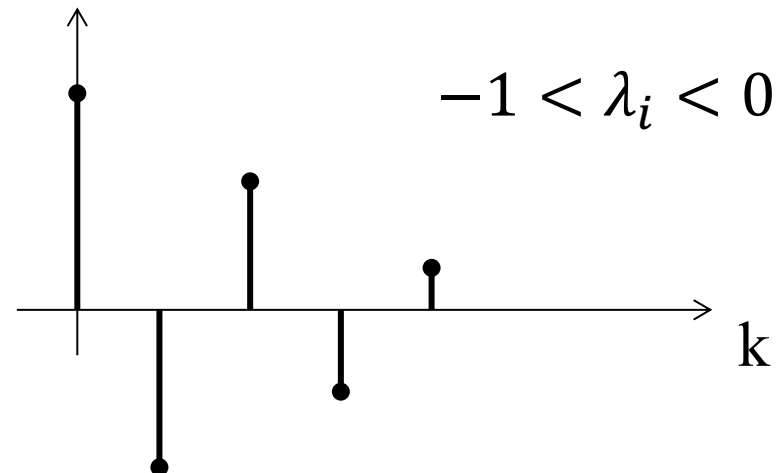
✧ *Convergent mode in one step*



✧ In the case of negative eigenvalues, the mode exhibits an alternating behavior

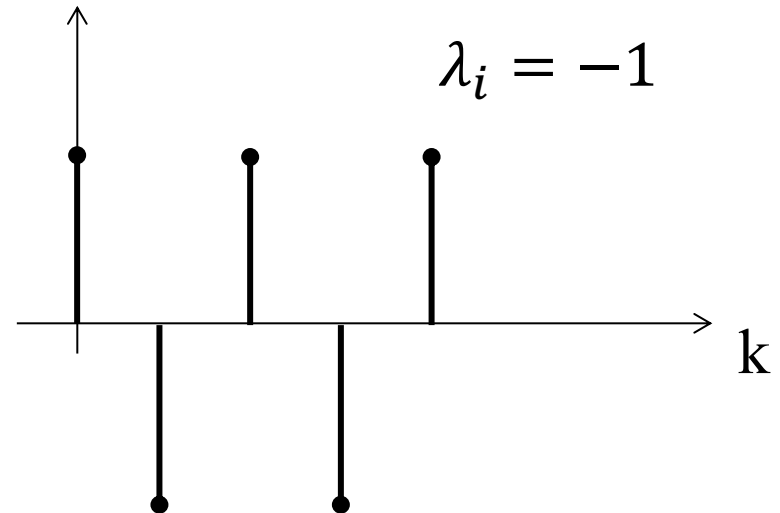
$$\lambda^k = (-|\lambda|)^k = (-1)^k |\lambda|^k$$

✧ *Aperiodic convergent mode with oscillation behavior*

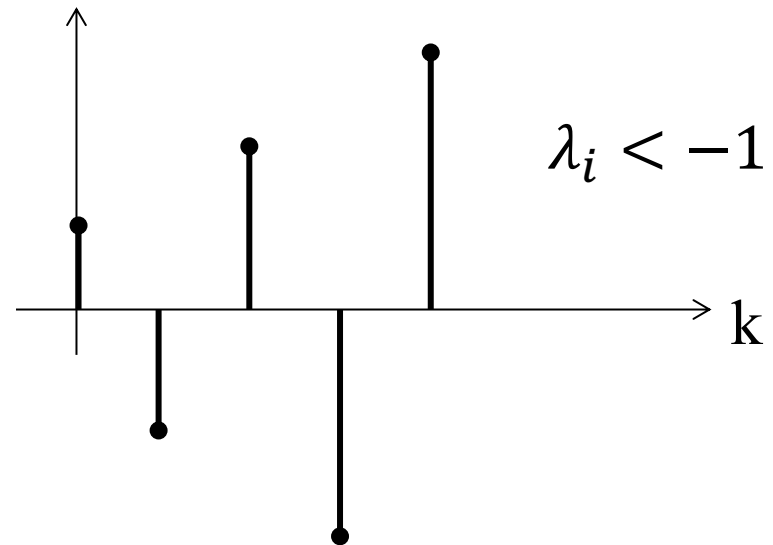


Aperiodic evolution modes (4/4)

✦ *Constant mode with oscillation behavior*



✦ *Divergent aperiodic mode with oscillation behavior*





Free evolution: complex eigenvalues

- ✧ In the case of complex conjugate eigenvalues $\rho_1 e^{\pm j\theta_1}$, $\rho_2 e^{\pm j\theta_2}$, ... $\rho_\nu e^{\pm j\theta_\nu}$, complex conjugate eigenvalues, the free evolution exhibits a pseudo-periodic mode
- ✧ Convergent iff $\rho_i < 1$.

Pseudo-periodic evolution modes

✧ *Convergent pseudo-periodic mode*

✧ *Constant pseudo-periodic mode*

✧ *Divergent pseudo-periodic mode*

