L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- **Brief notes on affine maps.**

Affine maps generalize **linear maps**, since they allow not only the basis but also the origin of the reference system to be changed.

Affine maps preserve collinearity, i.e. they map aligned points to aligned points and parallel lines to parallel lines*, but in general they **don't preserve** distances between points or angles between segments.

* this means that ∞ is a fixed point, i.e. it is mapped to itself

Recall that, for example, if $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation, then F is completely identified by its action on any basis of $\mathbb{R}^2$; that is, if you know $F(b_1)$ and $F(b_2)$ for some basis $\{b_1, b_2\}$ of $\mathbb{R}^2$, then you can find the matrix of F.

Similarly, an affine transformation on $\mathbb{R}^2$ is completely identified by its action on any three non-collinear points, that is, any three points that form a proper triangle (affinely independent points).

An elementary real affine transformation between two real affine spaces with finite dimensions

$$\Psi : P(x) \in \mathbb{R}^n \longrightarrow Q(y) \in \mathbb{R}^m \qquad P,Q \text{ points}$$

is described, in Cartesian coordinates, by the following matrix equation

$$Q = (y) = \Psi(P) = A(x) + (b)$$

coordinates of the image of Origin $O \in \mathbb{R}^n$

translation

cartesian coordinates

where $P \equiv (x)$ and $Q \equiv (y)$,

and in Homogeneous coordinates by

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{pmatrix} A & & & b \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

augmented vectors

augmented matrix

Any linear map is also an affine map.

Linear maps preserve linear combinations, while **affine maps** usually do not preserve linear combinations; however, there is a particular combination that is preserved.

Theor. **Affine maps** preserve linear combinations of points in which the sum of coefficients is 1; these are said affine combinations:

$$P=\sum_{i=1}^{n}\alpha_i P_i \quad \text{affine combination*} \quad \Longleftrightarrow \quad \sum_{i=1}^{n}\alpha_i = 1$$

* if, in addition, $\alpha_i > 0 \; \forall i$, it is said a convex combination

Proof:

$$Q = AP + b = A\sum_{i=1}^{n}\alpha_i P_i + b(1) = A\sum_{i=1}^{n}\alpha_i P_i + b\left(\sum_{i=1}^{n}\alpha_i\right) = \sum_{i=1}^{n}\alpha_i\left(AP_i + b\right)$$

Example of an affine map

$$\Psi : P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mapsto Q = \Psi(P) = \alpha x_1 + \beta x_2 + \gamma \in \mathbb{R}$$

in cartesian coordinates

$$Q = \Psi(P) = \alpha x_1 + \beta x_2 + \gamma = \begin{pmatrix} \alpha & \beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \gamma$$

in homogeneous coordinates

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \left[\begin{array}{cc|c} \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{array}\right]\begin{bmatrix} x \\ 1 \end{bmatrix}$$

it is an affine map $y = A(x) + (b)$

In general, any linear function $f(x)=\alpha^T x+\beta=\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n+\beta$ can be considered as an affine map between $\mathbb{R}^n$ and $\mathbb{R}$.

in homogeneous coordinates

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & A & & b \\ 0 & \dots & 0 & 1 \end{array}\right]\begin{bmatrix} x \\ 1 \end{bmatrix}$$

augmented vectors

augmented matrix or affine transformation matrix

In the general case, when the last row vector is not restricted to be $[\,0\ \dots\ 0 \mid 1\,]$, the matrix is used to perform projective transformations, and it is called a projective transformation matrix.

Example of a non-affine map in $\mathbb{R}^2$

The 2D projective transformation (or homography)

$$\Phi : P(x,y)^{\mathsf{T}} \in \mathbb{R}^2 \mapsto \Phi(P) = Q\left(\frac{ax+by+c}{gx+hy+k}, \frac{dx+ey+f}{gx+hy+k}\right)^{\mathsf{T}} \in \mathbb{R}^2$$

can be described as a matrix-vector product by resorting to homogeneous coordinates:

$$P(x,y) = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} x^* \\ y^* \\ w^* \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow Q(X,Y) = \left(x^*/w^*, \; y^*/w^* \right)$$

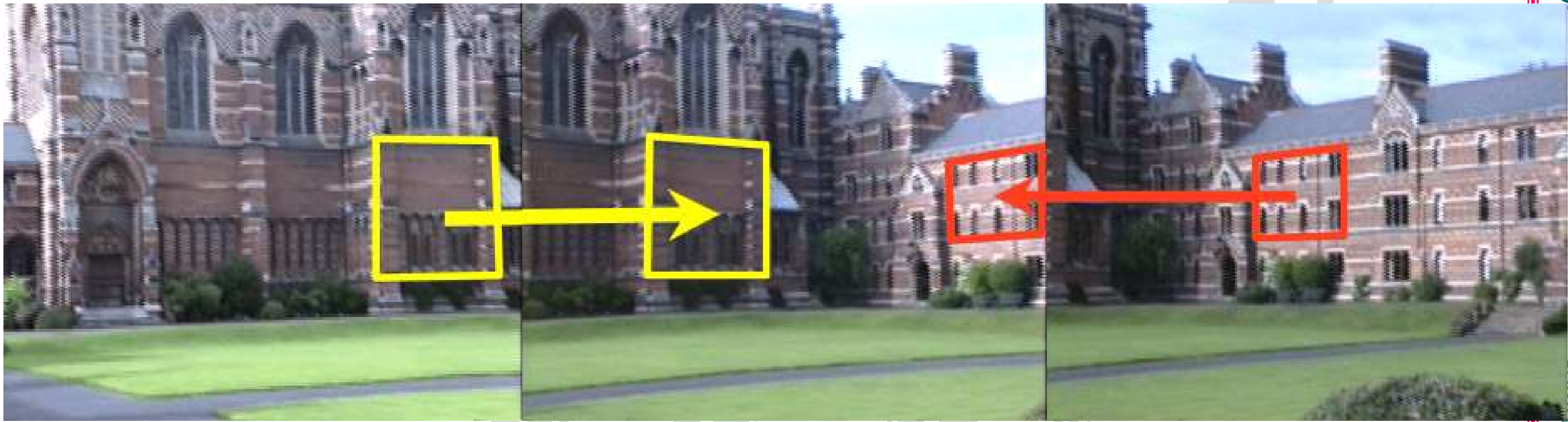
we can set $k=1$, by scaling all the matrix elements, so that we have 8 parameters

Indeed

$$\begin{pmatrix} x^* \\ y^* \\ w^* \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+c \\ dx+ey+f \\ gx+hy+k \end{pmatrix} \Rightarrow \begin{pmatrix} x^*/w^* \\ y^*/w^* \end{pmatrix} = \begin{pmatrix} \dfrac{ax+by+c}{gx+hy+k} \\ \dfrac{dx+ey+f}{gx+hy+k} \end{pmatrix} \begin{matrix} \leftarrow \mathbf{X} \\ \leftarrow \mathbf{Y} \end{matrix} \; Q(\mathbf{X},\mathbf{Y})$$

If $g=h=0$ then the projective transformation reduces to an affine map (based on 6 parameters).

Application example of a homography: mosaicing



Example of a linear map as an affinity: horizontal shear in $\mathbb{R}^2$

$$\Psi : x \in \mathbb{R}^2 \longrightarrow y = A\,x \in \mathbb{R}^2 \qquad A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \; r \in \mathbb{R}$$

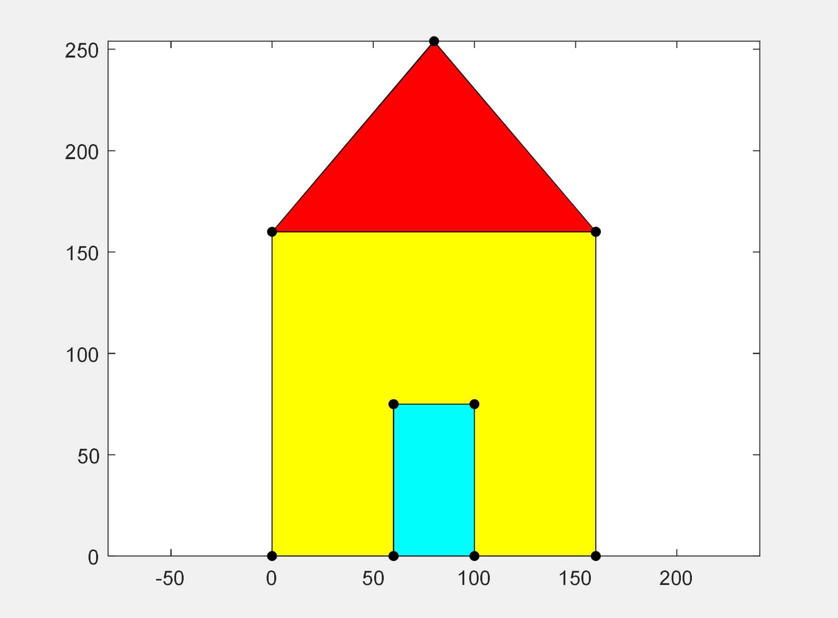
$r = \tan(\theta)$

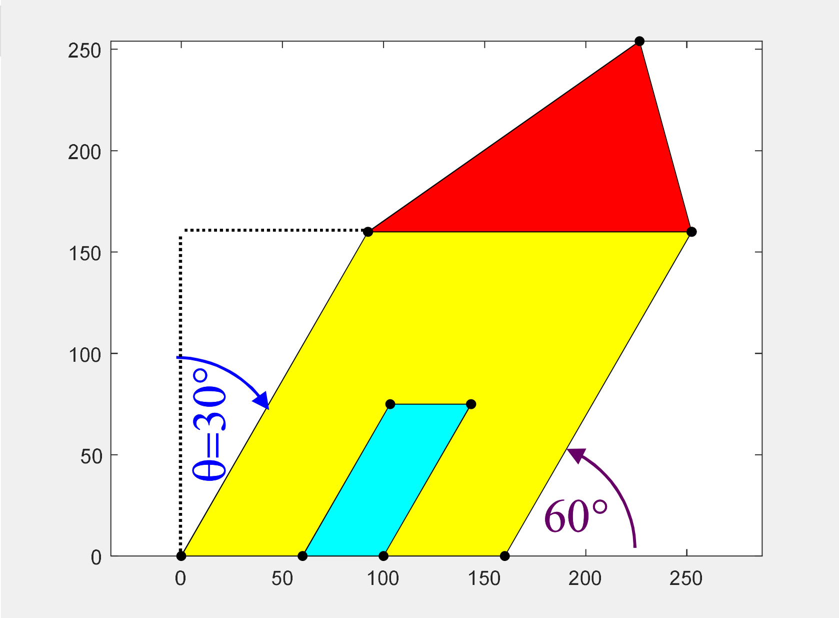
colorful little house drawn by a child

```
X1=[0 160 160 0];   Y1=[0 0 160 160]; P1=[X1;Y1]; % yellow
X2=[0 160 80];      Y2=[160 160 254]; P2=[X2;Y2]; % red
X3=[60 100 100 60]; Y3=[ 0 0 75 75];  P3=[X3;Y3]; % cyan
patch(X1,Y1,'y'); axis equal; hold on; box on; patch(X2,Y2,'r'); patch(X3,Y3,'c')
P=[P1 P2 P3]; plot(P(1,:),P(2,:),'.k','MarkerSize',15)
```

colorful little house transformed by the horizontal shear

```
th=pi/6; r=tan(th); A=[1 r;0 1]; Q1=A*P1; Q2=A*P2; Q3=A*P3; Q=[Q1 Q2 Q3];
patch(Q1(1,:),Q1(2,:),'y'); axis equal; hold on; box on
patch(Q2(1,:),Q2(2,:),'r'); patch(Q3(1,:),Q3(2,:),'c')
plot(Q(1,:),Q(2,:),'.k','MarkerSize',15)
```





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Example of a linear map as an affinity: vertical shear in $\mathbb{R}^2$

$$\Psi : x \in \mathbb{R}^2 \longrightarrow y = A\,x \in \mathbb{R}^2 \qquad A = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix},\ r \in \mathbb{R}$$

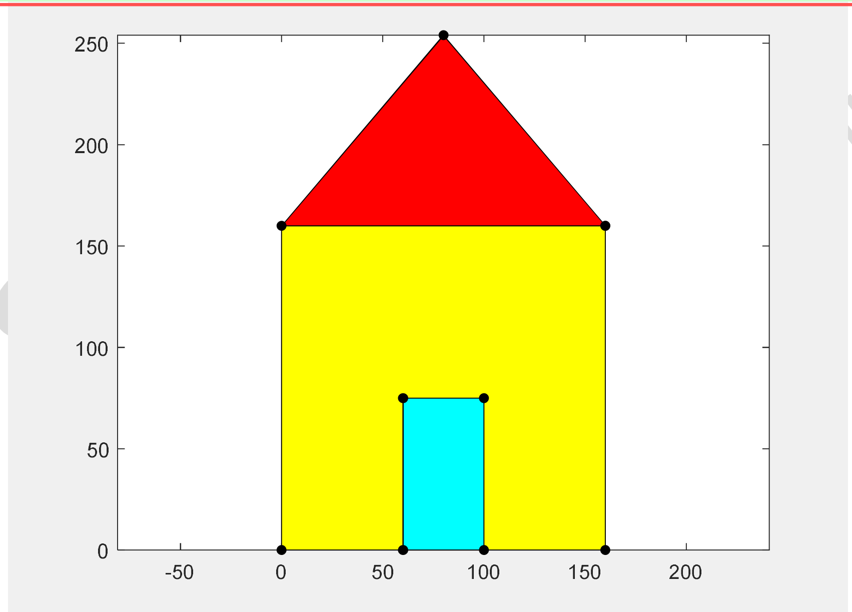
$r = \tan(\theta)$

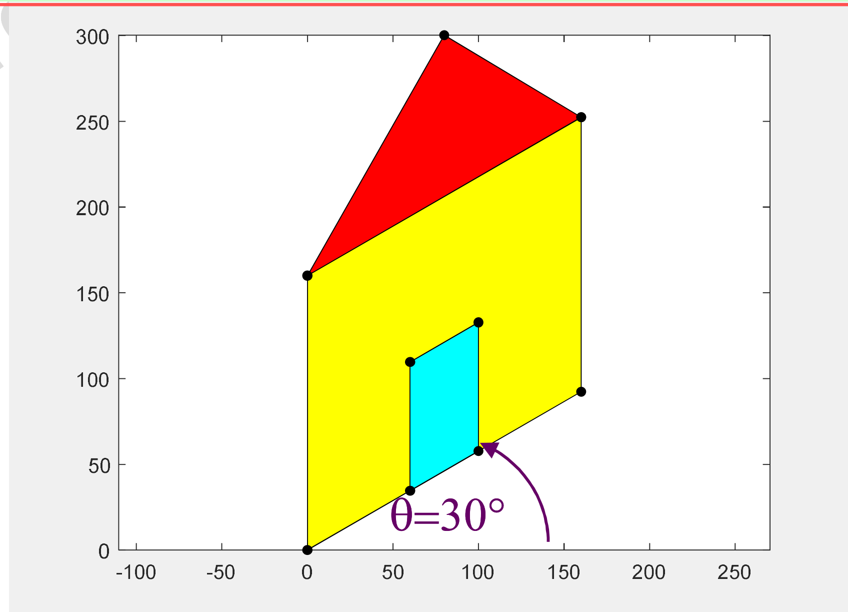
colorful little house drawn by a child

```
X1=[0 160 160 0];   Y1=[0 0 160 160]; P1=[X1;Y1]; % yellow
X2=[0 160 80];      Y2=[160 160 254]; P2=[X2;Y2]; % red
X3=[60 100 100 60]; Y3=[ 0 0 75 75];  P3=[X3;Y3]; % cyan
patch(X1,Y1,'y'); axis equal; hold on; box on; patch(X2,Y2,'r'); patch(X3,Y3,'c')
P=[P1 P2 P3]; plot(P(1,:),P(2,:),'.k','MarkerSize',15)
```

colorful little house transformed by the vertical shear

```
th=pi/6; r=tan(th); A=[1 0;r 1]; Q1=A*P1; Q2=A*P2; Q3=A*P3; Q=[Q1 Q2 Q3];
patch(Q1(1,:),Q1(2,:),'y'); axis equal; hold on; box on
patch(Q2(1,:),Q2(2,:),'r'); patch(Q3(1,:),Q3(2,:),'c')
plot(Q(1,:),Q(2,:),'.k','MarkerSize',15)
```





A Translation in $\mathbb{R}^n$, w.r.t. the Cartesian coordinates, is not a linear map

(because it cannot be written as a matrix-vector product, i.e. it cannot be expressed as a map induced by a matrix).

Example

$$T \quad : \quad x \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} \in \mathbb{R}^2$$

No matrix $A(2\times 2)$ exists such that $y = T(x) = Ax$

a translation is an affine transformation

T can be written as $y = T(x) = \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} = Ix + \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$$y = Ax + v$$

Example: Translation in homogeneus coordinates

The translation $T \; : \; x \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} \in \mathbb{R}^2$

is written in homogeneus coordinates as

$$y = T(x) = \begin{pmatrix} \dfrac{X_1}{X_3} - 2 \\ \dfrac{X_2}{X_3} - 1 \end{pmatrix} = \begin{pmatrix} \dfrac{X_1 - 2X_3}{X_3} \\ \dfrac{X_2 - X_3}{X_3} \end{pmatrix} \equiv \begin{pmatrix} X_1 - 2X_3 \\ X_2 - X_3 \\ X_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Therefore it becomes

$$T = t_A \; : \; X \in \mathbb{R}^3 \longrightarrow Y = \mathbf{A}X = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathbb{R}^3$$

origin displacement

identity matrix

$$T = \begin{pmatrix} I & \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

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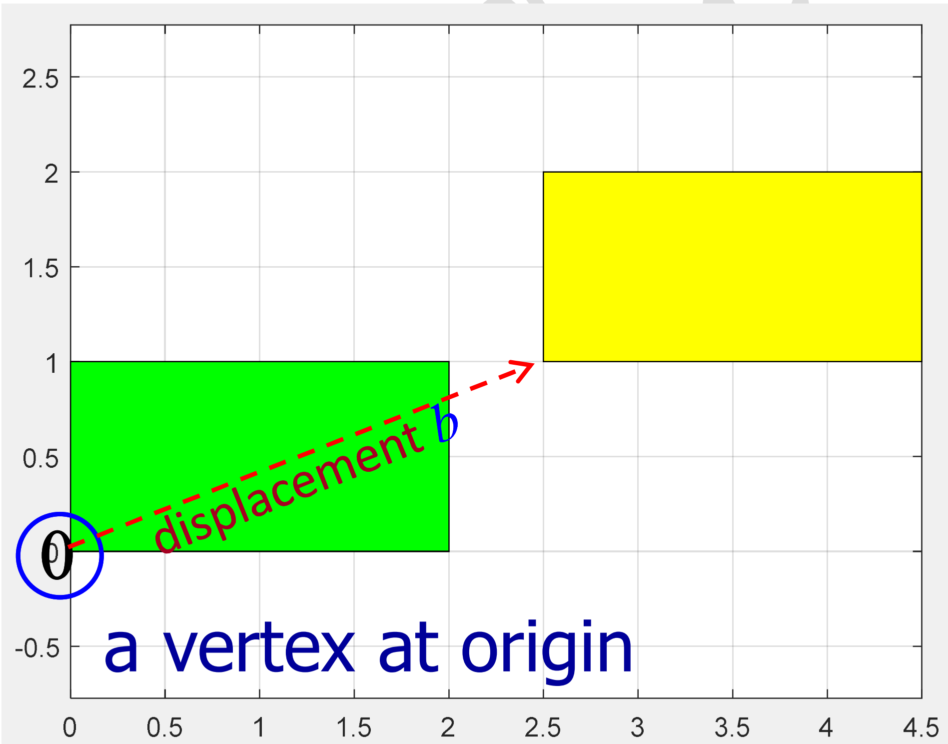
Example of affinity: Translation in $\mathbb{R}^2$

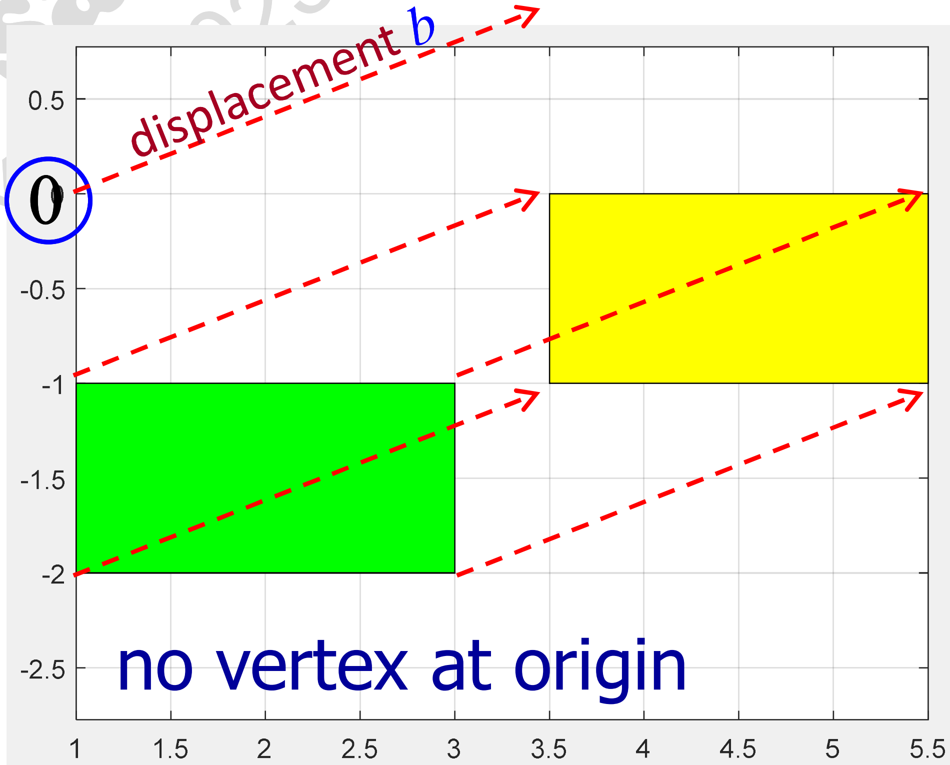
$$\Psi \; : \; x \in \mathbb{R}^2 \longrightarrow y = Ix + b \in \mathbb{R}^2$$

```
b=[2.5 1]';
X=[0 2 2 0]; Y=[0 0 1 1];
patch(X,Y,'g'); axis equal; hold on
B=[X;Y] + repmat(b,1,numel(X));
patch(B(1,:),B(2,:),'y')
```

```
b=[2.5 1]';
X=1+[0 2 2 0]; Y=-2+[0 0 1 1];
patch(X,Y,'g'); axis equal; hold on
B=[X;Y] + repmat(b,1,numel(X));
patch(B(1,:),B(2,:),'y')
```

isometry

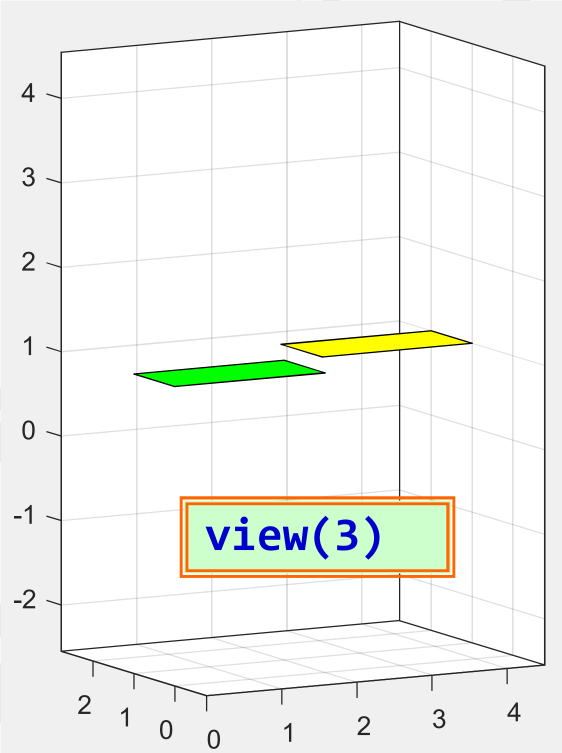


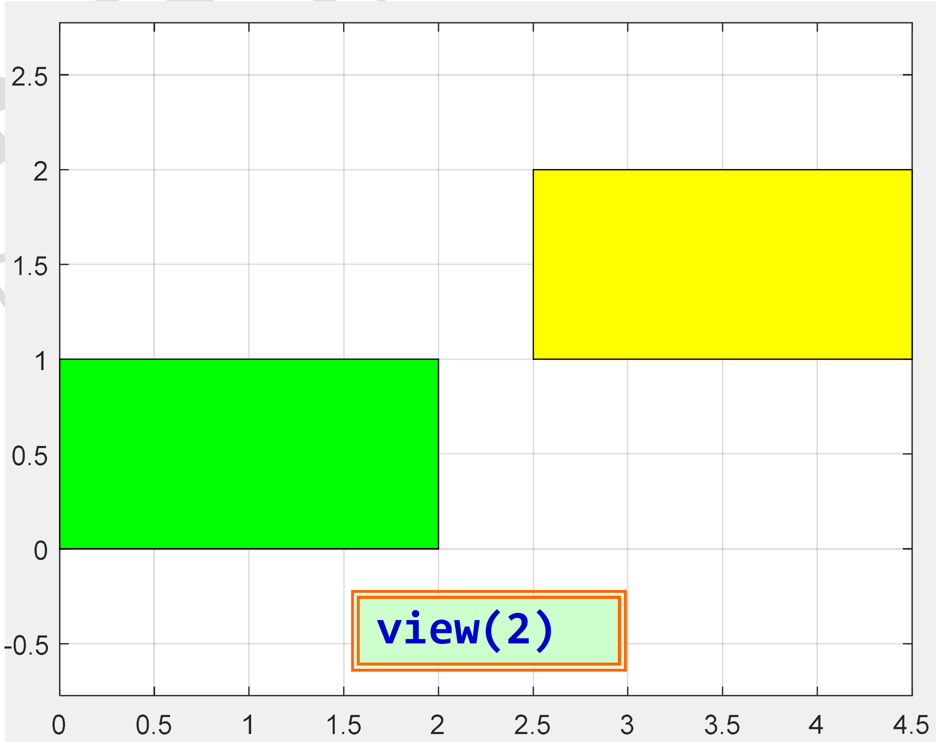


Example of affinity: Translation in $\mathbb{R}^2$ in homogeneous coordinates

$$\Psi \;:\; x \in \mathbb{R}^3 \longrightarrow y = Ax \in \mathbb{R}^3 \qquad A = \begin{pmatrix} I_2 & \vec{b} \\ \vec{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

```
b=[2.5 1]'; A=[eye(2) b;0 0 1];
X=[0 2 2 0];Y=[0 0 1 1]; P=[X;Y;ones(size(X))];
patch(P(1,:),P(2,:),P(3,:),'g'); axis('equal'); hold on
Q=A*P;     translation as matrix-vector product
patch(Q(1,:),Q(2,:),Q(3,:),'y')
```

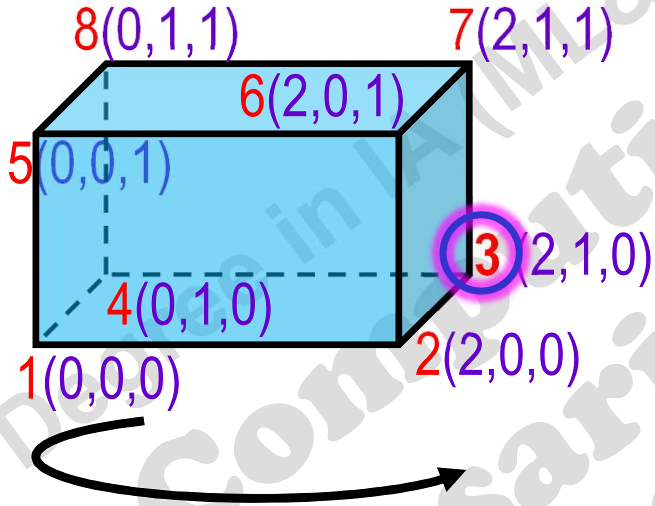




How to use MATLAB **patch()** to draw 3D objects

1) Set object vertex coordinates

```
vert=[0 0 0;   % 1
      2 0 0;   % 2
      2 1 0;   % 3
      0 1 0;   % 4
      0 0 1;   % 5
      2 0 1;   % 6
      2 1 1;   % 7
      0 1 1]   % 8
vert =
     0     0     0
     2     0     0
     2     1     0
     0     1     0
     0     0     1
     2     0     1
     2     1     1
     0     1     1
```



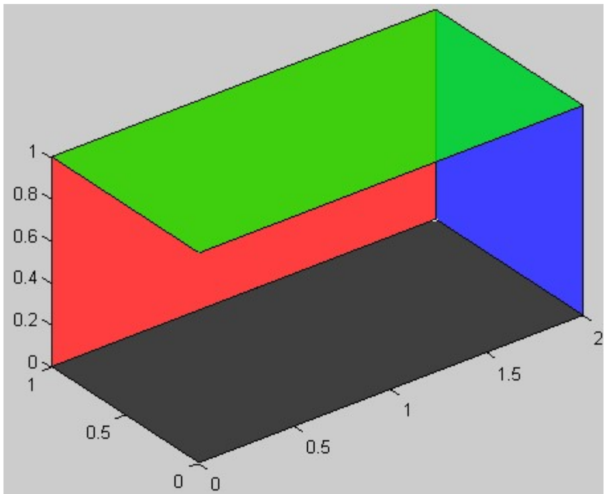
0) Enumerate vertices

2) Set object faces

```
face=[1 2 3 4;   % 1
      3 7 8 4;   % 2
      2 3 7 6;   % 3
      5 6 7 8];  % 4
```

3) Set face colors in RGB

```
col=[0 0 0;   % 1
     1 0 0;   % 2
     0 0 1;   % 3
     0 1 0];  % 4
```



4) Draw with parameters "Name"-"Value"

```
patch('Faces',face,'Vertices',vert, ...
      'FaceVertexCData',col, ...
      'FaceColor','flat', ...
      'FaceAlpha',0.5) % trasparency α∈[0,1]
view(3); axis equal; axis tight
```

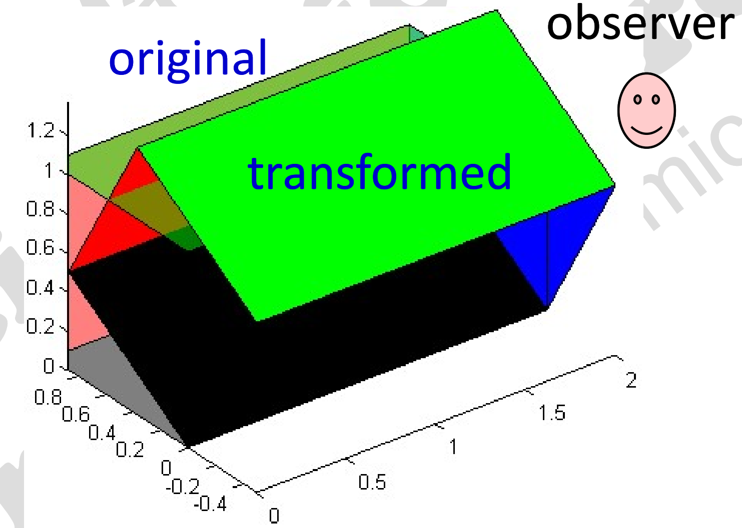
5) If you want, add light effects

Affinity example: rotation around X-axis by $\alpha=\pi/6$ in $\mathbb{R}^3$

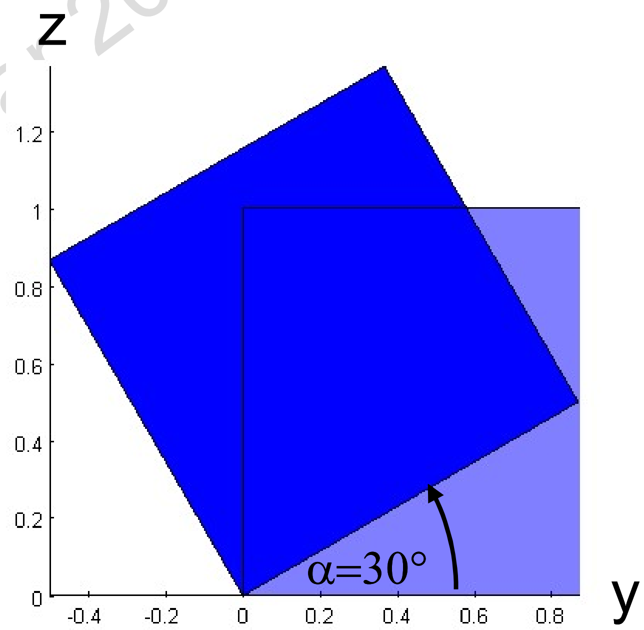
$$\Psi \; : \; x \in \mathbb{R}^3 \longrightarrow y = Ax \in \mathbb{R}^3$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

```
vert=...; fac=...; col=...;
alpha=pi/6; A=blkdiag(1,[cos(alpha) -sin(alpha);sin(alpha) cos(alpha)]);
B=A*vert';
patch('Faces',face,'Vertices',B','FaceVertexCData',col,'FaceColor','flat')
view(3); axis equal; axis tight; hold on
patch('Faces',face,'Vertices',vert, ...
```



a vertex at origin

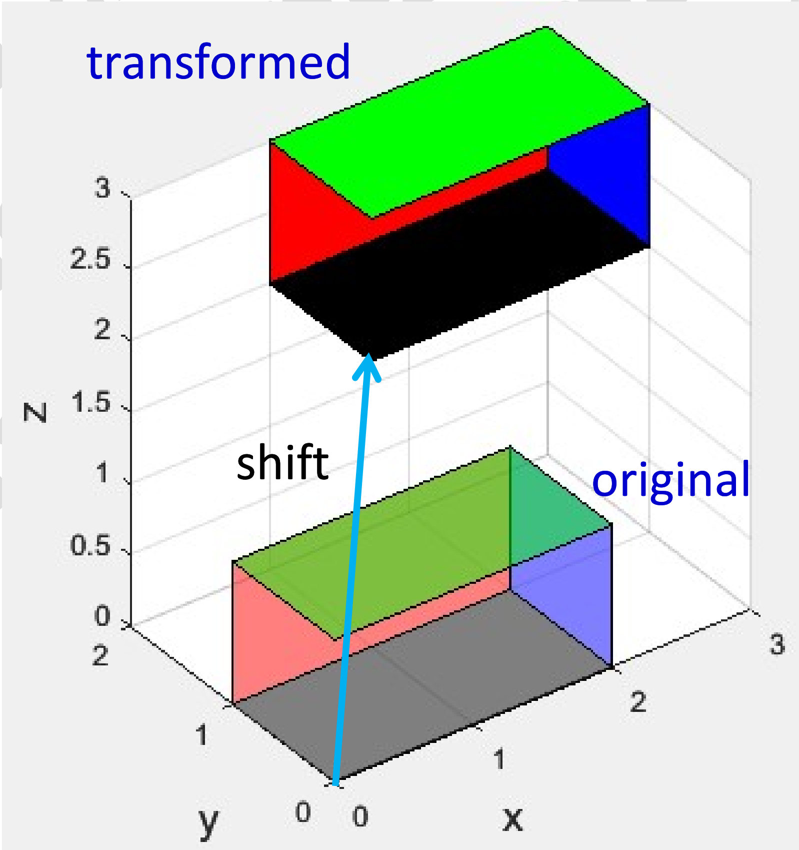


observer's viewpoint

Affinity example: translation of shift $(1,1,2)^T$ in $\mathbb{R}^3$

$$\Psi \; : \; x \in \mathbb{R}^3 \longrightarrow y = x + b \in \mathbb{R}^3$$

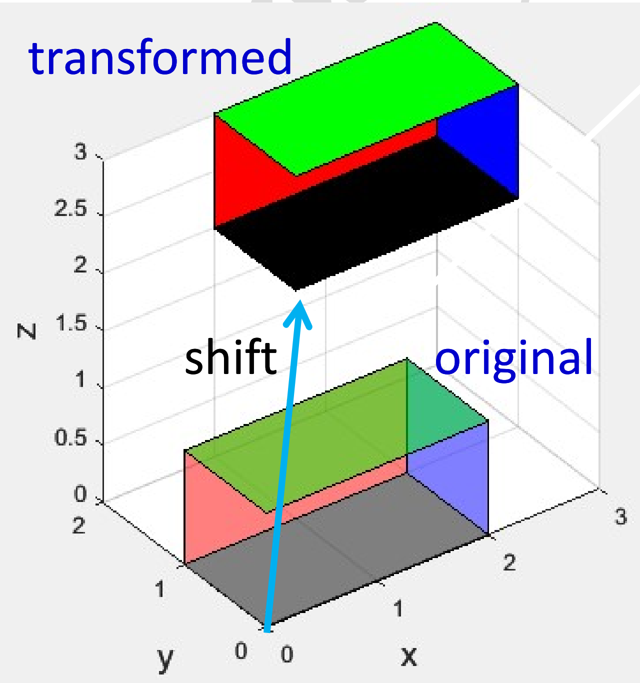
```
vert=...; fac=...; col=...;
b=[1;1;2]; B=vert' + b;
patch('Faces',face,'Vertices',B','FaceVertexCData',col,'FaceColor','flat')
view(3); axis equal; axis tight; hold on
patch('Faces',face,'Vertices',vert, ...
```



Affinity example: translation of shift $(1,1,2)^T$ in $\mathbb{R}^3$ in homogeneous coordinates

$$\Psi \;:\; x \in \mathbb{R}^4 \longrightarrow y = Ax \in \mathbb{R}^4 \qquad A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
vert=...; fac=...; col=...;
b=[1;1;2]; A=[eye(3) b;0 0 0 1];
B=A*[vert';ones(1,size(vert,1))]; B=B(1:3,:);
patch('Faces',face,'Vertices',B','FaceVertexCData',col,'FaceColor','flat')
view(3); axis equal; axis tight; hold on
patch('Faces',face,'Vertices',vert, ...
```



Exercise: Establish whether the following maps are linear or affine, and check if they are invertible. For an invertible map, find its inverse. Display in MATLAB the effect of the map and of its inverse on a unitary square.

$$\phi \; : \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} 2x_1 - 2x_2 + 1 \\ -x_1 + x_2 - 1 \end{pmatrix} \in \mathbb{R}^2$$

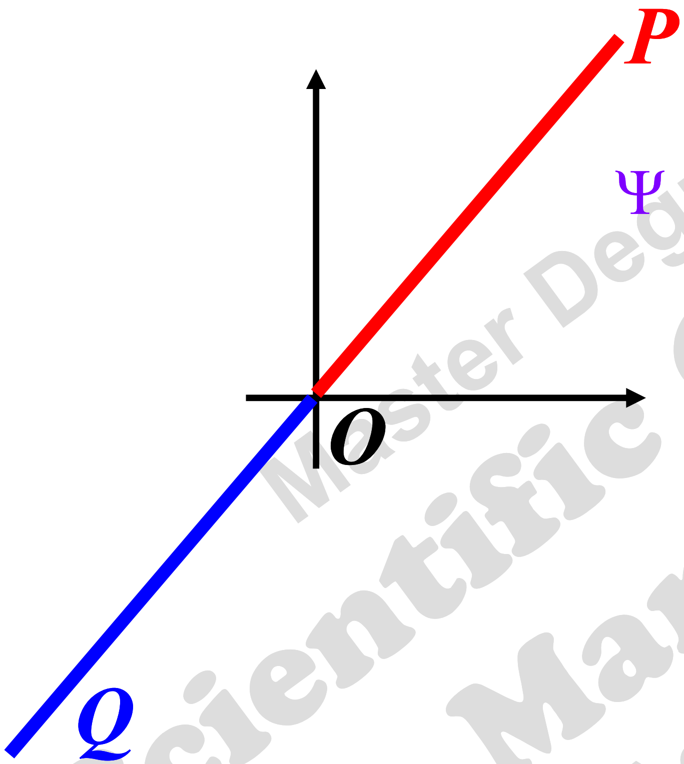
$$\phi \; : \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} 2x_1 - x_2 + 1 \\ -x_1 + x_2 - 1 \end{pmatrix} \in \mathbb{R}^2$$

$$\phi \; : \; x = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \longrightarrow y = \begin{pmatrix} 9 & 3 & -2 \\ 3 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} x \in \mathbb{R}^3$$

$$\phi \; : \; x = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \longrightarrow y = \begin{pmatrix} -1 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} x \in \mathbb{R}^3$$

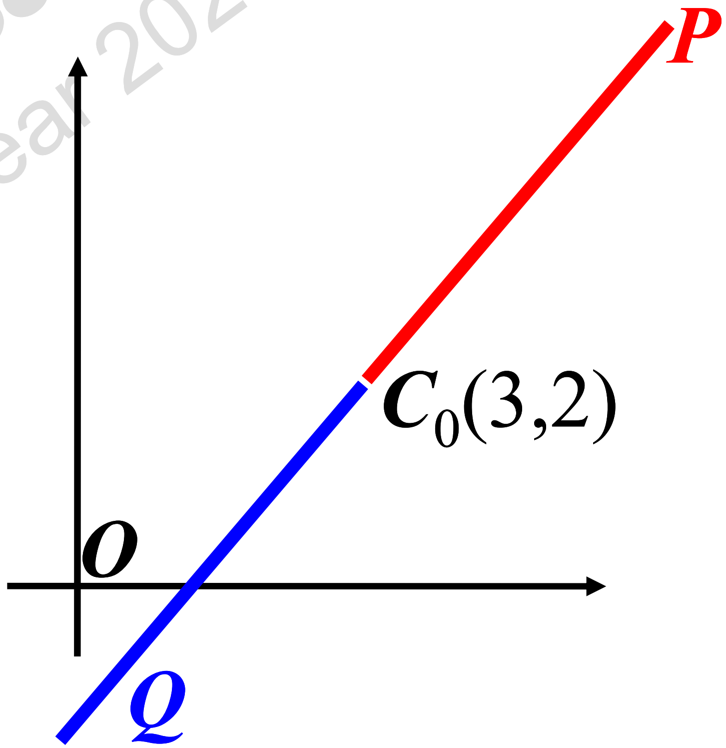
Exercise: Find the matrix equation of a central reflection with respect to the origin of the real plane (isomorphism), and that one of a central reflection with respect to a generic center $C_0(x_0,y_0)$ (affinity).

Write also the matrix equation in homogeneous coordinates.



$$\Psi \; : \; P = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow Q = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \in \mathbb{R}^2$$

$$\Psi \; : \; P = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow Q = \textbf{?} \in \mathbb{R}^2$$



Exercises on affine maps

Find the matrix equation of the rotation (of the real plane) by an angle θ around a point $P_0(x_0,y_0)$ non at origin.

Find the matrix equation of the rotation (of the real 3D space) by an angle θ around an axis passing through a point $P_0(x_0,y_0)$ and parallel to a direction (α,β,γ).