Dato un sistema del II ordine definito dalla seguente f.d.T

$$G(s)=\frac{b}{s^2+a_1s+a_0},$$

nelle hp di poli complessi coniugati, ovvero

$$s^2+a_1s+a_0=0 \iff (s-p)(s-\bar{p})=0$$

dove $p=\alpha+j\omega$, $\bar{p}=\alpha-j\omega$,

calcolare la risposta al gradino unitario.

$$Y(s)=G(s)U(s)=\frac{G(s)}{s}=\frac{b}{s^2+a_1s+a_0}\cdot\frac{1}{s}=$$

$$=\frac{b}{(s-p)(s-\bar{p})}\cdot\frac{1}{s}=\frac{b}{(s-\alpha-j\omega)(s-\alpha+j\omega)}\cdot\frac{1}{s}$$

$$=\frac{b}{(s-\alpha)^2+\omega^2}\cdot\frac{1}{s}=\frac{A}{s}+\frac{Bs+C}{(s-\alpha)^2+\omega^2}$$

$$=\frac{A(s^2-2\alpha s+\alpha^2+\omega^2)+Bs^2+Cs}{s\left[(s-\alpha)^2+\omega^2\right]}=\frac{(A+B)s^2+(C-2\alpha A)s+(\alpha^2+\omega^2)A}{s\left[(s-\alpha)^2+\omega^2\right]}=$$

- $$A=\lim_{s\to 0} s\,Y(s)=\lim_{s\to 0}\cancel{s}\frac{G(s)}{\cancel{s}}=G(0)=G_0$$

$$=\frac{b}{a_0}=\frac{b}{\alpha^2+\omega^2}$$

- $$\begin{cases} A+B=0 \\ C-2\alpha A=0 \\ A(\alpha^2+\omega^2)=b \end{cases} \overset{\text{già noto}}{\Rightarrow} A=\frac{b}{\alpha^2+\omega^2}=G_0 \quad \Rightarrow \begin{cases} A=\frac{b}{\alpha^2+\omega^2}=G_0 \\ B=-A=-G_0 \\ C=2\alpha A=2\alpha G_0 \end{cases}$$

$$Y(s)=\frac{A}{s}+\frac{Bs+C}{(s-\alpha)^2+\omega^2}=\frac{G_0}{s}+\frac{-G_0s+2\alpha G_0}{(s-\alpha)^2+\omega^2}=$$

$$=G_0\Big(\underbrace{1}_{Y_1(s)}-\underbrace{\frac{s-2\alpha}{(s-\alpha)^2+\omega^2}}_{Y_2(s)}\Big)$$

Si ricorda che

$$\mathcal{L}^{-1}\left(\frac{s-\alpha}{(s-\alpha)^2+\omega^2}\right)=e^{\alpha t}\cos(\omega t)\,1(t)$$

$$\mathcal{L}^{-1}\left(\frac{\omega}{(s-\alpha)^2+\omega^2}\right)=e^{\alpha t}\sin(\omega t)\,1(t)$$

Quindi

$$Y_2(s)=\frac{s-2\alpha}{(s-\alpha)^2+\omega^2}=\frac{s-\alpha-\alpha}{(s-\alpha)^2+\omega^2}=\frac{s-\alpha}{(s-\alpha)^2+\omega^2}-\frac{\alpha}{\omega}\cdot\frac{\omega}{(s-\alpha)^2+\omega^2}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y_2(t)=e^{\alpha t}\cos(\omega t)-\frac{\alpha}{\omega}e^{\alpha t}\sin(\omega t)$$

$$Y_1(s)=1 \xrightarrow{\mathcal{L}^{-1}} y_1(t)=1(t)$$

$$y(t)=G_0\big(y_1(t)-y_2(t)\big)=$$

$$=G_0\left(1-e^{\alpha t}\cos(\omega t)+\frac{\alpha}{\omega}e^{\alpha t}\sin(\omega t)\right)1(t)$$

Si ricorda che

$$K_1e^{\alpha t}\sin(\omega t)+K_2e^{\alpha t}\cos(\omega t)=$$

$$=Me^{\alpha t}\cdot\sin(\omega t+\theta),$$

dove $M=\sqrt{K_1^2+K_2^2}$ e $\theta=tg^{-1}\left(\frac{K_2}{K_1}\right)$

Pertanto $\theta=tg^{-1}\left(-\frac{\omega}{\alpha}\right)$, $M=\sqrt{\frac{\alpha^2}{\omega^2}+1}=\frac{\sqrt{\alpha^2+\omega^2}}{\omega}=-\frac{1}{\sin\theta}$

Quindi,

$$y(t)=G_0\left(1-\frac{1}{\sin\theta}e^{\alpha t}\cdot\sin(\omega t+\theta)\right)\cdot 1(t)$$

Se si vuole calcolare la risposta, $y^*(t)$, al gradino di ampiezza U_0, $u(t)=U_0\cdot 1(t)$,

$$y^*(t)=U_0\,y(t)=G_0U_0\left(1-\frac{1}{\sin\theta}e^{\alpha t}\cdot\sin(\omega t+\theta)\right)1(t).$$