Course of “Industrial Automation”
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Discrete systems analysis: introduction

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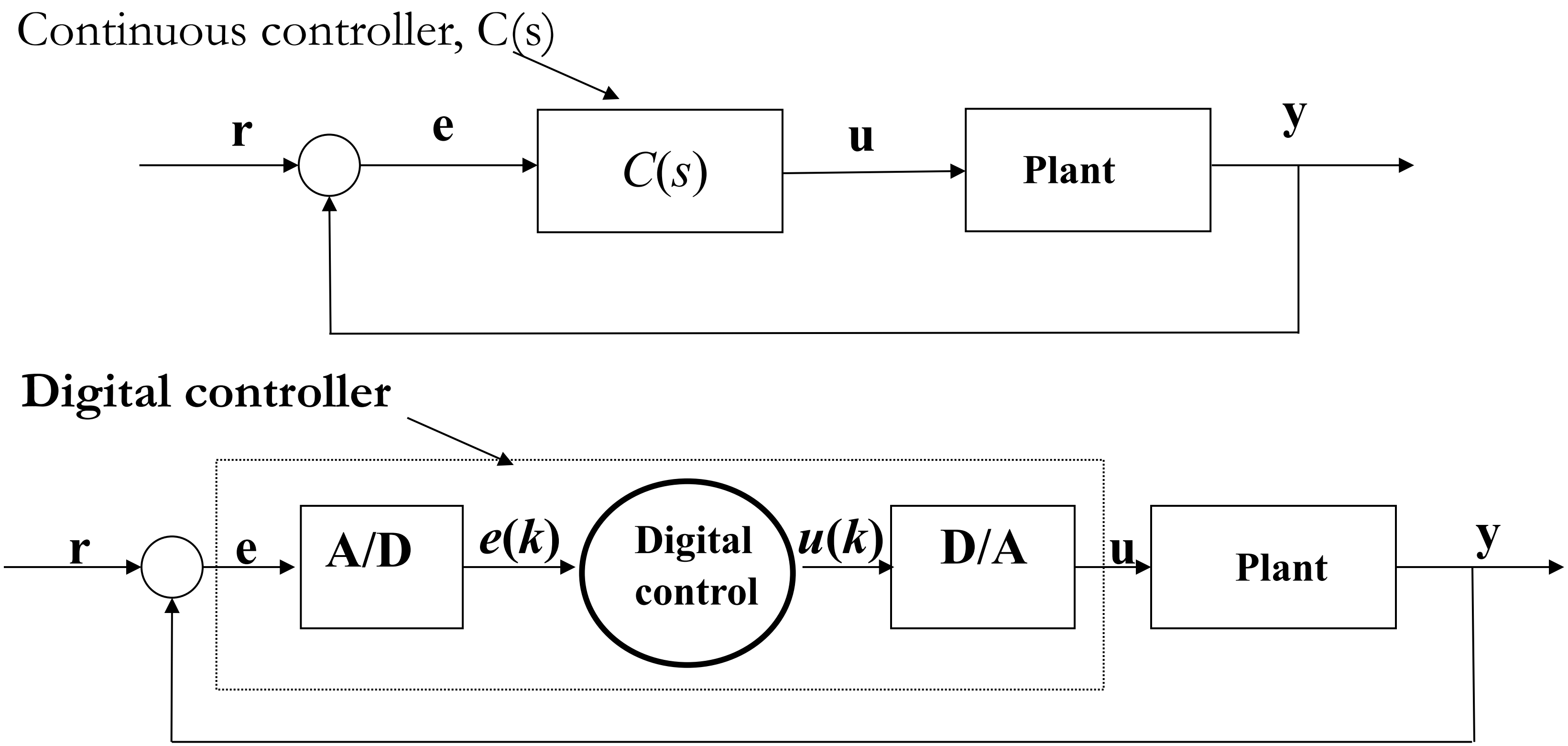
Team code: **5qf4mll**

Difference equations



Assume that A/D takes sample of continuous signal $e(t)$ at discrete times and passes them to the controller/computer: $\hat{e}(kT) = e(kT) = e(k)$.

The controller/computer elaborates these signals in some fashion and sends to D/A.

Inputs signals $e(0), e(1), \dots, e(k)$

Outputs signals $u(0), u(1), \dots, u(k)$

$$u(k) = f\big(e(0), \dots, e(k); u(0), \dots, u(k-1)\big)$$

The machine computes *f*

$$u(k) = f\big(e(0), \dots, e(k); u(0), \dots, u(k-1)\big)$$

If ***f*** is linear and dipends on only a finite number of past *e*'s and *u*'s, (***m+1*** and ***n*** samples, respectively):

$$u(k) = -a_1 u(k-1) - a_2 u(k-2) - \cdots - a_n u(k-n) + b_0 e(k) + b_1 e(k-1) + \ldots + b_m e(k-m)$$

This is an example of linear recurrence equation.
If a's and b's are constant, the computer is solving a constant-coefficient difference equation (CCDE).

↓

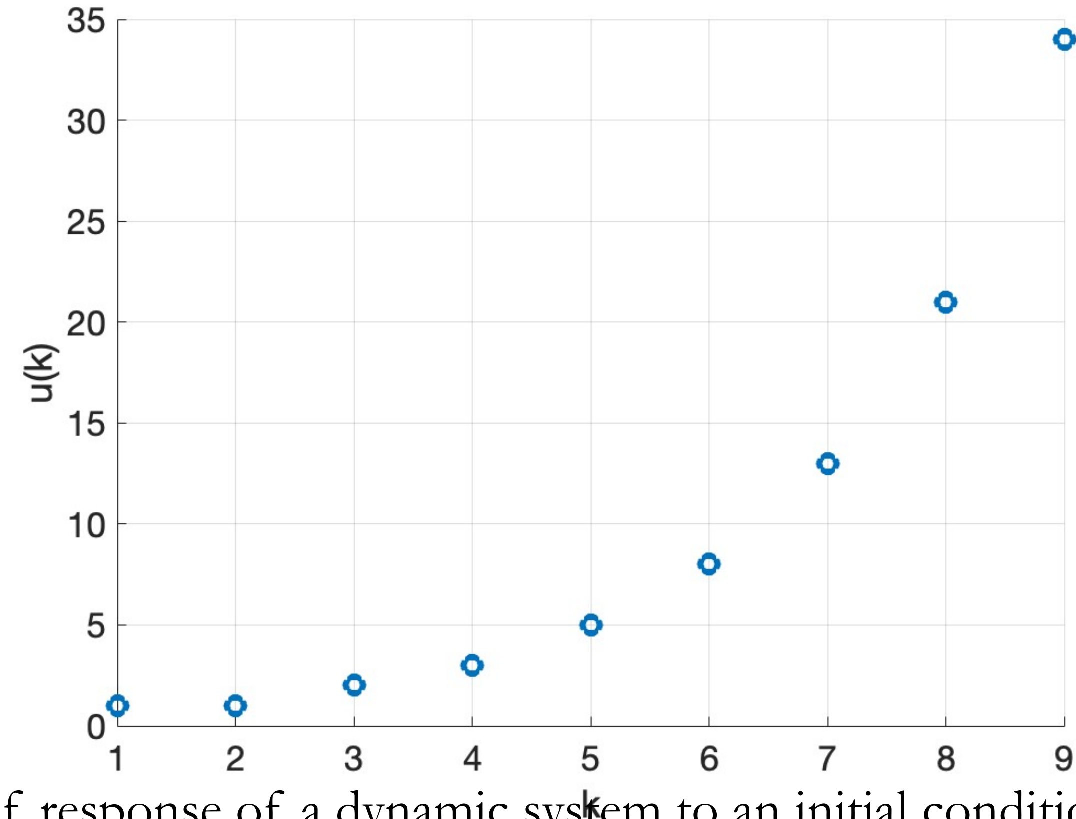
Methods for achieving solutions of CCDEs.

Example – Linear difference equation

$$u(k) = u(k-1) + u(k-2)$$

We start at k=3. There are no inputs, and $u(1) = u(2) = 1$

The results, the Fibonacci numbers



Example of response of a dynamic system to an initial condition growing without bound **(unstable system)**

Assuming difference equations as

$u(k) = -a_1u(k-1) - a_2u(k-2) - \cdots - a_nu(k-n) + b_0e(k) + b_1e(k-1) + \ldots + b_me(k-m)$

without solving them explicitely,

- determine if they are stable or unstable
- understand the general shape of the solution

One approach to solving this problem is to assume a form for the solution with unknown constants and then solve for the constants to match the given intial conditions

Example – Linear difference equation

Consider the **Fibonacci sequence:**

$$u(k) = u(k-1) + u(k-2)$$

If we assume $u(k) = Az^k$, then the sequence becomes

$$Az^k = Az^{k-1} + Az^{k-2}$$

By dividing by A ($\neq 0$) and by multiplying by z^{-k}

$$z^2 = z + 1 \quad \Rightarrow \quad z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Since the equation is linear, a sum of the individual solutions is also solution:

$$u(k) = A_1 z_1^k + A_2 z_2^k$$

By satisfying the intial conditions, $u(0) = u(1) = 1$:

$$\begin{cases} \mathbf{1} = \boldsymbol{A_1} + \boldsymbol{A_2} \\ \mathbf{1} = \boldsymbol{A_1 z_1} + \boldsymbol{A_2 z_2} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{A_1} = (\sqrt{\mathbf{5}} + \mathbf{1})/(\mathbf{2}\sqrt{\mathbf{5}}) \\ \boldsymbol{A_2} = (\sqrt{\mathbf{5}} - \mathbf{1})/(\mathbf{2}\sqrt{\mathbf{5}}) \end{cases}$$

Linear difference equations: stability

We got a solution in a closed form for the Fibonacci sequence.

Since $z_1 = \frac{1+\sqrt{5}}{2} > 1$**, then the equation represents an unstable system (**z_1^k will grow without bound)

In general, by substituting $u = z^k$**, we get a polynomial in** *z* known as the characteristic equation of the difference equation. If any solution of this equation is outside the unit circle (i.e. magnitude greater than one) the corresponding difference equation is unstable

If all the roots of the characteristic equation are inside the unit circle, then the corresponding difference equation is stable.

Example – Discrete stability

Consider the following difference equation:

$$u(k) = 0.8u(k-1) - 0.12u(k-2)$$

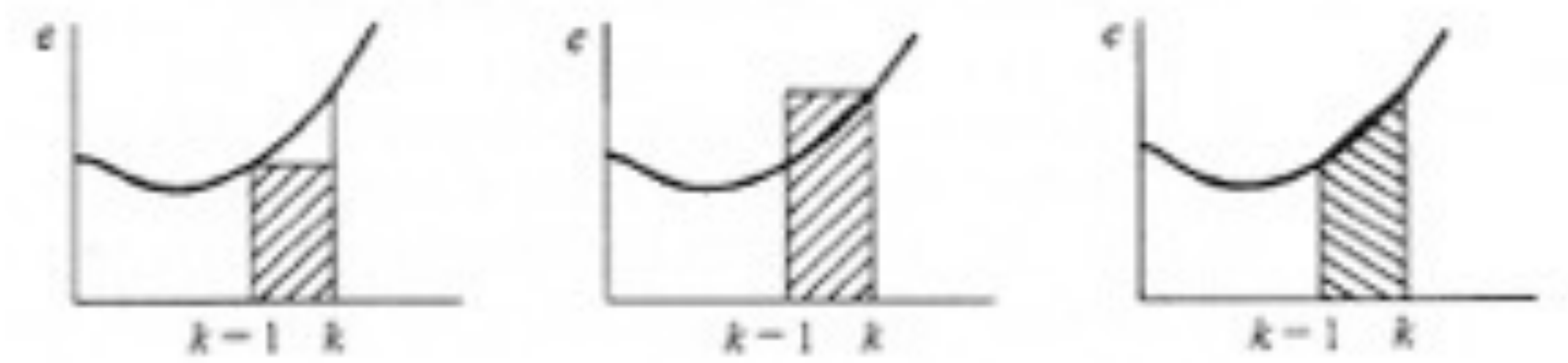
Determine the stability

Example – Linear difference equation with external input

We wish to compute an approximation for the integral of $\boldsymbol{e}(t)$ (numerical algorithms as discrete time systems).

Assume that we have an approximation for the integral from zero to the time t_{k-1} (i.e. k-1) that is u_{k-1}.

We need an approximation of the area under the curve $\boldsymbol{e}(t)$ between k-1 and k (sampling period constant, i.e. $\boldsymbol{T}$)



$$u(k) = u(k-1) + Te(k-1)$$

rectangle of height $e(k-1)$

called **forward rectangular rule (or Euler's method)**

$$u(k) = u(k-1) + Te(k)$$

rectangle of height $e(k)$

called **backward rectangular rule (or Euler's method)**

$$u(k) = u(k-1) + \frac{T}{2}\left(e(k) + e(k-1)\right)$$

Trapezoid rule

Introduction to z-transform

Given a sequence of discrete values, $f(0), \dots, f(k), \dots$, we define the z-transform as the function

$$\mathcal{Z}[f(k)] = F(z) = \sum_{k=0}^{+\infty} z^{-k} f(k)$$

with $r < |z| < R_0$, i.e. r and R_0 bounds on the magnitude of z, for which the series converges

By applying z-transform in the linear difference equations, we can find a relationship between the z-transform of the input and output sequences that allows the rapid solution of linear, constant, difference equations

Introduction to z-transform

For the differential equation approximating the integral of $\boldsymbol{e}$, by exploiting the Trapezoid rule, we have

$$u(k) = u(k-1) + \frac{T}{2}\left(e(k) + e(k-1)\right) \quad \mathbf{(1)}$$

By defining the z-transforms of the input and output sequences, e(k) and u(k), respectively

$$\mathcal{Z}[e(k)] = E(z) = \sum_{k=0}^{+\infty} z^{-k} e(k); \qquad \mathcal{Z}[u(k)] = U(z) = \sum_{k=0}^{+\infty} z^{-k} u(k)$$

By multiplying (1) by z^{-k} and sum over k, we get

$$\sum_{k=0}^{+\infty} z^{-k} u(k) = \sum_{k=0}^{+\infty} z^{-k} u(k-1) + \frac{T}{2}\left(\sum_{k=0}^{+\infty} z^{-k} e(k) + \sum_{k=0}^{+\infty} z^{-k} e(k-1)\right)$$

Z-Transfer Function

$$\sum_{k=0}^{+\infty} z^{-k}u(k) = \sum_{k=0}^{+\infty} z^{-k}u(k-1) + \frac{T}{2}\left(\sum_{k=0}^{+\infty} z^{-k}e(k) + \sum_{k=0}^{+\infty} z^{-k}e(k-1)\right)$$

$k-1 = j$

$$\sum_{k=0}^{+\infty} z^{-k}u(k-1) = \sum_{j=0}^{+\infty} z^{-(j+1)}u(j) = z^{-1}\sum_{j=0}^{+\infty} z^{-j}u(j) = z^{-1}U(z)$$

⇕

$$U(z) = z^{-1}U(z) + \frac{T}{2}\left[E(z) + z^{-1}E(z)\right]$$

⇕

Transfer function

$$U(z)(1-z^{-1}) = \frac{T}{2}(1+z^{-1})E(z) \Rightarrow \frac{U(z)}{E(z)} = H(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T}{2}\frac{z+1}{z-1}$$

For the more general relation

$$u(k) = -a_1 u(k-1) - a_2 u(k-2) - \cdots - a_n u(k-n) + b_0 e(k) + b_1 e(k-1) + \ldots + b_m e(k-m)$$

By operating z-transform

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

And if $\boldsymbol{n \geq m}$

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \ldots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \ldots + a_n}$$

z^{-1} delay of one time unit

For the relation

$$u(k) = e(k-1)$$

the output u(k) equals the input delayed by one period

By operating z-transform

$$U(z) = z^{-1}E(z)$$

$$\downarrow$$

$$\frac{U(z)}{E(z)} = H(z) = \frac{1}{z}$$