Course of “Industrial Automation”
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Root locus

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

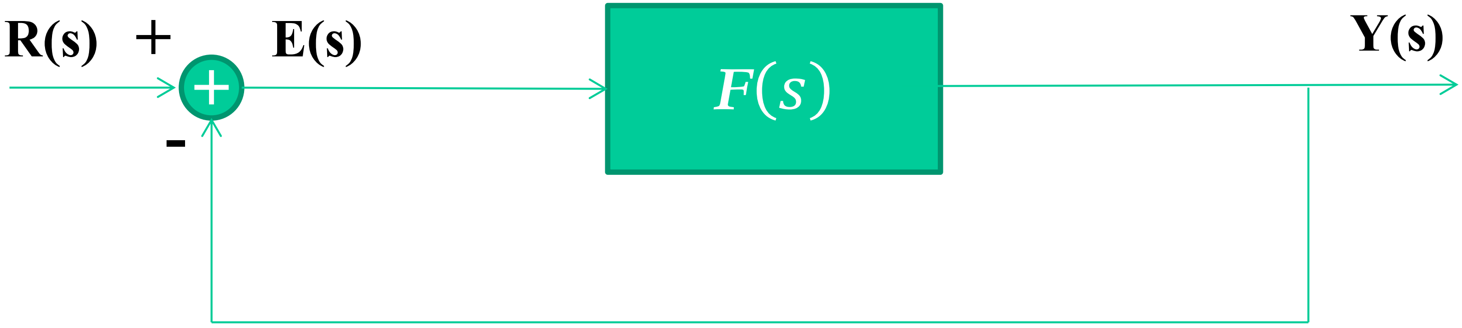
Università degli studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **5qf4mll**

Closed loop system

✓ Let us consider the $R(s) \rightarrow Y(s)$ closed loop system



The closed loop function W(s) is given by

$$W(s) = \frac{Y(s)}{R(s)} = \frac{F(s)}{1+F(s)}$$

where,

$$F(s) = \frac{N(s)}{D(s)} = \rho \frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)}$$

Definition of the root-locus

The **characteristic equation**, $\mathbf{1+F(s)=0,}$ provides the closed-loop poles.

The root-locus method is a technique in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Usually, this parameter is the constant **ρ of** ***F*(*s*)** defined by

$$F(s) = \rho \frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)}$$

ρ can be positive or negative, then, we need to plot the locus for each case. Here, we assume **ρ>0.** If **ρ<0,** it corresponds to the positive feedback case.

Angle and magnitude conditions

$$\begin{cases} 1 + F(s) = \mathbf{0} \\ F(s) = \rho \dfrac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} \end{cases} \Longrightarrow \frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} = -\frac{1}{\rho}$$

- **Angle condition:**

$$\sum_{i=1}^{m} \arg(s - z_i) - \sum_{j=1}^{n} \arg(s - p_j) = \begin{cases} (2k+1)\pi, & \rho > 0 \\ 2k\pi, & \rho < 0 \end{cases}$$

- **Magnitude condition:**

$$\frac{\prod_i^m |s - z_i|}{\prod_j^n |s - p_j|} = \frac{1}{|\rho|}$$

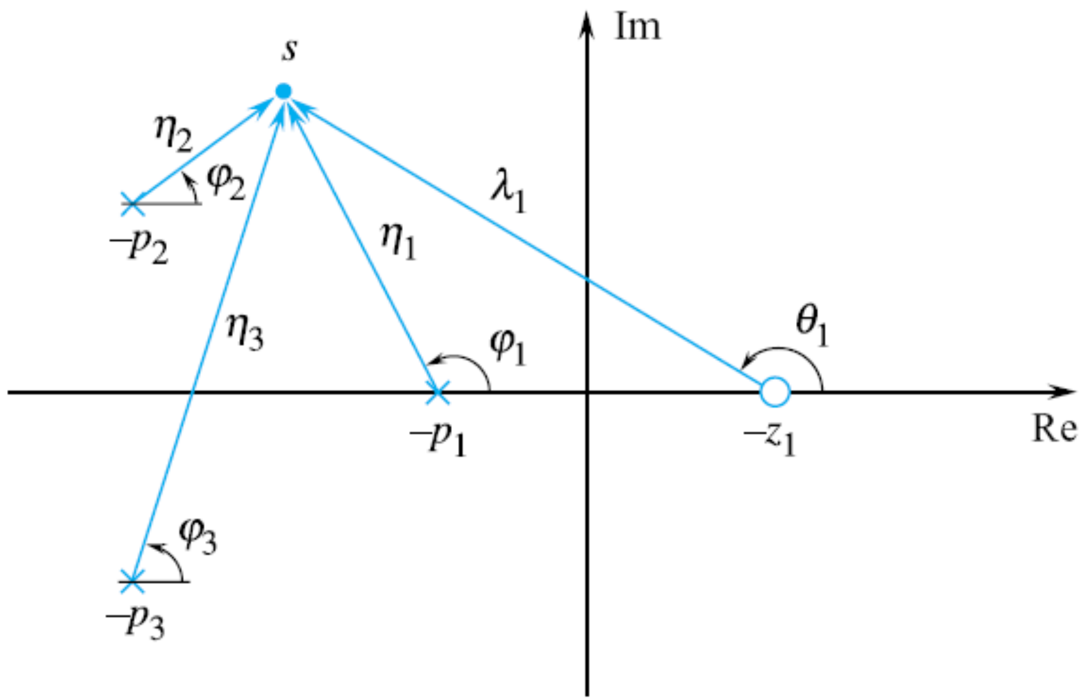
Angle and magnitude evaluation

One zero:

z_1

Three poles:

p_1, p_2 **and** p_3



$$\sum_{i=1}^{m} \arg(s - z_i) - \sum_{j=1}^{n} \arg(s - p_j) = \theta_1 - \varphi_1 - \varphi_2 - \varphi_3$$

$$\frac{\prod_i^m |s - z_i|}{\prod_j^n |s - p_j|} = \frac{\lambda_1}{\eta_1 \eta_2 \eta_3}$$

Example: computing analitically the roots as function of ρ

Given,

$$F(s) = \frac{\rho}{(s^2 + 3s + 2)} = \frac{\rho}{(s+1)(s+2)}$$

the **characteristic equation is**

$$1 + F(s) = \mathbf{0} \Rightarrow s^2 + 3s + 2 + \rho = \mathbf{0}$$

and the roots are: $s = \dfrac{-3 \pm \sqrt{1 - 4\rho}}{2}$

By varying ρ,

- $\rho = 0$ ⇒ **Two poles in -1 and -2**
- $0 < \rho < 1/4$ ⇒ **Two negative real poles**
- $\rho = 1/4$ ⇒ **Two equal poles in -3/2**
- $\rho > 1/4$ ⇒ **Two complex poles with real part -3/2**
- $\rho < 0$ ⇒ **Two real poles**

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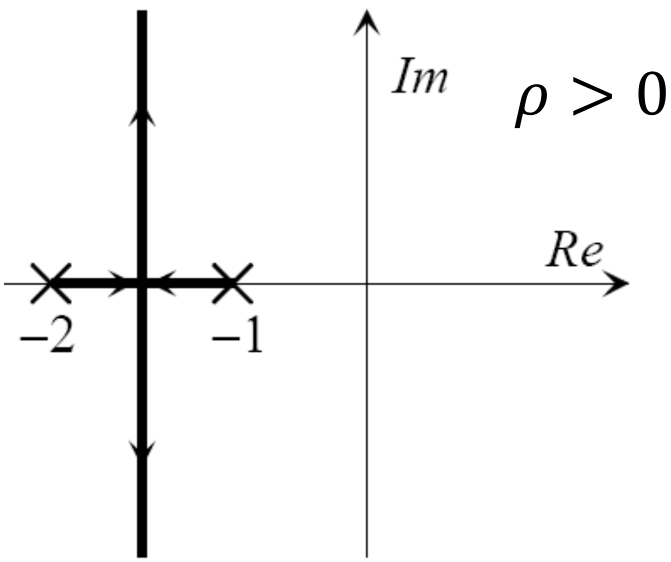
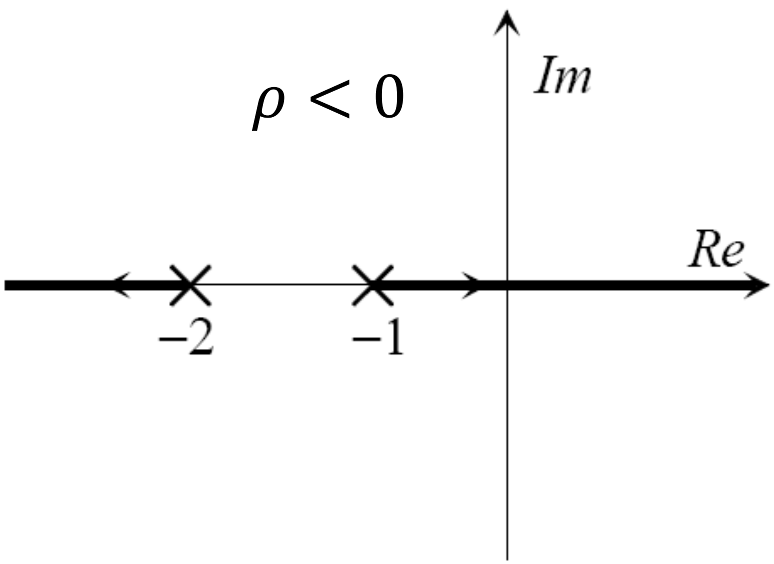
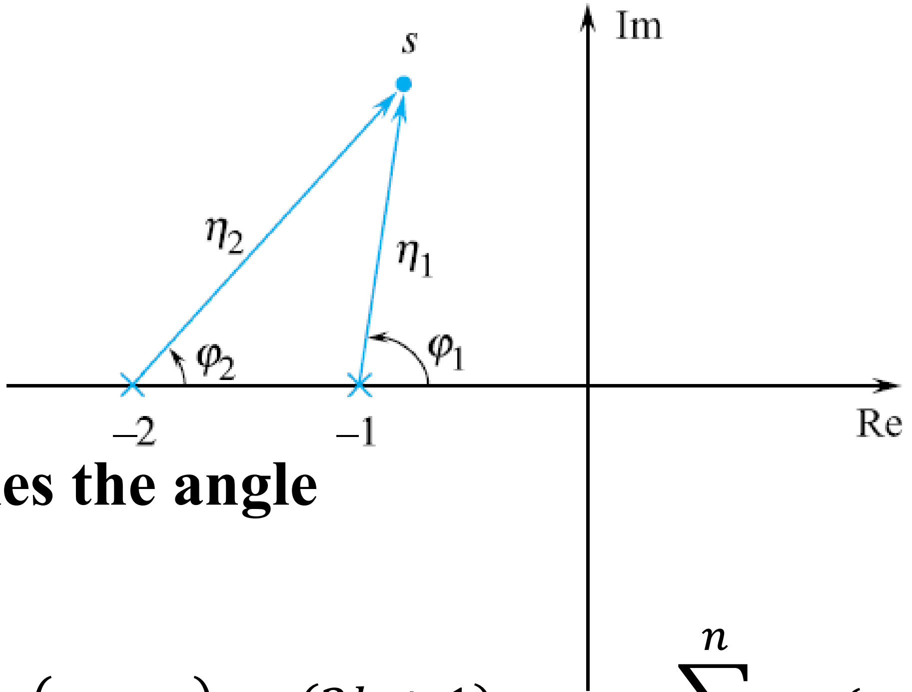




- $\rho > 0$



A test point *s* satisfies the angle condition if:

$$\sum_{i=1}^{m} \arg(s - z_i) - \sum_{j=1}^{n} \arg(s - p_j) = (2k+1)\pi \Rightarrow -\sum_{j=1}^{n} \arg(s - p_j) = -\varphi_1 - \varphi_2 = (2k+1)\pi$$

- **Then, the negative real axis between -2 and -1 satisfies the angle condition (i.e., $\varphi_1 = \pi, \varphi_2 = 0$) .**
- **Also, all the points belonging to the bisector of the segment (-2, -1), i.e. for which φ_1 and φ_2 are supplementary angles**

Example – graphical computation for constructing the root-locus

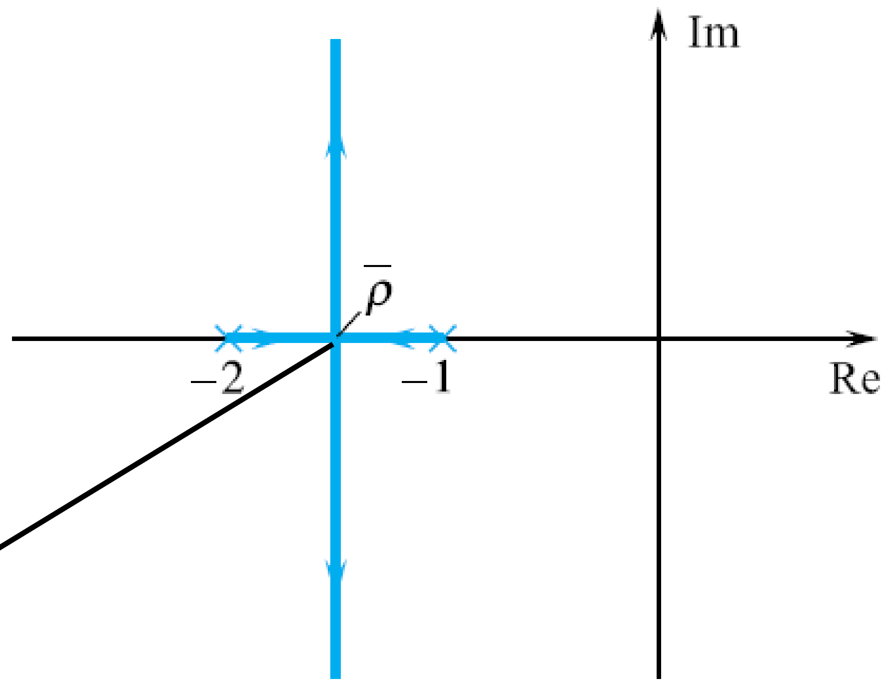
Then, for $\rho > 0$, the angle condition is satisfied by

- **the points belonging to (-2 -1)**
- **the points of the bisector of the segment (-2, -1),**

We use the magnitude condition,

$$\frac{\prod_i^m |s-z_i|}{\prod_j^n |s-p_j|} = \frac{1}{|\rho|},$$

for evaluating ρ in a given point of the root-locus; in this way, we can understand the orientation of the branches by increasing ρ:



For this example, the magnitude condition is:

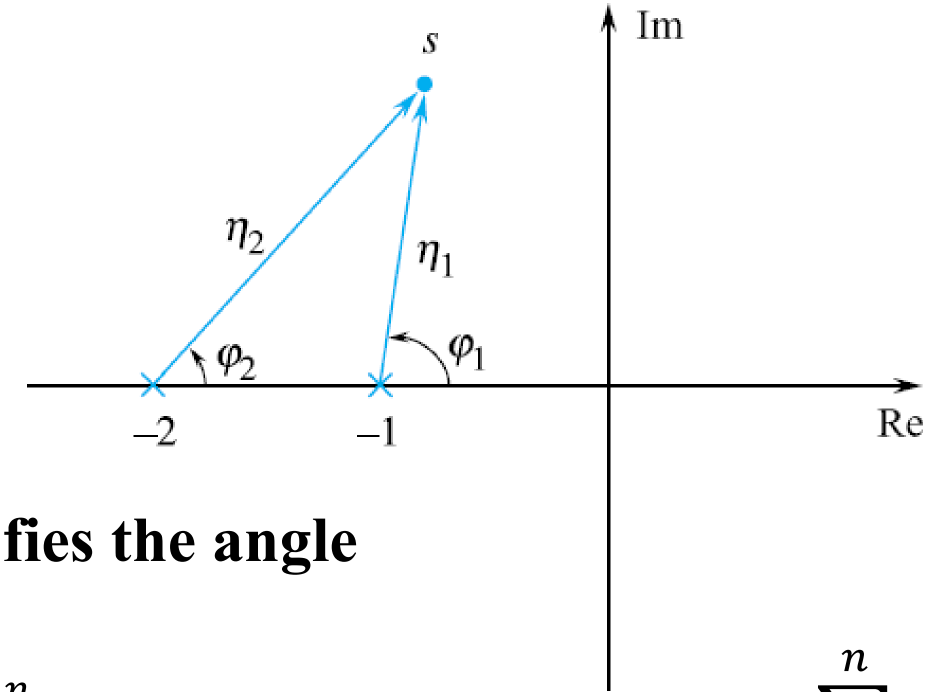
$$|s+1||s+2| = \rho$$

For $s = -3/2$, a multiple root

$$\Rightarrow \quad |-3/2+1||-3/2+2| = 0.5 \cdot 0.5 = 0.25 = \bar{\rho}$$

Example – graphical computation for constructing the root-locus

- $\rho < 0$



A test point *s* satisfies the angle condition if:

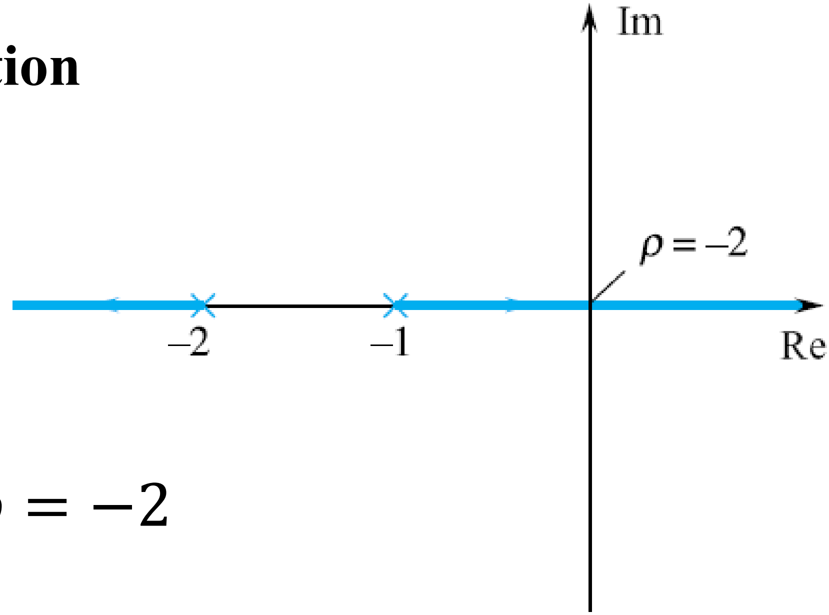
$$\sum_{i=1}^{m} \arg(s - z_i) - \sum_{j=1}^{n} \arg(s - p_j) = 2k\pi \quad \Rightarrow \quad -\sum_{j=1}^{n} \arg(s - p_j) = -\varphi_1 - \varphi_2 = 2k\pi$$

- **Then, the negative real axis between $-\infty$ and -2 satisfies the angle condition (i.e., $\varphi_1 = \pi$, $\varphi_2 = \pi$) and the points from -1 to $+\infty$ (i.e., $\varphi_1 = 0$, $\varphi_2 = 0$).**

Example – graphical computation for constructing the root-locus

- $\rho < 0$
- **The negative real axis between $-\infty$ and -2 satisfies the angle condition (i.e., $\varphi_1 = \pi$, $\varphi_2 = \pi$) and the points from -1 to $+\infty$ (i.e., $\varphi_1 = 0$, $\varphi_2 = 0$).**
- **We use the magnitude condition for evaluating ρ in $s = 0$:**

$$|s+1||s+2| = |\rho|$$



$s = 0$ ⇒ $|1||2| = |\rho|$ ⇒ $\rho = -2$

Procedure for root-locus plot: general rules (ρ>0)

1. Locate the open loop poles (by crosses) and the open loop zeros (by small circles) in the complex plane.
2. The root-loci will have just as many branches as there are roots of the charateristic equation:

$$1 + F(s) = 0 \Leftrightarrow 1 + \frac{N(s)}{D(s)} = 0 \Leftrightarrow 1 + \rho \frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} = 0 \Leftrightarrow$$

$$\prod_j^n (s - p_j) + \rho \prod_i^m (s - z_i) = 0.$$

   Then, the roots are n ($n \geq m$), so **the number of branches of the root-loci is equal to *n***, i.e the number of open loop poles.

3. The ***n* branches start from the open-loop poles** and ***m* branches terminate at *m* open-loop zeros**, the remaining ***n-m branches* terminate at infinity** (*n-m* implicit zeros at infinity).

$$\frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} = -\frac{1}{\rho}, \quad \textbf{if} \quad \begin{cases} \boldsymbol{s} = p_j \Rightarrow \dfrac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} \to \infty \Rightarrow \rho \to 0 \\ \boldsymbol{s} = z_i \Rightarrow \dfrac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} \to 0 \Rightarrow \rho \to \infty \end{cases}$$

4. **Determinate the root loci on the real axis**: the loci on the real axis is determined by the open-loop poles and zero lying on it; the complex-conjugate poles and zeros have no effect on the location of the root loci on the real axis because the angle contribution of a pair complex-conjugate poles or zeros is 360º on the real axis. **A ponit of the real axis lies on a root locus if the total number of real poles and real zeros to the right of this point is odd** (Vice versa for the complement locus, i.e. ρ<0).
5. **The root loci are symmetrical about the real axis** of the *s* plane: the roots of the characteristic equation are real or complex in coniugate pairs.
6. The ***n-m*** **branches terminating at infinity must be asymptotic to straight lines** which intersect on the real axis; this point, denoted with by $\boldsymbol{s = \sigma_a}$is given by

$$\boldsymbol{\sigma_a = \frac{\sum p_j - \sum z_i}{n-m}.}$$

The angles (slopes) of these lines (asymptotes) are given by (by applying the phase condition)

$$\textbf{angles of asymptotes} = \frac{(2k+1)\pi}{n-m} \quad (\rho<0,\ \textbf{angles of asymptotes} = \frac{2k\pi}{n-m})$$

for k=0, …,*n-m*-1

Procedure for root-locus plot: general rules (ρ>0)

- All the asymptotes intersect on the real axis. This point is obtained as it follows:

By expanding $$\frac{\prod_i^m (s - z_i)}{\prod_j^n (s - p_j)} = -\frac{1}{\rho}$$

$$\Rightarrow \quad \frac{s^m - \left(\sum z_i\right)s^{m-1} + \cdots}{s^n - \left(\sum p_j\right)s^{n-1} + \cdots} = -\frac{1}{\rho}$$

Then by dividing the denominator by the numerator

$$\frac{1}{s^{n-m} - \left(\sum p_j - \sum z_i\right)s^{n-m-1} + \cdots} = -\frac{1}{\rho}$$

that can be approximated for a large value of *s* as

$$\frac{1}{\left(s - \frac{\sum p_j - \sum z_i}{n - m}\right)^{n-m}} = -\frac{1}{\rho}$$

By denoting with $\sigma_a = \frac{\sum p_j - \sum z_i}{n-m}$

Procedure for root-locus plot: general rules (ρ>0)

- Then, we can approximate the characteristic eqaution for large values of s by

$$\frac{1}{(s-\sigma_a)^{n-m}} = -\frac{1}{\rho}$$

with

$$\sigma_a = \frac{\sum p_j - \sum z_i}{n-m}$$

Then, we represent a numebr of poles and zeros with an **unique pole $\boldsymbol{\sigma_a}$ with multiplicity *n-m*,** from which starts ***n-m branches*** terminating at infinity.

By applying the angle condition we get

$$\arg\left(\frac{1}{(s-\sigma_a)^{n-m}}\right) = (2k+1)\pi \iff -(n-m)\arg(s-\sigma_a) = (2k+1)\pi$$

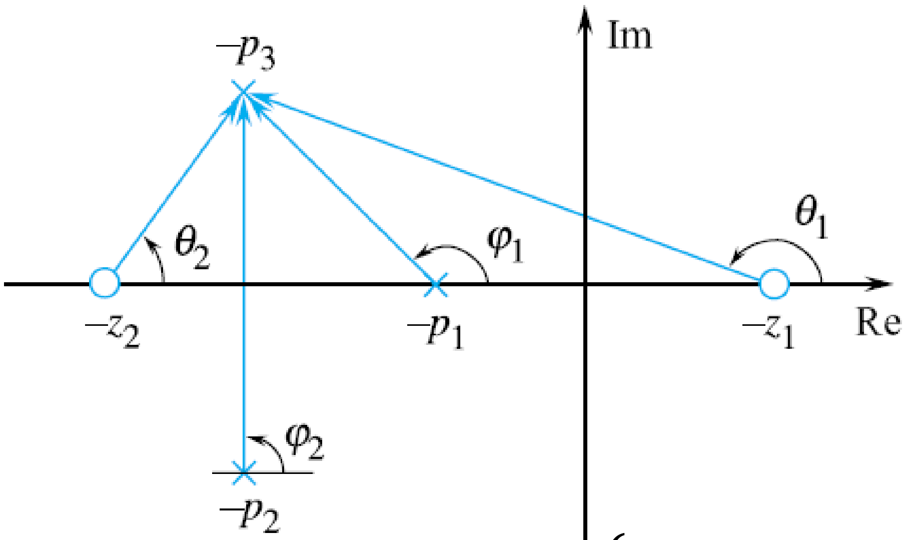
$$\iff \arg(s-\sigma_a) = \frac{(2k+1)\pi}{n-m},$$

i.e., **asymptotes** with **angles** (slopes) multiple of ***π*/(*n-m*), for k=0, ...,*n-m*-1**

- **For ρ<0,** $\arg(s-\sigma_a) = \dfrac{2k\pi}{n-m},$ i.e., **asymptotes** with **angles** (slopes) multiple of **2*π*/(*n-m*), for k=0, ...,*n-m*-1**

7. Determinate the angle of departure of the roots locus from a complex pole

Angle of departure from a complex pole, ϕ_{pj} =
π - (sum of the angles of vectors, φ_i, from all other poles to the complex pole in question)
+ (sum of the angles of vectors, θ_i, from the zeros to the complex pole in question)



Example: in this figure, the angle of departure from p_3 (denoted with ϕ_{p3}), is given by:

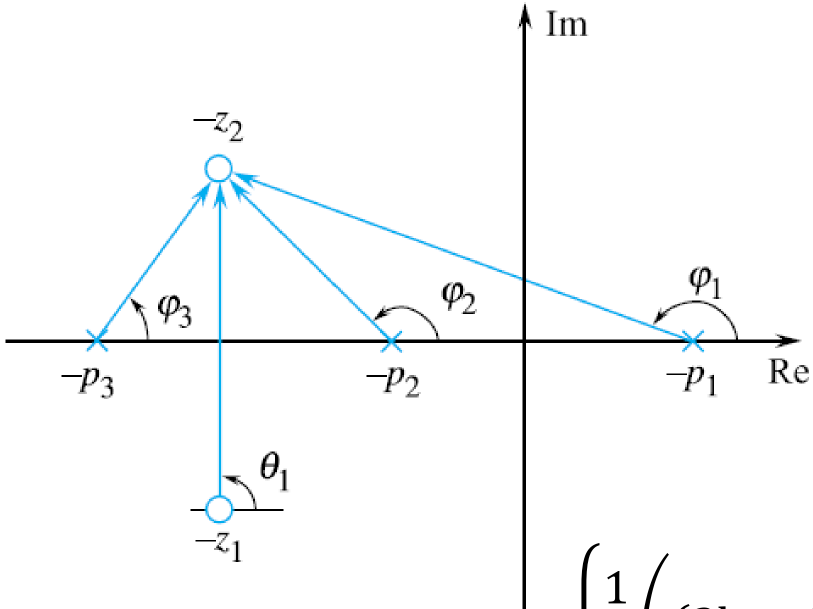
$$\phi_{p3} = \pi - (\varphi_1 + \varphi_2) + (\theta_1 + \theta_2)$$

In general, in the case of a complex pole, p_j, with multiplicity μ

$$\phi_{p_j} = \begin{cases} \dfrac{1}{\mu}\left((2k+1)\pi - \sum_{i\neq j}\varphi_i + \sum \theta_i\right), k = 0, \dots, \mu - 1, & \rho > 0 \\ \dfrac{1}{\mu}\left(2k\pi - \sum_{i\neq j}\varphi_i + \sum \theta_i\right), k = 0, \dots, \mu - 1, & \rho < 0 \end{cases}$$

8. Determinate the angle of arrival of the roots locus at a complex zero

Angle of arrival at a complex zero=
π - (sum of the angles of vectors from all other zeros to the complex zero in question)
+ (sum of the angles of vectors from poles to the complex pole in question)



Example: in this figure, the angle of departure from p_3 (denoted with ϕ_{z2}), is given by:

$$\phi_{z2} = \pi - (\theta_1) + (\varphi_1 + \varphi_2 + \varphi_3)$$

In general, in the case of a complex zero, z_j, with multiplicity μ

$$\phi_{z_j} = \begin{cases} \frac{1}{\mu}\left((2k+1)\pi - \sum_{i\neq j}\theta_i + \sum \varphi_i\right), k = 0, \dots, \mu-1, & \rho > 0 \\ \frac{1}{\mu}\left(2k\pi - \sum_{i\neq j}\theta_i + \sum \varphi_i\right), k = 0, \dots, \mu-1, & \rho < 0 \end{cases}$$

9. Find the break-in and breakaway points on the real axis

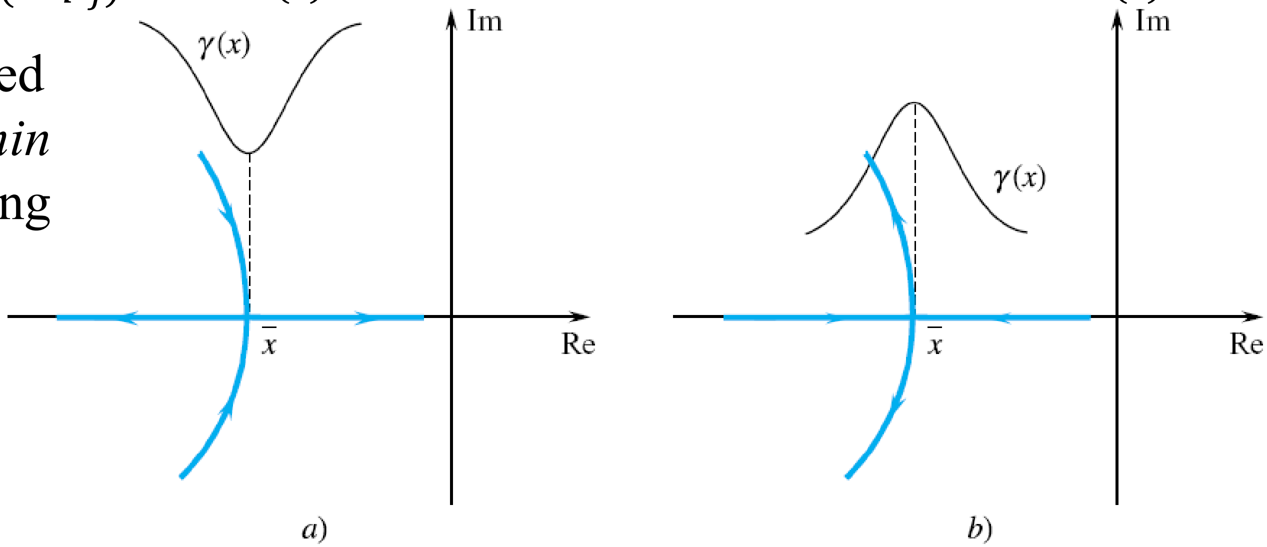
(see figure below, a) and b), respectively)

If a root locus lies between two adjacent open-loop poles on the real axis, then there is at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at -∞) on the real axis, then there always exists one break-in point between the two zeros…

By $F(s) = \frac{N(s)}{D(s)} = \rho \frac{\prod_i^m (s-z_i)}{\prod_j^n (s-p_j)} = \rho \frac{N^*(s)}{D(s)}$, the charac. eq. $1 + F(s) = 0 \Leftrightarrow \frac{N^*(s)}{D(s)} = -\frac{1}{\rho}$

These points are obtained by computing the *min* and *max* of the following function:

$$\gamma(x) = -\frac{D(x)}{N^*(x)}$$



Or by solving: $\sum_{i=1}^m \frac{1}{s-z_i} - \sum_{i=1}^n \frac{1}{s-p_i} = 0, s = \bar{x}$, for $\rho \in (-\infty, +\infty)$

without providing the type of points (break-in or breakaway)

10. Determine closed-loop poles

A particular point on each root-locus branch will be a closed–loop pole if the value of ρ satisfes the magnitude condition at that point. Conversely, the magnitude condition allows us to determine the value of ρ at any specif root location, $s = \bar{s}$, on the locus:

$$\frac{\prod_i^m |s - z_i|}{\prod_j^n |s - p_j|} = \frac{1}{|\rho|} \quad \xrightarrow{s = \bar{s}} \quad |\rho| = \frac{\prod_j^n |\bar{s} - p_j|}{\prod_i^m |\bar{s} - z_i|}$$

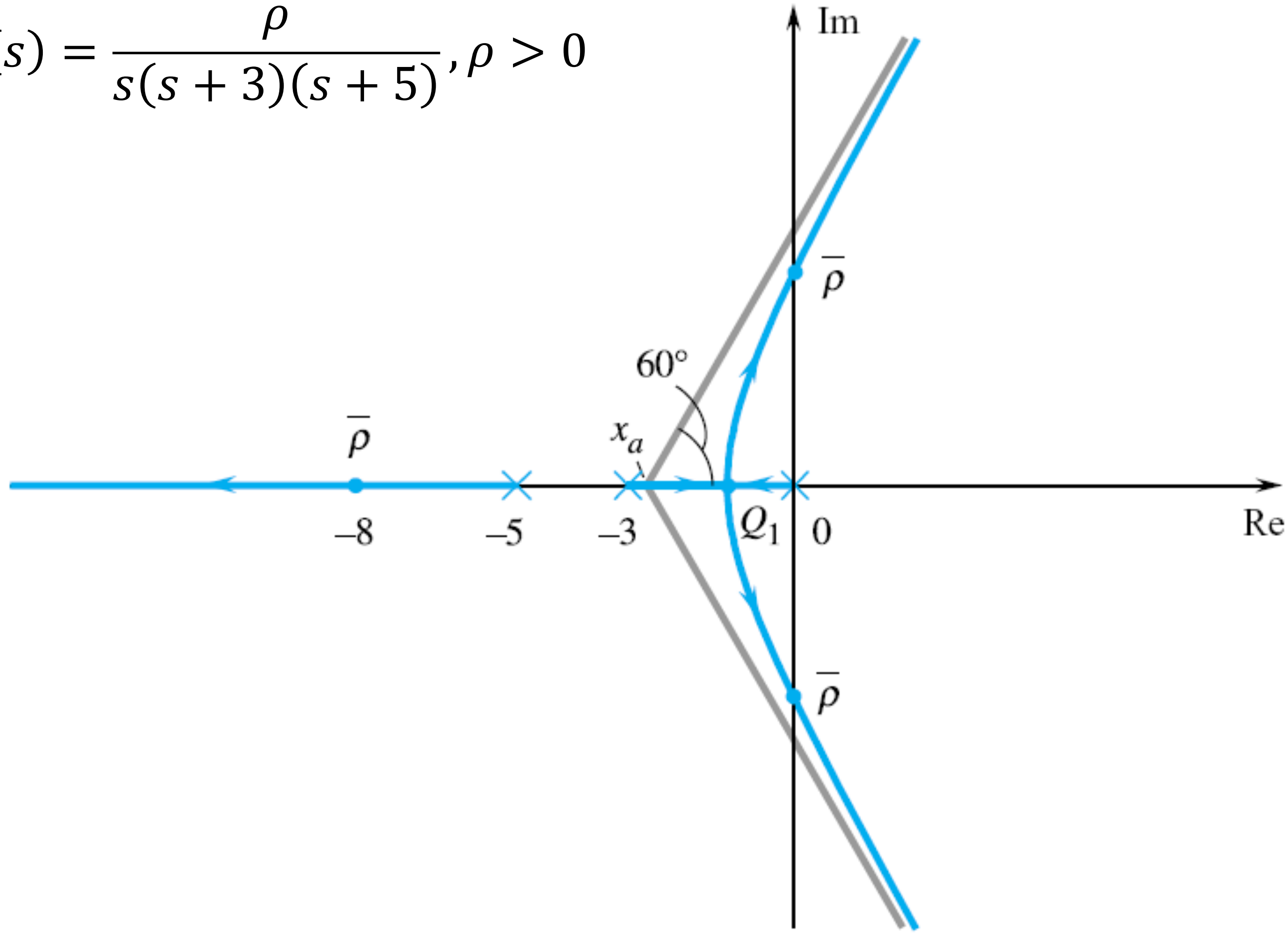
11. If $\boldsymbol{n - m \geq 2}$, the sum of the real parts of the poles of the closed loop system does not change by varying ρ

Then, for $\rho = 0$, such sum is equal to $x_b = \sum_{i=1}^{n} p_i$

Note: this rule can be useful for determining the points where the root loci cross the imaginary axis

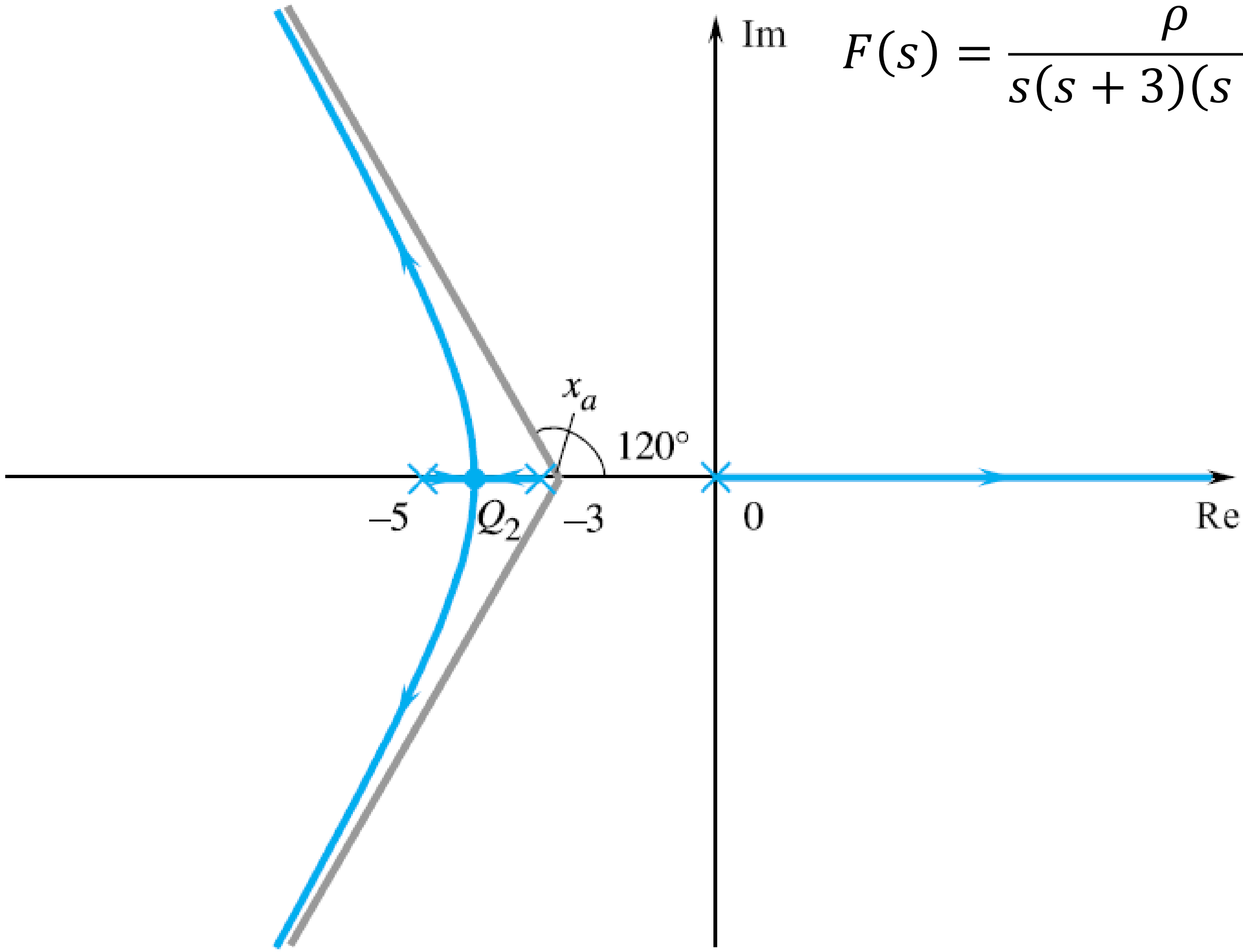
Example 1 (ρ>0)

$$F(s) = \frac{\rho}{s(s+3)(s+5)}, \rho > 0$$



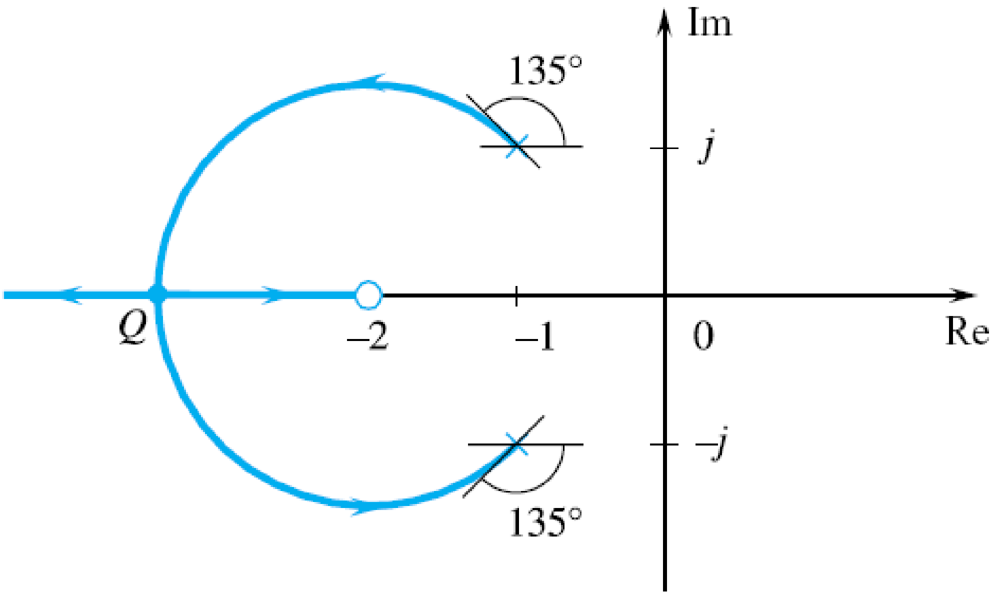
Example 1 (ρ <0)

$$F(s) = \frac{\rho}{s(s+3)(s+5)}, \rho < 0$$



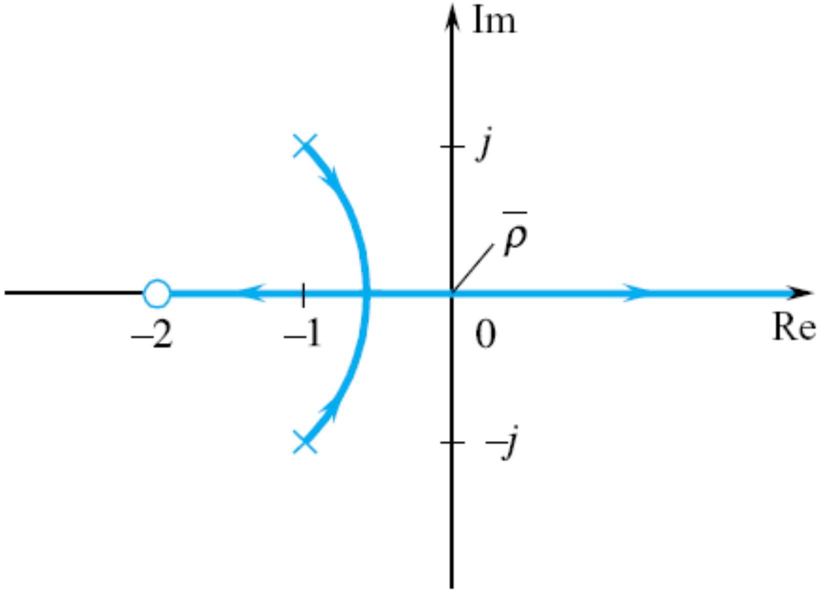
Example 2 (ρ >0)

$$F(s) = \rho \frac{s+2}{s^2+2s+2}, \rho > 0$$



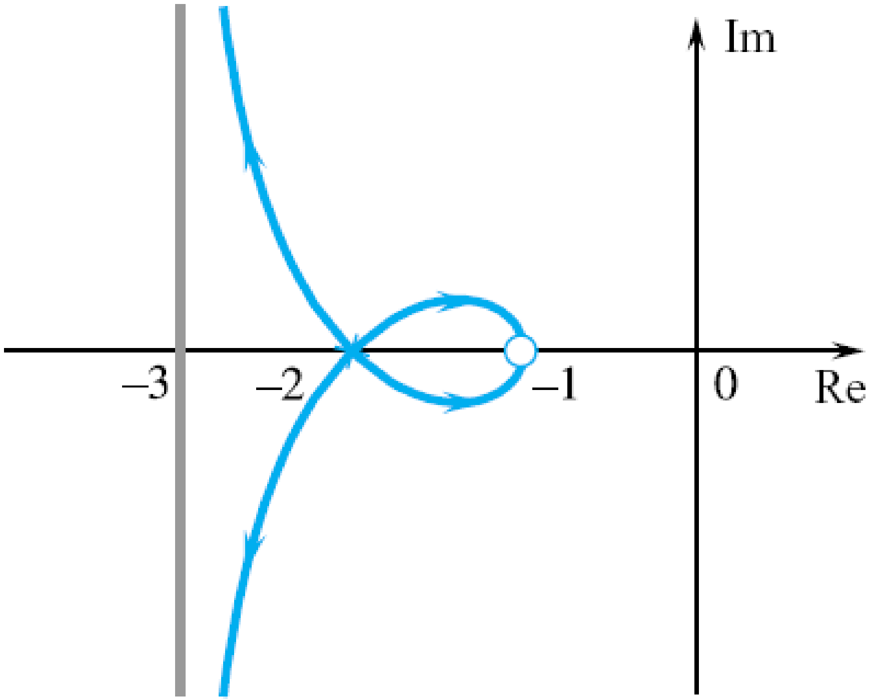
Example 2 (ρ <0)

$$F(s) = \rho \frac{s+2}{s^2+2s+2}, \rho < 0$$



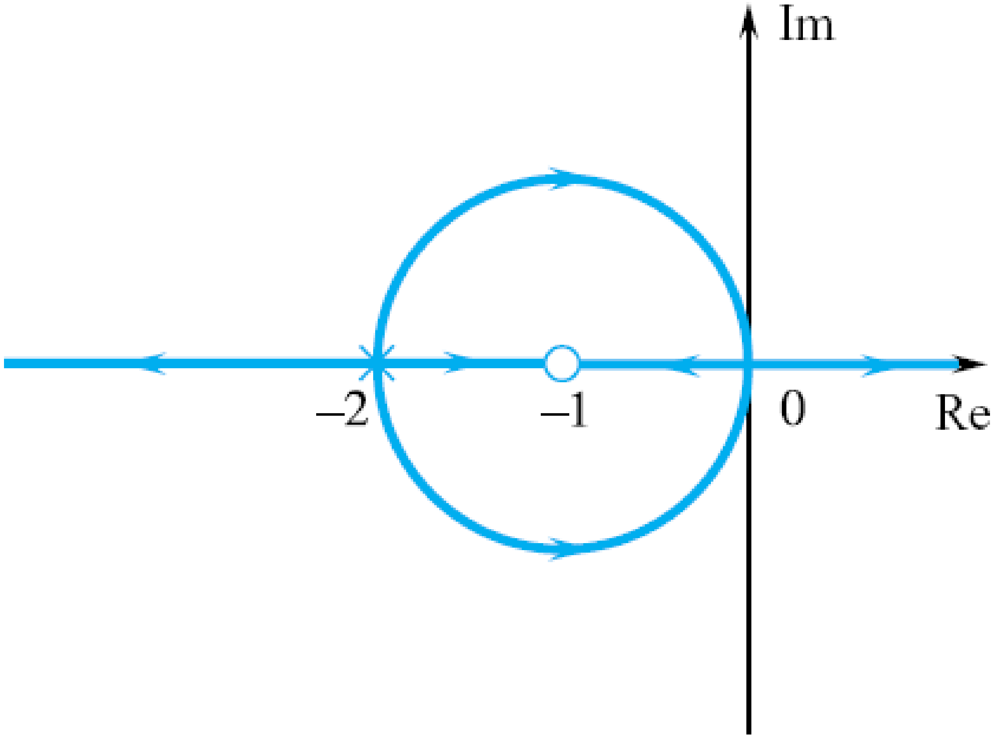
Example 3 (ρ >0)

$$F(s) = \rho \frac{(s+1)^2}{(s+2)^4}, \rho > 0$$

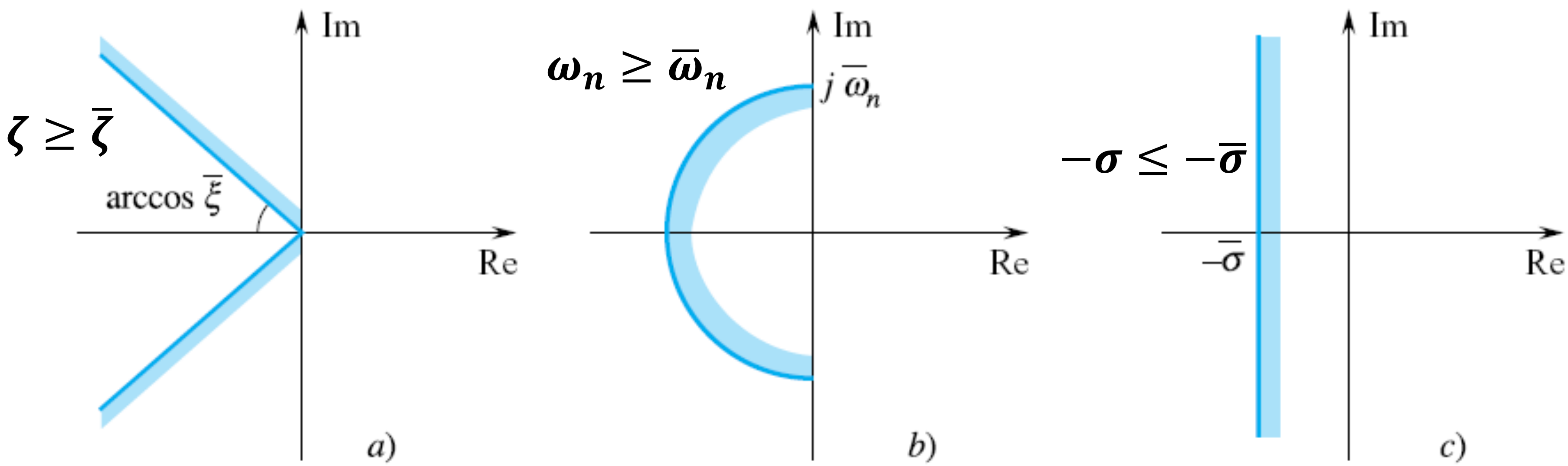


Example 3 (ρ <0)

$$F(s) = \rho \frac{(s+1)^2}{(s+2)^4}, \rho < 0$$

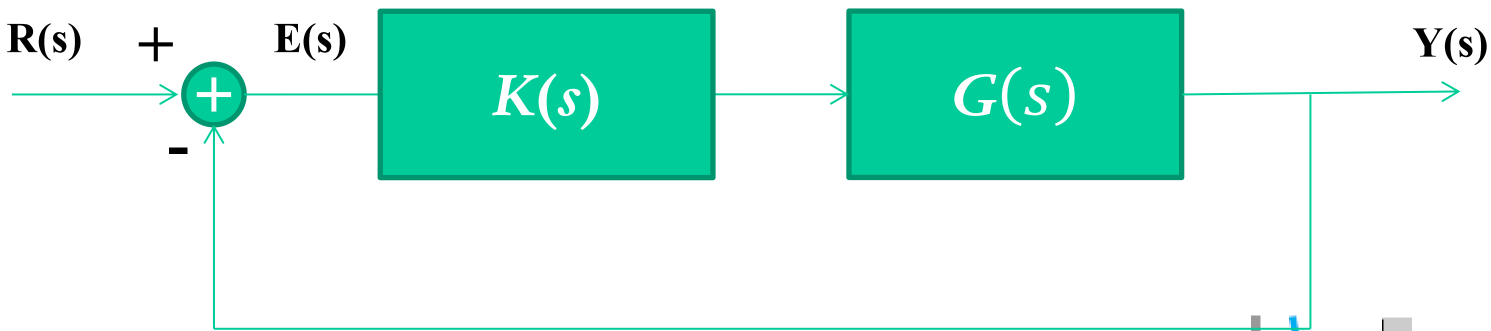


Controller design by using root-locus



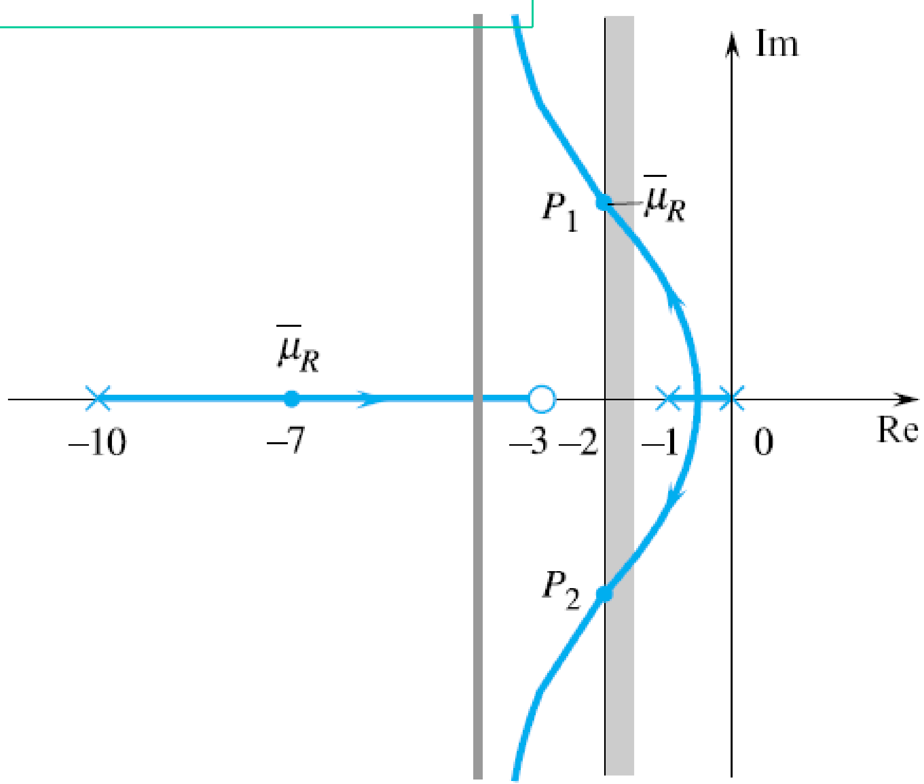
Example: design a controller by using root-locus

- Let us consider a closed loop system with a proportional control action ($K(s) = \mu_R$)



- where $G(s) = \frac{s+3}{s(s+1)(s+10)}$

- **Find μ_R such that $t_{a1} \le 2.5\ s$ of the output y in response to a step signal r**



Appendix - Routh table

- Let us consider the polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0$$

- ***The Routh table*** is defined as follows

n	a_n	a_{n-2}	a_{n-4}	$\ldots$
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	$\ldots$
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	$\ldots$
$n-3$	c_{n-3}	c_{n-5}	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	

$$b_{n-2} = -\frac{1}{a_{n-1}} \det\begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}$$

$$b_{n-4} = -\frac{1}{a_{n-1}} \det\begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}$$

$$c_{n-3} = -\frac{1}{b_{n-2}} \det\begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{pmatrix}$$