## L. Magistrale in IA (ML&BD)

# Scientific Computing
# (part 2 – 6 credits)

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# Contents

- **Linear Mappings (homomorphisms) and some properties.**
- **Iso-, endo-, auto-morphisms.**
- **Kernel (or Null Space) $\mathscr{N}(F)$ and Range (or Image Space) $\mathscr{R}(F)$ of a linear mapping.**

# Definition of Linear Mappings

Let $U$ and $V$ be two linear spaces with a field $K$.

A ***mapping*** (or ***transformation***) $F$

$$F \quad : \quad U \longrightarrow V$$

is said a **linear map** if it preserves the vector space structure, i.e. if:

♣ $\forall\, u,v \in U \qquad F(u+v) = F(u) + F(v)$

♣ $\forall\, u \in U,\ \forall\, \alpha \in K \qquad F(\alpha u) = \alpha\, F(u)$

… to summarize, in practice, the linearity rule holds:

$$\forall\, u,v \in U,\ \forall\, \alpha,\beta \in K \quad F(\alpha u+\beta v)=\alpha F(u)+\beta F(v)$$

# Example 1

- The mapping

$$\Phi \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \longrightarrow \begin{pmatrix} 4x+y \\ y-x \end{pmatrix} \in \mathbb{R}^2$$

is linear.

## Proof

Let us consider: $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ : $u = \begin{pmatrix} x_u \\ y_u \end{pmatrix}, \quad v = \begin{pmatrix} x_v \\ y_v \end{pmatrix}$

By the definition of $\Phi$, we get:

$$\Phi(\alpha u + \beta v) = \Phi \begin{pmatrix} \alpha x_u + \beta x_v \\ \alpha y_u + \beta y_v \end{pmatrix} = \begin{pmatrix} 4(\alpha x_u + \beta x_v) + (\alpha y_u + \beta y_v) \\ (\alpha y_u + \beta y_v) - (\alpha x_u + \beta x_v) \end{pmatrix} =$$

$$= \begin{pmatrix} 4\alpha x_u + \alpha y_u \\ \alpha y_u - \alpha x_u \end{pmatrix} + \begin{pmatrix} 4\beta x_v + \beta y_v \\ \beta y_v - \beta x_v \end{pmatrix} = \alpha \begin{pmatrix} 4x_u + y_u \\ y_u - x_u \end{pmatrix} + \beta \begin{pmatrix} 4x_v + y_v \\ y_v - x_v \end{pmatrix} =$$

$$= \; \longrightarrow \; = \alpha \Phi(u) + \beta \Phi(v) \quad \Longrightarrow \quad \Phi \text{ is a linear mapping}$$

# Example 2

- **The mapping**

$$\Phi \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \longrightarrow \begin{pmatrix} x+1 \\ y-x \end{pmatrix} \in \mathbb{R}^2$$

is **not** linear.

## Proof

Let us consider: $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$ : $u = \begin{pmatrix} x_u \\ y_u \end{pmatrix}, \quad v = \begin{pmatrix} x_v \\ y_v \end{pmatrix}$

$$\Phi(\alpha u + \beta u) = \Phi \begin{pmatrix} \alpha x_u + \beta x_v \\ \alpha y_u + \beta y_v \end{pmatrix} = \begin{pmatrix} (\alpha x_u + \beta x_v) + 1 \\ (\alpha y_u + \beta y_v) - (\alpha x_u + \beta x_v) \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha x_u + 1 \\ \alpha y_u - \alpha x_u \end{pmatrix} + \begin{pmatrix} 4\beta x_v + 1 \\ \beta y_v - \beta x_v \end{pmatrix}$$

$$\Big\updownarrow \neq$$

$$\alpha \Phi(u) + \beta \Phi(v) = \alpha \begin{pmatrix} x_u + 1 \\ y_u - x_u \end{pmatrix} + \beta \begin{pmatrix} x_v + 1 \\ y_v - x_v \end{pmatrix} \Longrightarrow$$

$\Phi$ is not linear

# Example 3

- The following transformation is linear

$$F \quad : \quad \underbrace{a_0 + a_1 x + a_2 x^2}_{\text{2nd degree polynomial}} \in \Pi_2 \longrightarrow (a_0, a_1, a_2)^T \in \mathbb{R}^3$$

The linearity rule follows from the polynomial addition and the polynomial multiplication by a scalar.

$$\left.\begin{array}{l} P(x) = a_0 + a_1 x + a_2 x^2 \\ Q(x) = b_0 + b_1 x + b_2 x^2 \end{array}\right. \Rightarrow$$

$$F\left(\alpha P(x) + \beta Q(x)\right) = \left(\alpha a_0 + \beta b_0 + (\alpha a_1 + \beta b_1) x + (\alpha a_2 + \beta b_2) x^2\right) =$$

$$= \left(\alpha a_0 + \beta b_0, \quad \alpha a_1 + \beta b_1, \quad \alpha a_2 + \beta b_2\right)^T$$

$$\uparrow \; = \; \downarrow$$

$$\alpha F\left(P(x)\right) + \beta F\left(Q(x)\right) = \alpha \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \beta \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} \Longrightarrow \Phi \text{ is a linear mapping}$$

# Properties

- Any transformation that comes from a matrix, $A$(m×n), is a linear transformation $t_A$.
- $t_A$ is said transformation induced by the $A$ matrix

$$t_A \quad : \quad x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^m$$

The linearity rule follows from the matrix multiplication.

- Conversely, any linear transformation such as

$$F \quad : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

can always be written as a transformation induced by a suitable $A$ matrix of size (m×n)

Example:

$$F \quad : \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow \begin{pmatrix} 4x_1 + x_2 \\ x_2 - x_1 \end{pmatrix} \in \mathbb{R}^2$$

$$t_A \quad : \quad x \in \mathbb{R}^2 \longrightarrow t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}$$

# To detect the matrix which induces the map $F$

$$F \;:\; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow \begin{pmatrix} 4x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y \in \mathbb{R}^2$$

we think to the image vector $y=Ax$ as a linear combination of the columns in $A$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4x_1 + 1x_2 \\ -1x_1 + 1x_2 \end{pmatrix} = x_1 \begin{pmatrix} 4 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

# Some definitions

Let $F$ be a linear transformation

$F \; : \; U \longrightarrow U$

the same space

def ⟺ **endomorphism**

$F \; : \; U \longrightarrow V$

$F$ is bijective

def ⟺ **isomorphism**

$F \; : \; U \longrightarrow U$

the same space

$F$ is bijective

def ⟺ **automorphism**

# Examples

♣The following linear transformation is an ***automorphism***

$$F \quad : \quad (x, y)^T \in \mathbb{R}^2 \longrightarrow (4x + y, y - x)^T \in \mathbb{R}^2$$

♣The following linear transformation is an ***endomorphism***

$$t_A \quad : \quad x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^n$$

♣The following linear transformation is an ***automorphism***

$$t_A \quad : \quad x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^n$$

if, and only if, $A$ is invertible.

♣The following linear transformation is an ***isomorphism***

$$F \quad : \quad a_0 + a_1 x + a_2 x^2 \in \Pi_2 \longrightarrow (a_0, a_1, a_2)^T \in \mathbb{R}^3$$

# Kernel $\mathscr{N}$ and Range $\mathscr{R}$ of a linear mapping

Let $F \ : U \longrightarrow V$ be a linear transformation.

The **Kernel** (or **Null Space**) of the transformation is:

**DEF**

$$\mathscr{N}(F) = \{u \in U : F(u) = \underline{0} \in V\}$$

If $F$ is a $t_A$, i.e. the mapping induced by a matrix, $A$, then the Kernel of $F$ equals the **Null Space** of the matrix:

$$\mathscr{N}(F) = \mathscr{N}(A)$$

The **Range** (or **Image Space**) of the transformation is:

**DEF**

$$\mathscr{R}(F) = \{v \in V : \exists u \in U : F(u) = v\} = F(U)$$

If $F$ is a $t_A$, i.e. the mapping induced by a matrix, $A$, then the Range of $F$ equals the **Column Space** of the matrix:

$$\mathscr{R}(F) = \mathscr{R}(A)$$

# Theorem

## $\mathscr{N}(F) \subseteq U$ and $\mathscr{R}(F) \subseteq V$ are *Linear Subspaces*

Proof

(by the Theor. “a linear subspace contains all the linear combinations of its vectors”)

Since the transformation $F$ is linear, we have:

$$\forall u, v \in \mathscr{N}(F) \quad \Rightarrow \quad F(u) = F(v) = \underline{0}$$

$$\Rightarrow \quad \forall \alpha, \beta \in K, \quad F(\alpha u + \beta v) = \alpha F(u) + \beta F(v) = \underline{0}$$

and this means that $\alpha \boldsymbol{u} + \beta \boldsymbol{v} \in \mathscr{N}(\boldsymbol{F})$

Since the transformation $F$ is linear, we have:

$$\forall u, v \in \mathscr{R}(F) \quad \Rightarrow \quad \exists x, y \in U : F(x) = u \wedge F(y) = v$$

$$\Rightarrow \quad \forall \alpha, \beta \in K, \alpha u + \beta v = \alpha F(x) + \beta F(y) = F(\alpha x + \beta y)$$

and this means that $\alpha \boldsymbol{u} + \beta \boldsymbol{v} \in \mathscr{R}(\boldsymbol{F})$

# **Example 1**: Kernel and Range of $F$

$$F \;:\; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

$F$ corresponds to $t_A \;:\; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \longrightarrow Au = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^3$

rank($A$)=2

$$\mathscr{N}(F) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \;:\; F(u) = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = 0 \right\} \iff \mathscr{N}(F) = \{\underline{0}\}$$

$\mathscr{N}(F) = \mathscr{N}(A)$
dim $\mathscr{N}(A)$=0

$$\mathscr{R}(F) = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \right\} \iff \mathscr{R}(F) \text{ is the horizontal plane of } \mathbb{R}^3$$

$\mathscr{R}(F) = \mathscr{R}(A)$
dim $\mathscr{R}(A)$=2

# **Example 2**: Kernel and Range of $F$

$$F\ :\ A=\begin{pmatrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{pmatrix}\in M_{3\times 3}(\mathbb{R})\longrightarrow F(A)=A-A^{\mathrm{T}}\in M_{3\times 3}(\mathbb{R})$$

$$\mathcal{N}(F)=\left\{A:\ F(A)=A-A^{\mathrm{T}}=\underline{0}\right\}\quad\Longleftrightarrow$$

$\mathcal{N}(F)$ is the subspace of *symmetric matrices* of size (3×3)

$$\mathcal{R}(F)=\left\{A-A^{\mathrm{T}}=\boxed{\begin{pmatrix}0 & a_{12}-a_{21} & a_{13}-a_{31}\\ -(a_{12}-a_{21}) & 0 & a_{23}-a_{32}\\ -(a_{13}-a_{31}) & -(a_{23}-a_{32}) & 0\end{pmatrix}}\right\}\quad\Longleftrightarrow$$

$\mathcal{R}(F)$ is the subspace of *antisymmetric matrices* of size (3×3)

(or skew-symmetric matrices)

$$\boldsymbol{M=-M^{\mathrm{T}}}$$

**Exercise:** Why is $F$ a linear mapping?

# Exercise

By continuing the MATLAB code below, compute by means of *Symbolic Math Toolbox* the Kernel and the Range of $F$, where $F$ is the linear mapping of the Example 2 (previous page).

```
F=@(M) M - M.';
N=3; A=sym('a',N,'real')
A =
[a1_1, a1_2, a1_3]
[a2_1, a2_2, a2_3]
[a3_1, a3_2, a3_3]
assumptions
ans =
[in(a1_1, 'real'), in(a1_2, 'real'), ...
FA=F(A)
FA =
[          0, a1_2 - a2_1, a1_3 - a3_1]
[a2_1 - a1_2,           0, a2_3 - a3_2]
[a3_1 - a1_3, a3_2 - a2_3,           0]
```

```
isAlways(FA == FA.')
...
all(all(isAlways(FA == FA.')))
...
```

```
S=solve(tril(FA),'ReturnConditions',true)
...
```

⇒ $\mathcal{R}(FA)=$...

⇒ $\mathcal{N}(FA)=$...

# Example

- The following transformation

$$F\ :\ f(x)\ \text{[differentiable]} \longrightarrow f'(x)$$

is linear

The linearity rule follows from the properties of derivatives:

- ☐ The derivative of the sum of two differentiable functions is the sum of their derivatives.
- ☐ The derivative of the product of a scalar by a differentiable function is the product of the scalar and the derivative of the function.

# Exercise

What is the Kernel of this transformation?

# Contents

- **Injective and surjective linear maps.**
- **Theor.: $\mathscr{R}(A^\top)$ and $\mathscr{R}(A)$ are isomorphic.**
- **Automorphism as a change of basis.**
- **Example: advantage in using an orthonormal basis.**

$$\boldsymbol{F} : \overset{\text{domain}}{\boldsymbol{U}} \longrightarrow \overset{\text{codomain}}{\boldsymbol{V}}$$

$\boldsymbol{F}$ is a linear map between vector spaces.

## Theorem

$\boldsymbol{F}$ injective (injection) $\Longleftrightarrow$ $\mathcal{N}(\boldsymbol{F}) = \{\underline{0}\}$

and

$\boldsymbol{F}$ surjective (surjection) $\Longleftrightarrow$ $\mathcal{R}(\boldsymbol{F}) = \boldsymbol{V}$

$\boldsymbol{F}$ bijective (bijection) if $\boldsymbol{F}$ is both injective and surjective.

In particular, if it is induced by a matrix:

$$\boldsymbol{t}_{A(m\times n)} : x \in \mathbb{R}^{\boldsymbol{n}} \longrightarrow \boldsymbol{A}x \in \mathbb{R}^{\boldsymbol{m}}$$

$\boldsymbol{t}_A$ injective $\Longleftrightarrow$ rank$(\boldsymbol{A}) = \boldsymbol{n}$ $\quad \mathcal{N}(\boldsymbol{t}_A)=\mathcal{N}(\boldsymbol{A})$

and

$\boldsymbol{t}_A$ surjective $\Longleftrightarrow$ rank$(\boldsymbol{A}) = \boldsymbol{m}$ $\quad \mathcal{R}(\boldsymbol{t}_A)=\mathcal{R}(\boldsymbol{A})$

# Example

$$F \;:\; (u_1, u_2)^T \in \mathbb{R}^2 \longrightarrow (u_1, u_2, 0)^T \in \mathbb{R}^3$$

is an **injective** **linear map.**

It is linear since

$$F \equiv t_A : x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

and it is injective because

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longrightarrow F(u) = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \neq F(v) = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$$

$\mathscr{N}(F) = \{\underline{0}\}$ and $\mathscr{R}(F)$ = the horizontal plane in the 3D space, whose cartesian eq. is $z=0$.

**Example:** Given a non-zero vector, $\boldsymbol{v} \in \mathbb{R}^3$, the mapping

$$\boldsymbol{F} \;:\; \boldsymbol{u} \in \mathbb{R}^3 \longrightarrow \alpha = \langle \boldsymbol{u}, \boldsymbol{v} \rangle \in \mathbb{R}$$

is a surjective linear map, but it is not injective.

for istance, $\boldsymbol{v} = (3\ 2\ 1)^T$

$\boldsymbol{F}$ is linear because …

$$\boldsymbol{F} \equiv \boldsymbol{t}_A : \boldsymbol{x} \in \mathbb{R}^3 \longrightarrow \boldsymbol{y} = \boldsymbol{t}_A(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} \in \mathbb{R}$$

where $\boldsymbol{A} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

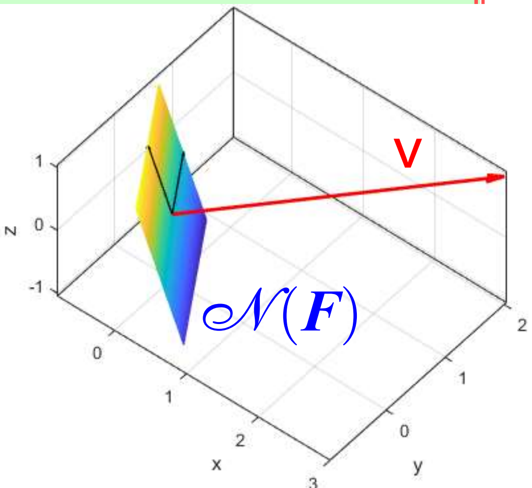
$\boldsymbol{F}$ is surjective because … $\forall \alpha \in \mathbb{R} \longrightarrow \exists ? u \in \mathbb{R}^3 : \langle u, v \rangle = \alpha \quad \Longleftrightarrow$

$$Au = \alpha \quad \Leftrightarrow \quad v_1 u_1 + v_2 u_2 + v_3 u_3 = \alpha \qquad A = [3\ 2\ 1] \Rightarrow$$

underdetermined linear system $\Rightarrow$ $\boldsymbol{F}$ is not injective

```
syms a b real; v=[3 2 1]'; N=numel(v);
syms x y [N 1] real
F=@(u) dot(v,u);
F1=F(a*x + b*y); F2=a*F(x) + b*F(y);
isAlways(F1 == F2) % Linearity
```

```
A=[3 2 1]; isAlways(F(x) == A*x)
ans =
  logical
   1
```



## Exercise

Given v=[3;2;1], by means of MATLAB Symbolic Math Toolbox:

- verify that $\boldsymbol{F}$ is a surjective map, but it is non-injective;
- compute (and display) the subspaces $\mathcal{N}(\boldsymbol{F})$ and $\mathcal{R}(\boldsymbol{A}^T)$.

# Graphical representation of the linear transformation $t_A$ associated with a matrix $A(m\times n)$

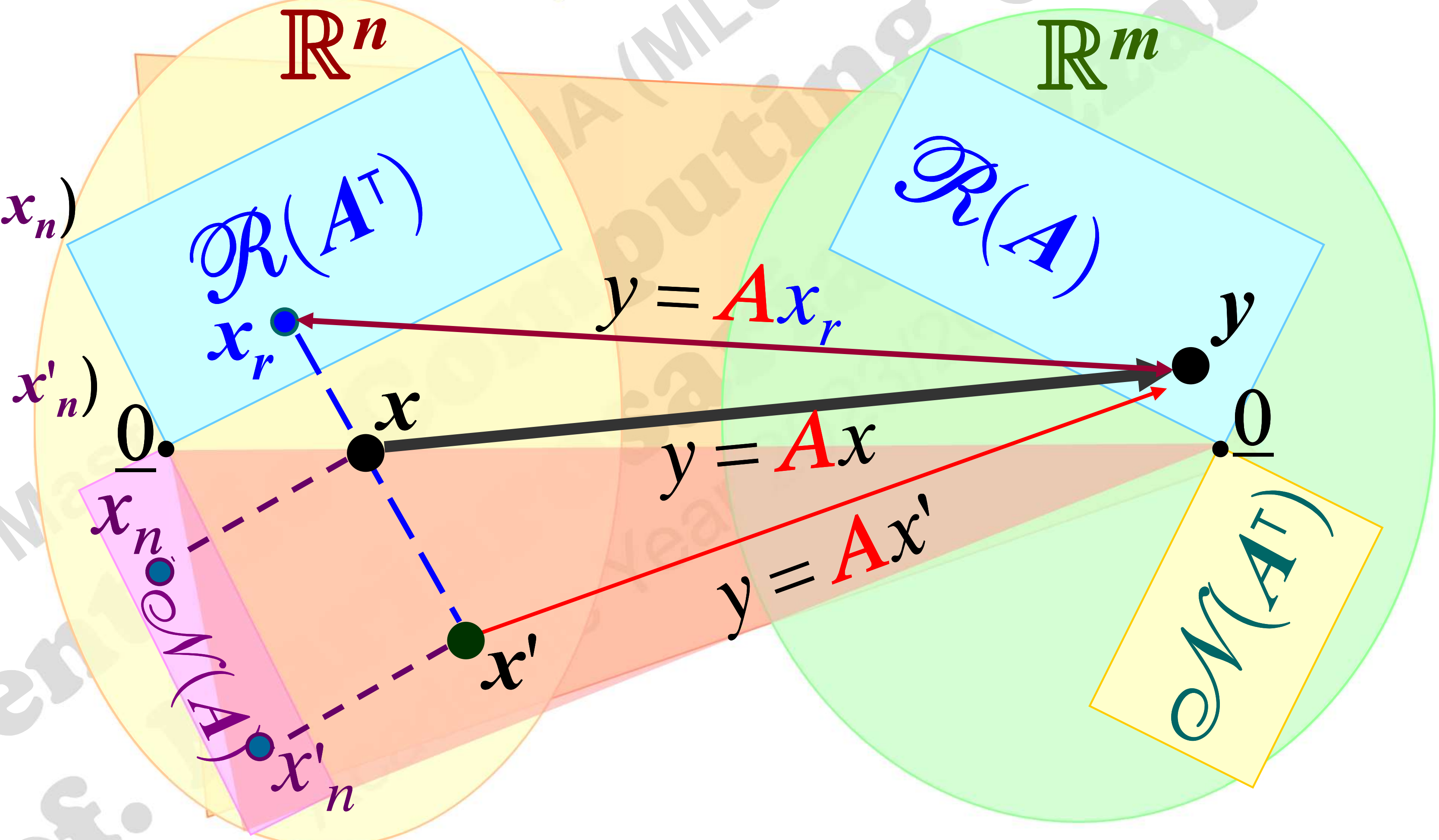
$$t_A \;:\; x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^m$$

$x = x_r + x_n$

$Ax = A(x_r + x_n)$

$x' = x_r + x'_n$

$Ax' = A(x_r + x'_n)$

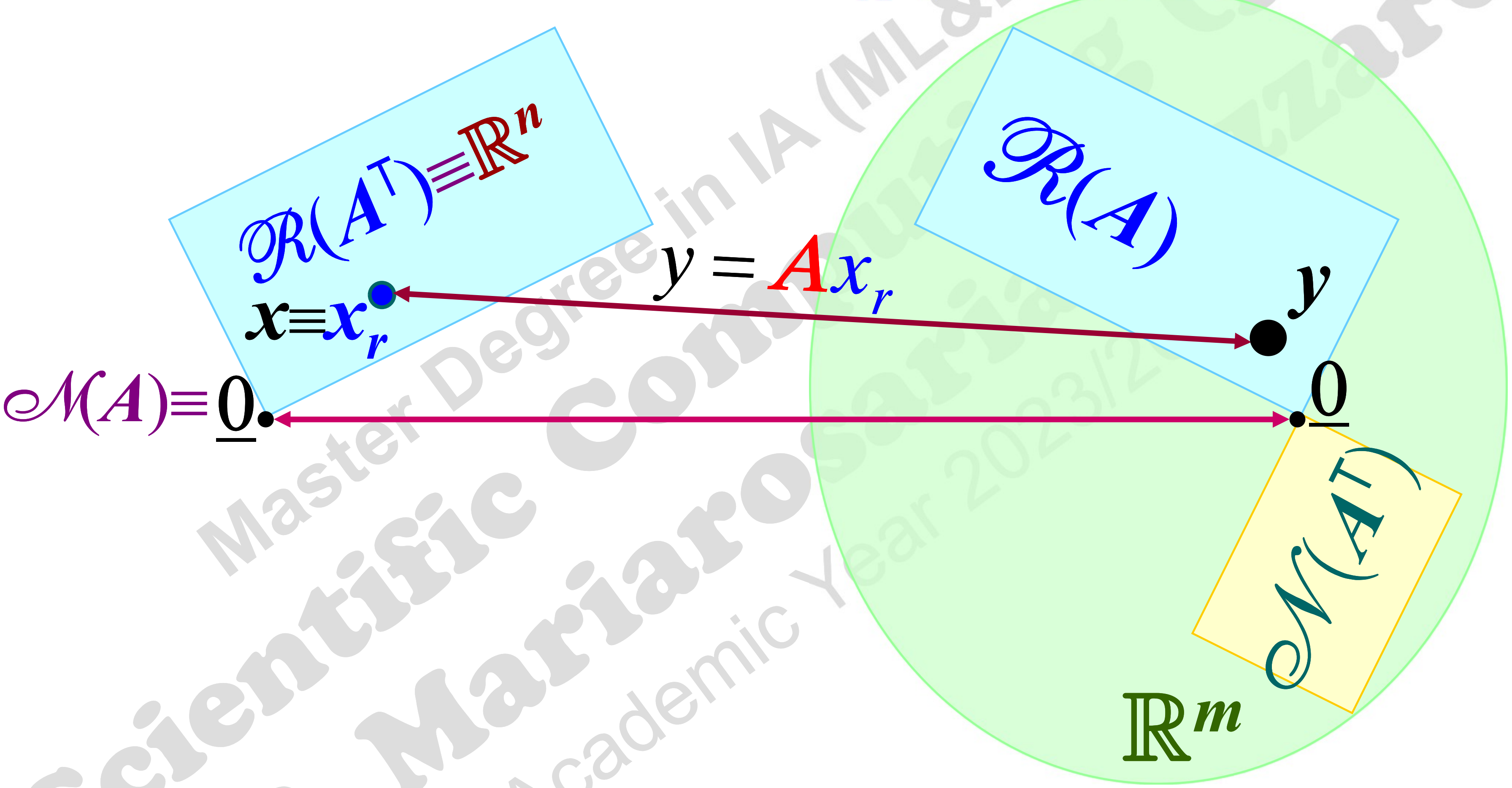


$$\mathbb{R}^n = \mathscr{N}(A) \oplus \mathscr{R}(A^T) \qquad \mathbb{R}^m = \mathscr{R}(A) \oplus \mathscr{N}(A^T)$$

$$\mathscr{N}(A) = \mathscr{R}(A^T)^{\perp} \qquad \mathscr{R}(A) = \mathscr{N}(A^T)^{\perp}$$

# Case of an injective map $t_A$

$$t_A \; : \; x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^m$$

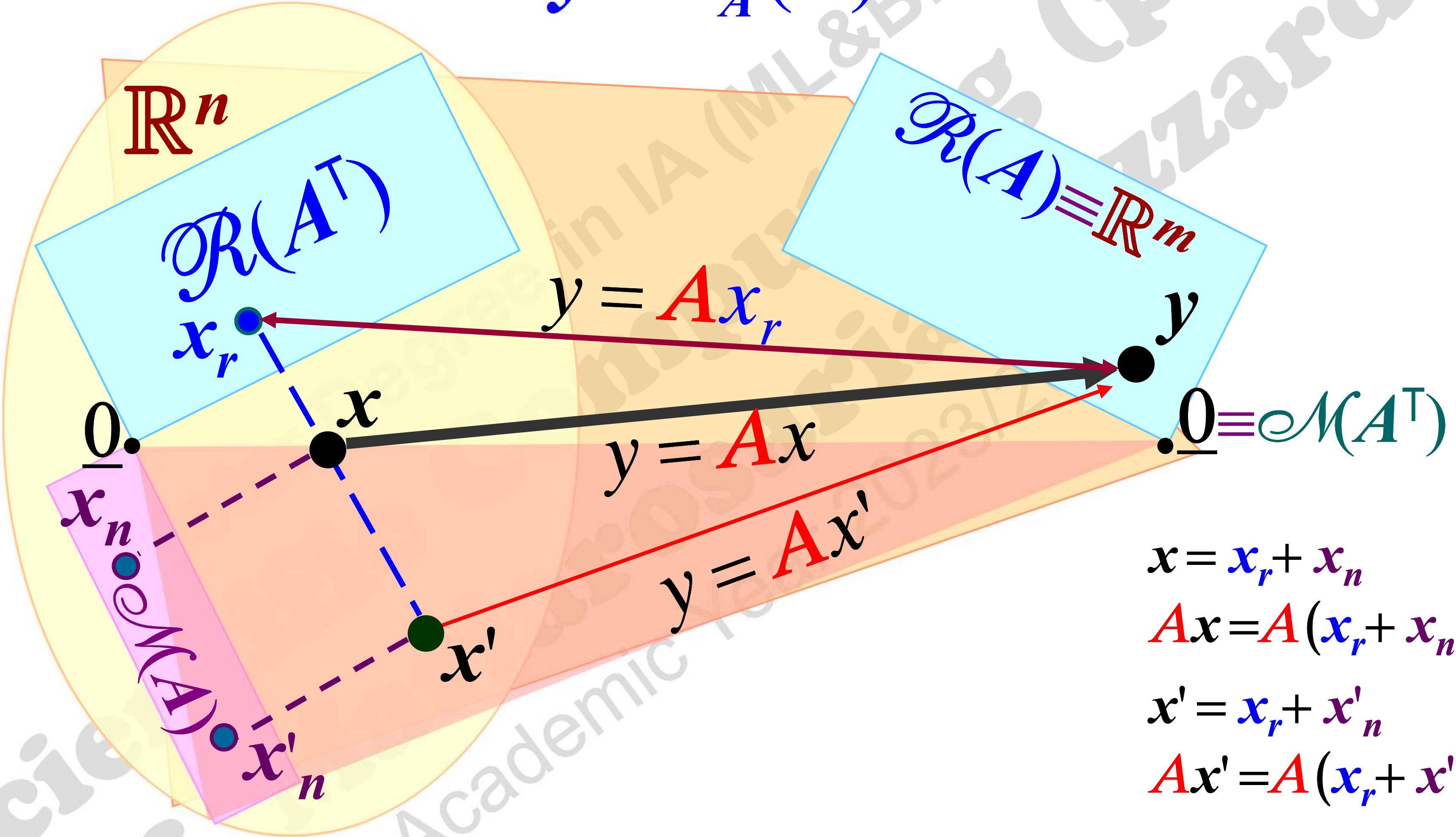


$$\mathbb{R}^n = \mathcal{R}(A^T) \qquad \mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T)$$

$$\mathcal{N}(A) = \{\underline{0}\} \qquad \mathcal{R}(A) = \mathcal{N}(A^T)^\perp$$

# Case of a surjective map $t_A$

$$t_A \; : \; \boldsymbol{x} \in \mathbb{R}^n \longrightarrow \boldsymbol{y} = t_A(\boldsymbol{x}) = A\boldsymbol{x} \in \mathbb{R}^m$$



$$\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$$
$$A\boldsymbol{x} = A(\boldsymbol{x}_r + \boldsymbol{x}_n)$$
$$\boldsymbol{x}' = \boldsymbol{x}_r + \boldsymbol{x}'_n$$
$$A\boldsymbol{x}' = A(\boldsymbol{x}_r + \boldsymbol{x}'_n)$$

$$\mathbb{R}^n = \mathscr{N}(A) \oplus \mathscr{R}(A^T) \qquad \mathbb{R}^m = \mathscr{R}(A)$$

$$\mathscr{N}(A) = \mathscr{R}(A^T)^\perp \qquad \mathscr{N}(A^T) = \{\underline{0}\}$$

$$t_A : x \in \mathbb{R}^n \longrightarrow Ax = y \in \mathbb{R}^m$$

$$\Phi = \boxed{t_{A/\mathscr{R}(A^T)}} : x_r \in \mathscr{R}(A^T) \longrightarrow Ax_r = y \in \mathscr{R}(A)$$

$\Phi$ is the restriction of $t_A$ to $\mathscr{R}(A^T)$

## Theorem

The restriction $\Phi$ (of the map $t_A$) between the **Row Space**, $\mathscr{R}(A^T)$, and the **Column Space**, $\mathscr{R}(A)$, is bijective.

This means that:

$\mathscr{R}(A^T)$ and $\mathscr{R}(A)$ are **isomorphic**.

In practice, $\mathscr{R}(A^T)$ and $\mathscr{R}(A)$ have the same geometric "shape"

# Proof

Th.: $A(m\times n)\ \forall y\in\mathcal{R}(A)\ \ \exists!\, x_r\in\mathcal{R}(A^T)\ :\ Ax_r = y$

$$\Phi\ :\ x_r\in\mathcal{R}(A^T)\subseteq\mathbb{R}^n \longrightarrow Ax_r = y\in\mathcal{R}(A)\subseteq\mathbb{R}^m$$

By definition of *Column Space*, $\forall y\in\mathcal{R}(A),\ \ \exists\ x\in\mathbb{R}^n\ :\ Ax = y$.

Since $\mathbb{R}^n=\mathcal{N}(A)\oplus\mathcal{R}(A^T)$, in general $\forall x\in\mathbb{R}^n$ can be written as:

$x = x_r + x_n\ :$ where $x_r\in\mathcal{R}(A^T)$ and $x_n\in\mathcal{N}(A)$

$\forall y\in\mathcal{R}(A) \ \Rightarrow\ \exists x_r : y = Ax = Ax_r + Ax_n = Ax_r,\ x_r\in\mathcal{R}(A^T)$

This proved that $\Phi$ is **surjective**, that is $\forall y$, $y$ is the image, by $\Phi$, of a vector $x_r\in\mathcal{R}(A^T)$.

To prove that $\Phi$ is **injective**, we assume that two vectors have the same image: $\exists\ x_r, x'_r\in\mathcal{R}(A^T) : Ax_r = Ax'_r = y$

$\Rightarrow\ A(x_r - x'_r) = \underline{0}\ \Rightarrow\ (x_r - x'_r)\in\mathcal{R}(A^T)\cap\mathcal{N}(A)$

But this is possible if, and only if, $x'_r=x_r$ since $\mathbb{R}^n=\mathcal{N}(A)\oplus\mathcal{R}(A^T)$.

# Attention!

$\boxed{1}$ $\Phi : x_r \in \mathscr{R}(A^T) \subseteq \mathbb{R}^n \longrightarrow \Phi(x_r) = Ax_r = y \in \mathscr{R}(A) \subseteq \mathbb{R}^m$

$\boxed{2}$ $\Phi^{-1} : y \in \mathscr{R}(A) \subseteq \mathbb{R}^m \longrightarrow \Phi^{-1}(y) = x_r \in \mathscr{R}(A^T) \subseteq \mathbb{R}^n$

$\boxed{3}$ $\Phi_{A^T} : y \in \mathscr{R}(A) \subseteq \mathbb{R}^m \longrightarrow A^T(y) = A^T Ax \in \mathscr{R}(A^T) \subseteq \mathbb{R}^n$

The first mapping is an isomorphism between $\mathscr{R}(A^T)$ and $\mathscr{R}(A)$; the other twos are isomorphisms between $\mathscr{R}(A)$ and $\mathscr{R}(A^T)$.

The second mapping is the inverse isomorphism of the first.

The last two mappings are generally different; they are the same only if A is an orthogonal matrix.

# **Example 1:** $t_A \; : \; x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 0 & 0 \end{pmatrix} \xrightarrow{G^{\downarrow}} \begin{pmatrix} \boxed{1} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = S$$

pivot

```
A=sym([…])
S=rref(A)
```

rank($A$)=1 Theor.

$t_A$ is neither injective nor surjective

```
RAT = colspace(A');
NA  = null(A);
```

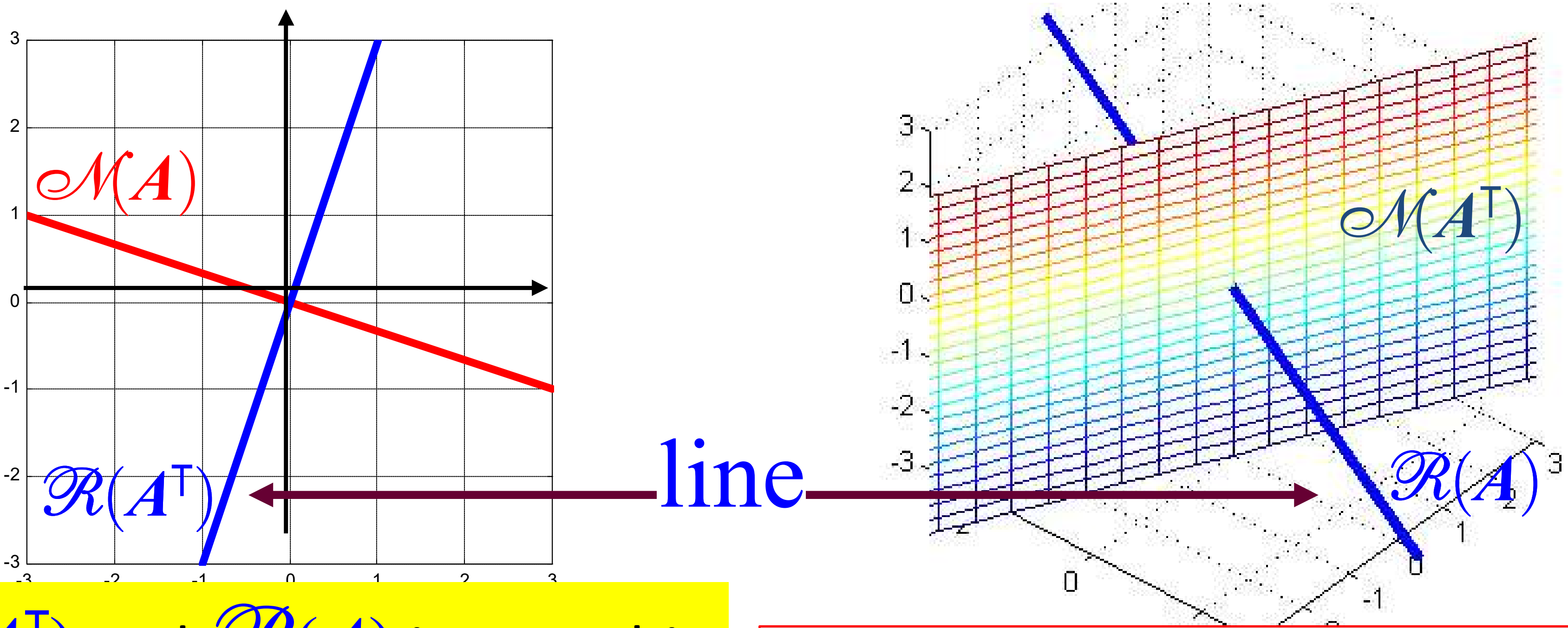
```
RA  = colspace(A);
NAT = null(A');
```

$\mathcal{R}(A^{\mathsf{T}})=\mathbf{span}\{(1,3)^{\mathsf{T}}\}$

$\mathcal{N}(A)=\mathbf{span}\{(-3,1)^{\mathsf{T}}\}$

$\mathcal{R}(A)=\mathbf{span}\{(1,2,0)^{\mathsf{T}}\}$

$\mathcal{N}(A^{\mathsf{T}})=\mathcal{R}(A)^{\perp}=\mathbf{span}\{(-2,1,0)^{\mathsf{T}},(0,0,1)^{\mathsf{T}}\}$



$\mathcal{R}(A^{\mathsf{T}})$ and $\mathcal{R}(A)$ isomorphic

Which isomorphism between $\mathcal{R}(A^{\mathsf{T}})$ and $\mathcal{R}(A)$?

$$A=\begin{pmatrix}1 & 3\\ 2 & 6\\ 0 & 0\end{pmatrix}$$

rank($\mathbf{A}$)=1

# Isomorphism between $\mathscr{R}(A^T)$ and $\mathscr{R}(A)$

$\mathscr{R}(A^T)=\textbf{span}\{(1,3)^T\}$ $\quad$ $\mathscr{R}(A)=\textbf{span}\{(1,2,0)^T\}$

$\mathscr{N}(A)=\textbf{span}\{(-3,1)^T$ $\quad$ $\mathscr{N}(A^T)=\textbf{span}\{(-2,1,0)^T, (0,0,1)^T\}$

$$\Phi\ :\ x_r\in\mathscr{R}(\mathbf{A}^T)\subset\mathbb{R}^2 \longrightarrow \Phi(x_r)=\mathbf{A}x_r=y\in\mathscr{R}(\mathbf{A})\subset\mathbb{R}^3$$

If $\Phi$ is bijective, we want to find a single $x_r$ ($\exists!\ x_r$) such that

$$\Phi^{-1}\ :\ y\in\mathscr{R}(\mathbf{A})\subseteq\mathbb{R}^3 \longrightarrow \Phi^{-1}(y)=\boxed{x_r}\in\mathscr{R}(\mathbf{A}^T)\subseteq\mathbb{R}^2$$

$\mathscr{R}(\mathbf{A})=\textbf{span}\{(1,2,0)^T\}$

⇩

$\forall y\in\mathscr{R}(\mathbf{A})\ \Leftrightarrow\ \exists\ \alpha\in\mathbb{R} : y=\alpha(1,2,0)^T$, and $\exists\ x\in\mathbb{R}^2 :\ \mathbf{A}x=y\ \Leftrightarrow\ \begin{cases}x_1+3x_2=\alpha\\ \cancel{2x_1+6x_2=2\alpha}\\ 0=0\end{cases}$

$\Leftrightarrow$ $x_1+3x_2=\alpha$ undetermined linear system ⇨ $\infty$ solutions

The general solution x of the indeterminate system

x can be written as $\Leftrightarrow$ $x=x_p+x_n$ where: $x_n$ is any vector in $\mathscr{N}(\mathbf{A})$

and $x_p$ is a particular solution of Ax=y.

$x_p=$ $\begin{matrix}x_1=\alpha\\ x_2=0\end{matrix}$ ⇦

```
syms a real; y=a*RA; xp=A\y;
```

In general $x_p\in\mathbb{R}^2$, but $x_p\notin\mathscr{R}(A^T)$. Since $\mathbb{R}^2=\mathscr{R}(A^T)\oplus\mathscr{N}(A)$ ⇨ $x_p=\boxed{x_r}+x'_n$

and this decomposition is unique.

```
B=[RAT NA]; % basis of R^2
coef=B\xp;   xr=RAT*coef(1)
```

# **Example 2:** $t_A \; : \; x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$

where $A = \begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 1 \end{pmatrix} = S \Rightarrow t_A$ surjective

pivots

in $\mathbb{R}^3$

$\mathcal{N}(A)=\text{span}\{(2,-1,1)^T\}$

$\mathcal{R}(A^T)=\text{span}\{(1,2,0)^T,(0,1,1)^T\}$

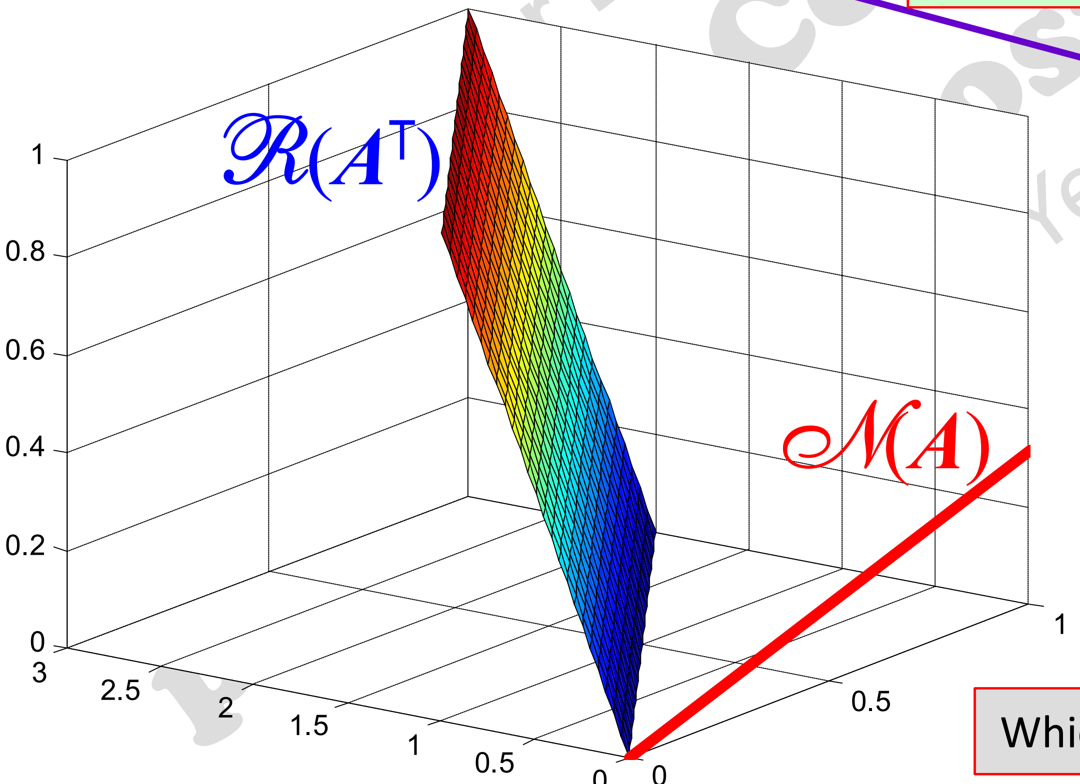
```
A   = sym([…]);
RAT = colspace(A');
NA  = null(A);
RA  = colspace(A);
NAT = null(A');
```

in $\mathbb{R}^2$

$\mathcal{N}(A^T)=\{\underline{0}\}$

$\mathcal{R}(A)=\mathbb{R}^2$

planes



$\mathcal{R}(A^T)$ and $\mathcal{R}(A)$ isomorphic

Which isomorphism between $\mathcal{R}(A^T)$ and $\mathcal{R}(A)$?

$A = \left( \boxed{\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}} \begin{matrix} 0 \\ 1 \end{matrix} \right)$

# Isomorphism between $\mathscr{R}(A^T)$ and $\mathscr{R}(A)$

$\mathscr{R}(A^T)=\text{span}\{(1,2,0)^T,(0,1,1)^T\}$ $\mathscr{R}(A)=\text{span}\{(1,0)^T,(2,1)^T\}$

$\mathscr{N}(A)=\text{span}\{(2,-1,1)^T\}$ $\mathscr{N}(A^T)=\{\underline{0}\}$

$$\Phi \;:\; x_r \in \mathscr{R}(\mathbf{A}^T) \subset \mathbb{R}^3 \longrightarrow \Phi(x_r) = \mathbf{A}x_r = y \in \mathscr{R}(\mathbf{A}) \subset \mathbb{R}^2$$

**Exercise:** Compute the isomorphism as seen before.

We can proceed as before, but now we can exploit the fact that $\mathbf{A}$ is a maximum rank matrix where rank($\mathbf{A}$) = number of rows.

$$\forall x_r \in \mathscr{R}(\mathbf{A}^T) \quad x_r = \mathbf{A}^T(\alpha,\beta)^T \longrightarrow \Phi(x_r) = y = \mathbf{A}\,x_r = \boxed{\mathbf{A}\mathbf{A}^T}(\alpha,\beta)^T \in \mathscr{R}(\mathbf{A})$$

invertible

$$\forall x_r \in \mathscr{R}(\mathbf{A}^T) \quad \Leftrightarrow \quad x_r = \mathbf{A}^T(\alpha,\beta)^T$$

$$t_M \;:\; (\alpha,\beta)^T \in \mathbb{R}^2 \longrightarrow y = \mathbf{A}\mathbf{A}^T(\alpha,\beta)^T \in \mathscr{R}(\mathbf{A}) = \mathbb{R}^2$$

$$t_M^{-1} : y \in \mathbb{R}^2 = \mathscr{R}(\mathbf{A}) \longrightarrow (\alpha,\beta)^T = [\mathbf{A}\mathbf{A}^T]^{-1} y \in \mathbb{R}^2$$

$$\Phi^{-1}(y) = x_r = \mathbf{A}^T(\alpha,\beta)^T \in \mathscr{R}(\mathbf{A}^T)$$

**Exercise:** Implement this algorithm and check the result.

# **Example 3:** $t_A \; : \; x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$

where $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad S = \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$

rank(A)=2

pivots

⇒ $t_A$ injective

```
A   = sym([…]);
RAT = colspace(A');
NA  = null(A);
RA  = colspace(A);
NAT = null(A');
```

in $\mathbb{R}^2$

$\mathcal{N}(A)=\{\underline{0}\}$

$\mathcal{R}(A^{\mathsf{T}})=\mathbb{R}^2$

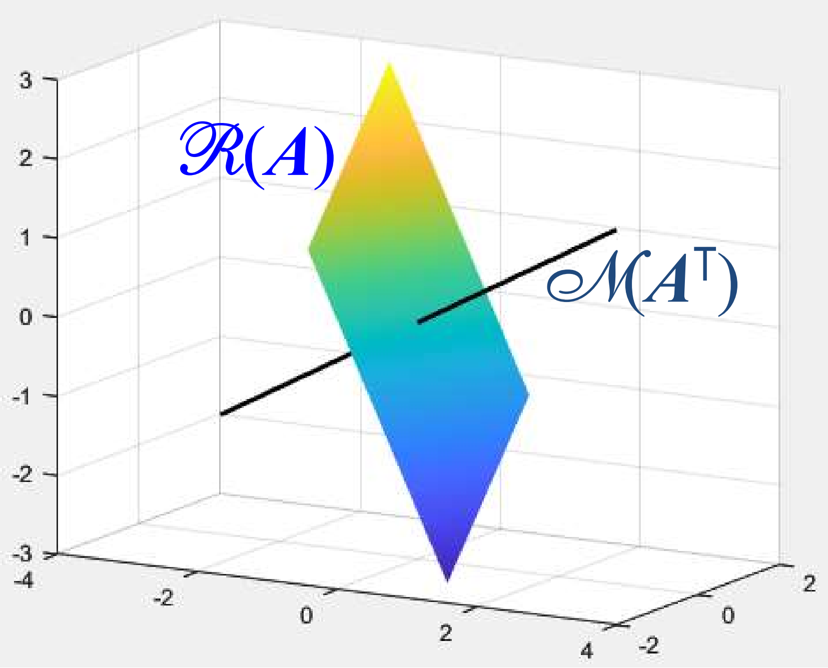
in $\mathbb{R}^3$

$\mathcal{N}(A^{\mathsf{T}})=\textbf{span}\{(2,-1,1)^{\mathsf{T}}\}$

$\mathcal{R}(A)=\textbf{span}\{(1,0,-2)^{\mathsf{T}},(0,1,1)^{\mathsf{T}}\}$

planes

$\mathcal{R}(A^{\mathsf{T}})$ and $\mathcal{R}(A)$ isomorphic



Which isomorphism between $\mathcal{R}(A^{\mathsf{T}})$ and $\mathcal{R}(A)$?

# Isomorphism between $\mathcal{R}(A^T)$ and $\mathcal{R}(A)$

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{N}(A)=\{\underline{0}\}$ $\quad$ $\mathcal{N}(A^T)=\mathrm{span}\{(2,-1,1)^T\}$

$\mathcal{R}(A^T)=\mathbb{R}^2$ $\quad$ $\mathcal{R}(A)=\mathrm{span}\{(1,0,-2)^T,(0,1,1)^T\}$

$$\Phi \;:\; x_r \in \mathcal{R}(\mathbf{A}^T) \subset \mathbb{R}^2 \longrightarrow \Phi(x_r) = \mathbf{A}x_r = y \in \mathcal{R}(\mathbf{A}) \subset \mathbb{R}^3$$

If $\Phi$ is bijective, we want to find a single $x_r$ ($\exists!\ x_r$) such that

$$\Phi^{-1} : y \in \mathcal{R}(\mathbf{A}) \subseteq \mathbb{R}^3 \longrightarrow \Phi^{-1}(y) = \boxed{x_r} \in \mathcal{R}(\mathbf{A}^T) \subseteq \mathbb{R}^2$$

Now we can exploit the fact that **A** is a maximum rank matrix where rank(**A**) = number of columns.

$$\forall y \in \mathcal{R}(\mathbf{A}) \iff \exists\, x \in \mathbb{R}^2 : \mathbf{A}x = y \iff \boxed{\mathbf{A}^T\mathbf{A}}\, x = \mathbf{A}^T y$$

invertible

$$\iff \boxed{x = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T y}$$

Ma $x \in \mathbb{R}^2 = \mathcal{R}(\mathbf{A}^T)$ $\Longrightarrow$ $\boxed{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T y$

**Exercise:** Implement this algorithm and check the result.

# Change of basis

$$t_A : x \in \mathbb{R}^n \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^n$$

If $A(n\times n)$ is a non-singular matrix, then the automorphism given by $t_A$ can be considered as a change of basis in $\mathbb{R}^n$

The columns in $U$ are a basis of $\mathbb{R}^n$

$$U = \begin{pmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_n \end{pmatrix}$$

The columns in $V$ are another basis of $\mathbb{R}^n$

$$V = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{pmatrix}$$

$$\forall x \in \mathbb{R}^n \longrightarrow x = U\underline{\alpha}$$

$\underline{\alpha}$: components of $\boldsymbol{x}$ w.r.t. $U$

$$\forall x \in \mathbb{R}^n \longrightarrow x = V\underline{\beta}$$

$\underline{\beta}$: components of $\boldsymbol{x}$ w.r.t. $V$

$$t_A : \underline{\alpha} \longrightarrow \underline{\beta}$$

$$\underline{x} = \boxed{U\underline{\alpha} = V\underline{\beta}} = \underline{x}$$

the vector $\boldsymbol{x}$ is the same, but it is written w.r.t. different bases

$$U\underline{\alpha} = V\underline{\beta} \quad \Leftrightarrow \quad \underline{\beta} = \boxed{V^{-1}U}\,\underline{\alpha} \longleftarrow A$$

change-of-basis matrix

# Example 3

In a previous example, we computed the components of $x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ w.r.t. the basis given by the columns of $V$:

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$x = U\underline{\alpha} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \text{ with respect to the standard basis } U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\parallel$$

$$x = V\underline{\beta} \in \mathbb{R}^3 \text{ with respect to the basis } V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \Longrightarrow \underline{\beta} = \boxed{V^{-1}U}\,\underline{\alpha}$$

*A*: change-of-basis matrix

```
U=eye(3); V=[1 1 1;0 1 2;0 0 3];
A=V\U; % ⇔ inv(V)*U change-of-basis matrix
alpha=[2;-1;1]; format rat; beta=A*alpha
beta =
      10/3
      -5/3
       1/3
```

```
disp(V*beta)
      2
     -1
      1
```

**OK!**

*We got the same result as before, when we solved a linear system*

**Advantage:** if we know the change-of-basis matrix, then we have not to solve a linear system for each vector $x$.

# Example 4

## Advantage in using an orthonormal basis

$U$ old basis

$$\underline{\beta} = \boxed{V^{-1}U}\ \underline{\alpha}$$

change-of-basis matrix

$V$ new basis

$V$ orthonormal basis

$$\Rightarrow \quad \underline{\beta} = \boxed{V^{\mathsf{T}}U}\ \underline{\alpha}$$

change-of-basis matrix

$$\Updownarrow$$

$$\boxed{V^{-1} = V^{\mathsf{T}}}$$

*faster* and *more accurate*, because we have not to compute the inverse matrix!

```
U=[1 0 1;1 1 0;1 2 3]'; [Q,R]=qr(U); disp([inv(Q) Q'])
   -0.7071   -0.0000   -0.7071             -0.7071         0   -0.7071
   -0.4082   -0.8165    0.4082             -0.4082   -0.8165    0.4082
   -0.5774    0.5774    0.5774             -0.5774    0.5774    0.5774
```

=

$Q$ orthogonal matrix

# Example 3 (contd.)

$$x=\begin{pmatrix}2\\-1\\1\end{pmatrix}\in\mathbb{R}^3=\operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\Rightarrow V=\begin{pmatrix}1&1&1\\1&2&0\\0&3&3\end{pmatrix}\quad x=\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}\in\mathbb{R}^3=\mathscr{R}(V)$$

```
U=eye(3);  alfa=[2;-1;1];  x1=U*alfa
x1 =
     2
    -1
     1
V=[1 1 1;1 2 0;0 3 0];  beta=(V\U)*alfa;
x2=V*beta
x2 =
     2
    -1
     1
[Q,R]=qr(V);  gamma=(Q'*U)*alfa;
x3=Q*gamma
x3 =
     2
    -1
     1
```

$\alpha$ : $x$ w.r.t. the standard basis (1st basis)

$\beta$ : $x$ w.r.t. the columns of $V$ (2nd basis: change-of-basis matrix)

$\gamma$ : $x$ w.r.t. the columns of $Q$ (3rd basis: orthonormal basis)

the vectors are the same

# Contents

- **Examples of 2D Linear Maps:**
  - ☐ uniform scaling, non-uniform scaling, reflections, rotations, particular shears.
  - ☐ translation in homogeneous coordinates, roto-translation, orthogonal projection onto a line.
- **Factorization of a 2D $t_A$ into elementary linear maps.**

# 2D Elementary Linear Maps

$$t_A \; : \; x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$$

($t_A$ automorphism)

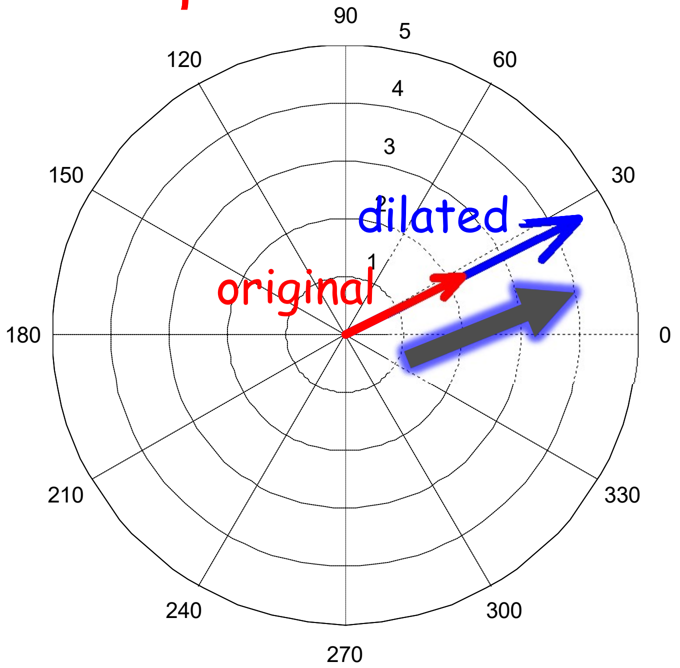
**Radial homothety** centered at O by a factor $\rho$
(or **uniform scaling** or **isotropic scaling**)

$0 < \rho < 1$ contraction
$1 < \rho$ dilation

**Example:** Homothety by factor $\rho=2$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

```
rho=2; A=rho*eye(2);
x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3)
hold on; axis('tight')
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```



# 2D Elementary Linear Maps

$$t_A \;:\; x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} \rho & 0 \\ 0 & \eta \end{pmatrix}, \quad \begin{matrix} \rho, \eta \in \mathbb{R} \\ \rho, \eta \neq 0 \end{matrix}$$

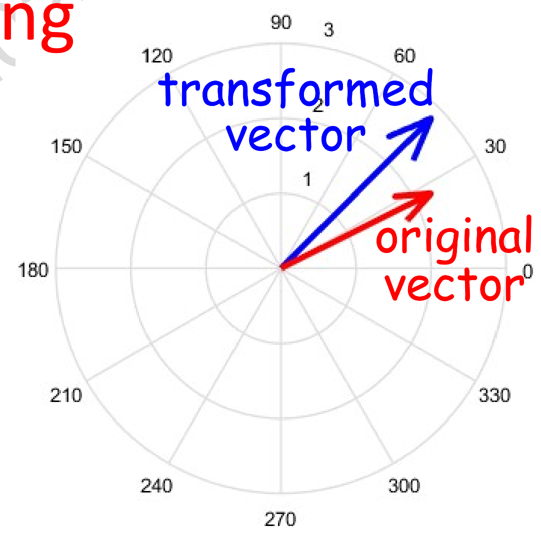
($t_A$ automorphism)

**Non-uniform Scaling** centered at O
(or **anisotropic scaling**)

## **Examples:** Non-uniform scaling

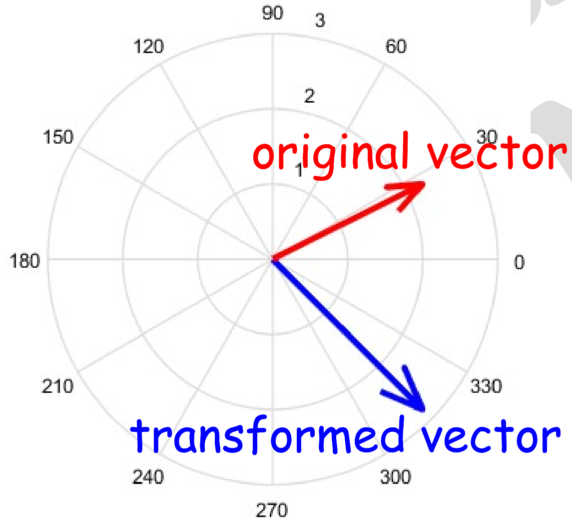
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

```
A=diag([1 2]);
x=[2 1]'; y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3); hold on
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```



transformed vector

original vector



original vector

transformed vector

```
A=diag([1 -2]);
x=[2 1]'; y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3); hold on
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

What does this matrix do?

# Particular reflections are described by a non-uniform scaling (or flippings)

$$t_A \ : \ x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

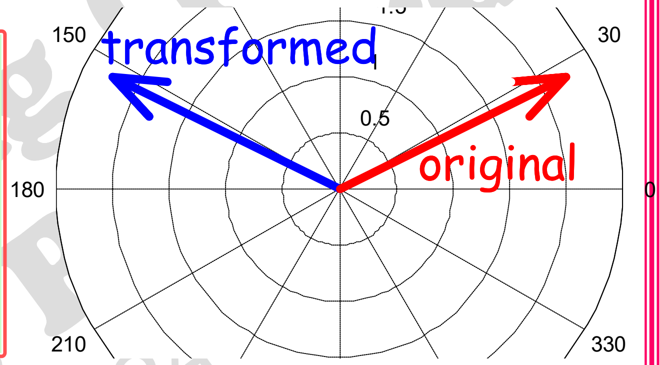
## Reflection over the y-axis

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
disp(det(A))
-1
```

```
disp(eig(A))
    -1
     1
```

```
A=diag([-1 1]);
x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3)
hold on; axis('tight')
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```



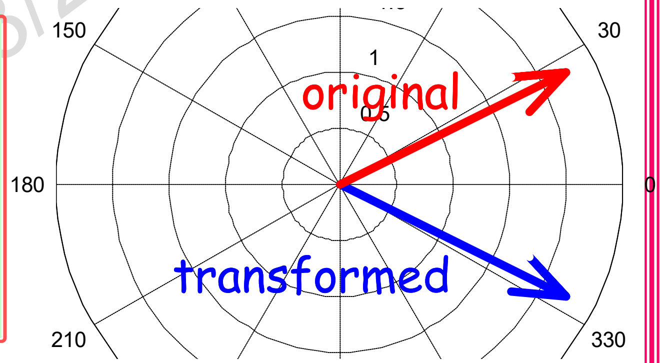
## Reflection over the x-axis

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
disp(det(A))
-1
```

```
disp(eig(A))
    -1
     1
```

```
A=diag([1 -1]);
x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3)
hold on; axis('tight')
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```



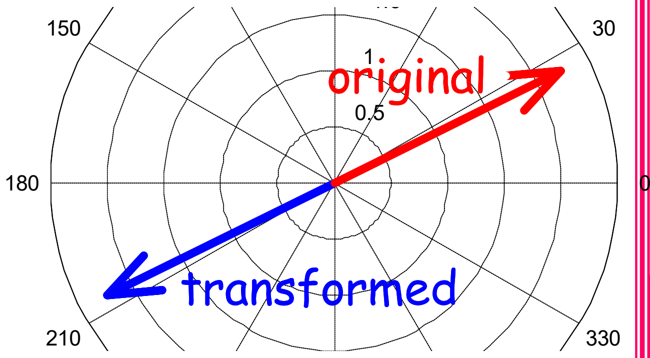
## reflection over the origin (previous reflections combined together)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

```
disp(det(A))
 1
```

```
disp(eig(A))
    -1
    -1
```

```
A=diag([-1 -1]);
x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b');
set(h,'LineWidth',3)
hold on; axis('tight')
h=compass(x(1),x(2),'r');
set(h,'LineWidth',3)
```

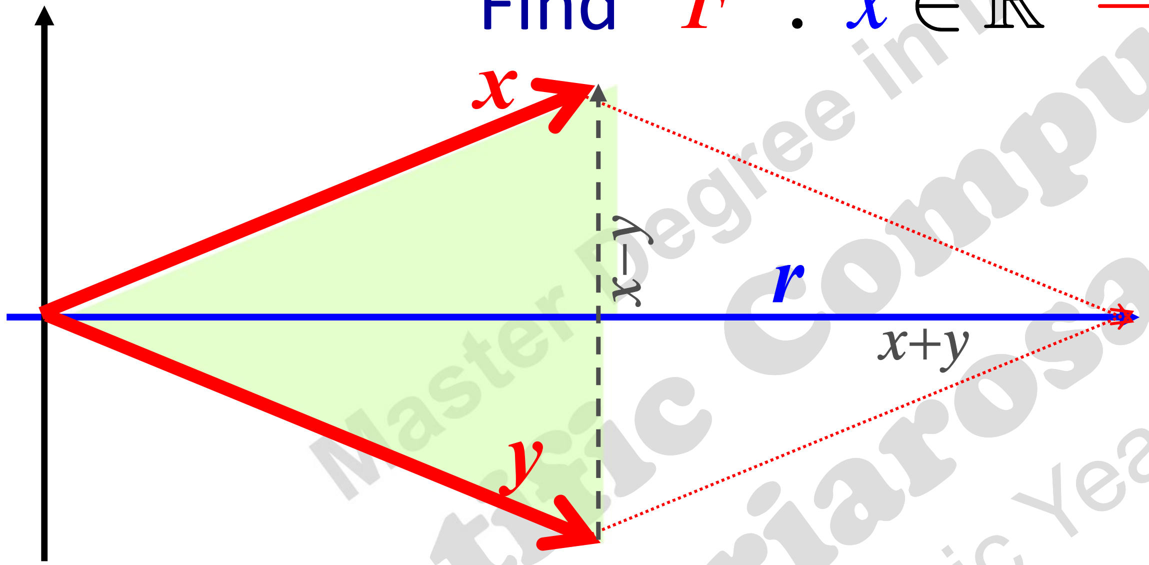


**Example:** Find* the automorphism in $\mathbb{R}^2$ corresponding to the Orthogonal Reflection over the $x$-axis:

$$r=\mathbf{span}\{a\}=\mathbf{span}\{(1,0)^{\mathsf{T}}\}.$$

* We have already identified it as a non unif. scaling $t_A$, where $A$ is $A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Find $F \; : \; x\in\mathbb{R}^2 \xrightarrow{\qquad} y\in\mathbb{R}^2$



such that

**Properties** $\begin{cases} x+y\in r \\ x-y\perp r \\ \|x\|=\|y\| \end{cases}$

$$\begin{cases} x+y=\lambda\begin{pmatrix}1\\0\end{pmatrix}, \quad \lambda\in\mathbb{R} \\ \left\langle x-y,\begin{pmatrix}1\\0\end{pmatrix}\right\rangle=0 \\ \langle x,x\rangle=\langle y,y\rangle \end{cases}$$

$$F=t_A : x\in\mathbb{R}^2 \longrightarrow y=Ax\in\mathbb{R}^2$$

$$\begin{cases} \boxed{x+y=\lambda\begin{pmatrix}1\\0\end{pmatrix} \quad \lambda\in\mathbb{R}} \\ \boxed{\left\langle x-y,\begin{pmatrix}1\\0\end{pmatrix}\right\rangle=0} \\ \langle x,x\rangle=\langle y,y\rangle \end{cases} \qquad \begin{cases} \boxed{\begin{matrix} x_1+y_1=\lambda \quad \lambda\in\mathbb{R} \\ x_2+y_2=0 \end{matrix}} \\ \boxed{x_1-y_1=0} \\ \langle x,x\rangle=\langle y,y\rangle \end{cases} \Rightarrow \begin{cases} y_2=-x_2 \\ x_1-y_1=0 \\ \langle x,x\rangle=\langle y,y\rangle \end{cases} \Leftrightarrow y=\begin{pmatrix} \textcircled{1}x_1 \\ \textcircled{-1}x_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} \textcircled{1} & 0 \\ 0 & \textcircled{-1} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

orthogonal reflection over $x$-axis

$A$

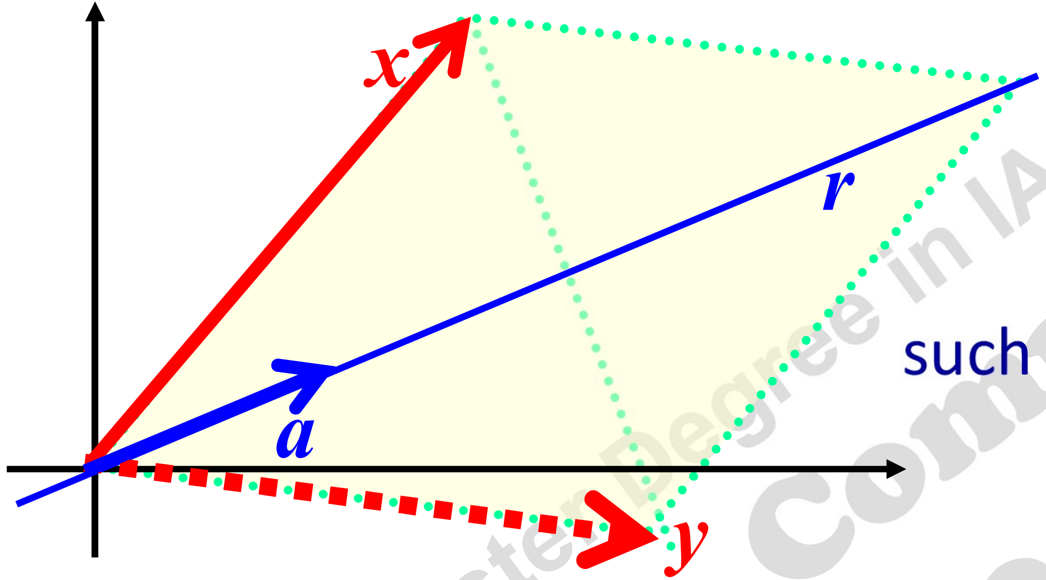
```
a=[1 0]'; syms lambda real
syms x y [2 1] real
P1=x+y-lambda*a;    P2=a'*(x-y);
Y=solve(P2,P1(2),y1,y2);
y=[Y.y1;Y.y2]
y                   %  y = A*x = x1*A(:,1) + x2*A(:,2)
  x1                %  y = A*x = x1* [1]   +  x2* [ 0]
 -x2                %                [0]   +      [-1]
A=zeros(2);    find the transformation matrix
for k=1:2
    if simplify(symvar(y(k)) == x1)
        A(k,1)=y(k)/x1;
    end
    if simplify(symvar(y(k)) == x2)
        A(k,2)=y(k)/x2;
    end
end
A
 A =
      1     0
      0    -1
```

more general algorithm

```
A=zeros(2);          find the transformation matrix
for k=1:2
    [c,t]=coeffs(y(k));
    for j=1:numel(t)
        if string(t(j)) == string(x(1))
            A(k,1)=c(j);
        end
        if string(t(j)) == string(x(2))
            A(k,2)=c(j);
        end
    end
end
```

# Orthogonal reflection over any line $r$=span{$a$}, $a\in\mathbb{R}^n$: how can we detect its transformation matrix?

$$t_A\ :\ x\in\mathbb{R}^2 \longrightarrow y=t_A(x)\in\mathbb{R}^2,\quad A(2\times 2)$$



such that $\begin{cases} x+y\in r \\ x-y\perp r \\ \boxed{\|x\|=\|y\| \text{ not used}} \end{cases}$ $\Rightarrow$ $\begin{cases} x+y=\lambda a \\ \langle x-y,a\rangle=0 \end{cases}$

$\Rightarrow$ $\begin{cases} y=\lambda a-x \\ \langle 2x-\lambda a,a\rangle=0 \end{cases}$ $\Leftrightarrow$ $\begin{cases} y=\lambda a-x \\ 2\langle x,a\rangle-\lambda\langle a,a\rangle=0 \end{cases}$ $\Leftrightarrow$ $\lambda=2\dfrac{\langle x,a\rangle}{\langle a,a\rangle}$

$$\Rightarrow\quad y=2a\frac{\langle a,x\rangle}{\|a\|^2}-x=\frac{2}{\|a\|_2^2}aa^{\mathrm{T}}\boxed{x}-\boxed{x}=\left(\frac{2}{\|a\|_2^2}aa^{\mathrm{T}}-I_2\right)x$$

$$\Rightarrow\quad \boxed{A=\left(\frac{2}{\|a\|_2^2}aa^{\mathrm{T}}-I_2\right)}$$

**Exercise**

What are $\mathcal{N}$(A), $\mathcal{R}$(A$^{\mathrm{T}}$), $\mathcal{N}$(A$^{\mathrm{T}}$) and $\mathcal{R}$(A)?
Is $t_A$ an automophism?

# MATLAB Lab: reflection over the line

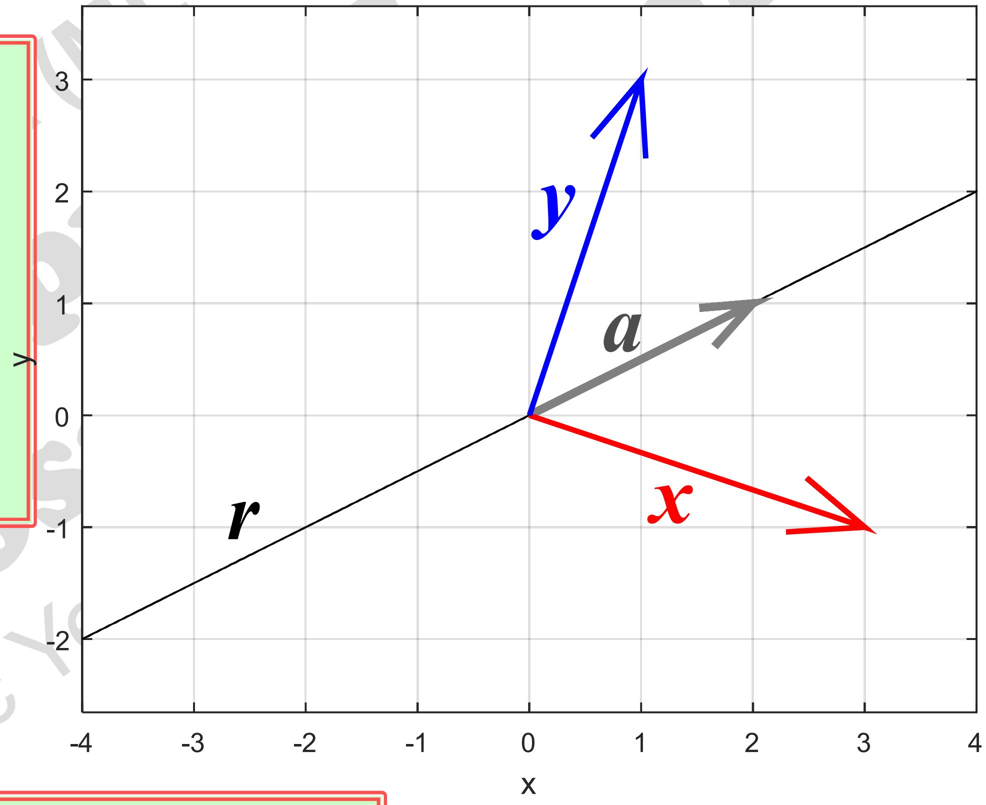
$$r = \text{span}\{a\} \quad : \quad a = (2,1)^T$$

## What are the properties of a 2D reflection matrix?

For this example

```
a=[2 1]'; syms t real; r=t*a;
fplot(r(1),r(2),[-2 2]); hold on
compass(a(1),a(2))
x=[3 -1]'; compass(x(1),x(2),'r')
A = 2/norm(a)^2*a*a'-eye(size(a,1))
A =
           0.6          0.8
           0.8         -0.6
y = A*x; compass(y(1),y(2),'b')
```

symmetric



```
disp(det(A))
-1
```

the determinant of A equals -1

```
disp(eig(A))
            -1
             1
```

its eigenvalues are -1, 1

```
disp(A*A)
             1   -3.8858e-16
  -3.8858e-16              1
```

the matrix equals its inverse

```
disp(A'*A)
             1   -3.8858e-16
  -3.8858e-16              1
```

```
disp(A*A')
             1   -3.8858e-16
  -3.8858e-16              1
```

orthogonal matrix

# Properties of 2D reflection matrices

over a line

matrix for the reflection
over a line span{a}

$$A = \frac{2}{\|a\|_2^2} aa^\top - I_2$$

1. The reflection matrix is symmetric.

2. The inverse of a reflection is the reflection itself.

Proof:

$$A \cdot A = \left[\frac{2}{\|a\|_2^2} aa^\top - I_2\right]\left[\frac{2}{\|a\|_2^2} aa^\top - I_2\right] = \frac{4}{\|a\|_2^{\cancel{4}\,2}} a\left[\cancel{a^\top a}\right]a^\top - \frac{4}{\|a\|_2^2} aa^\top + I_2 =$$

$$= \frac{4}{\|a\|_2^{\cancel{4}\,2}} a\left(\cancel{a^\top a}\right)a^\top - \frac{4}{\|a\|_2^2} aa^\top + I_2 = I_2$$

$$\boxed{a^\top a = \|a\|^2}$$

3. The reflection matrix is orthogonal.

4. Its determinant is -1 and eigenvalues are −1 and/or +1.

# Remark

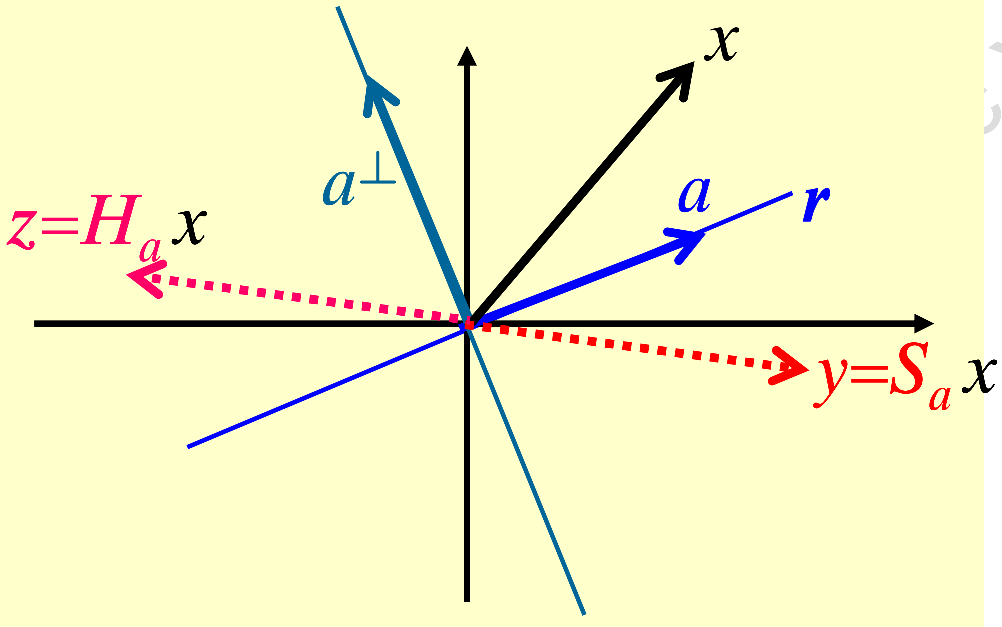
The matrix for a reflection over the line $r = \text{span}\{a\}$

$$\boxed{S(a) = \frac{2}{\|a\|^2} aa^{\mathsf{T}} - I_2}$$

is different from the following matrix:

$$\boxed{H(a) = I_2 - \frac{2}{\|a\|^2} aa^{\mathsf{T}}}$$

(Householder reflection)



$$\boxed{H(a) = -S(a)}$$

$H(a)$ represents an orthogonal reflection over the line:

$t = \text{span}\{a^\perp\}$

# The Householder reflector

The QR factorization algorithm makes use of a sequence of Householder reflectors to produce the upper triangular matrix R (in A=QR) starting from a matrix A.

For example, if we want to zero out all the components of a vector $x$ except the $k$th, to form the Householder reflector, we choose

$$a = x + \|x\|_2\, e_k$$

where $e_k$ is the versor of the $k$th axis, i.e. $e_k = (0,\dots,0,\overset{k}{1},0,\dots,0)^{\mathsf{T}}$.

Then we form the matrix $H$ $\quad H(a) = I_2 - \dfrac{2}{\|a\|^2} a a^{\mathsf{T}}$

Thus the vector $Hx$ is zero everywhere except possibly the $k$th component.

```
pH=@(a) sym(eye(numel(a)))-2/norm(a)^2*a*a'; x=sym([2 1 3 -2]');
e=sym([1 0 0 0]'); a=x+norm(x)*e; H=pH(a); y1=simplify(H*x)
y1 =
-3*2^(1/2)        only the 1st component is non-zero
         0
         0
         0
e=sym([0 0 1 0]'); a=x+norm(x)*e; H=pH(a); y3=simplify(H*x)
y3 =
         0
         0
-3*2^(1/2)        only the 3rd component is non-zero
         0
```

# 2D Elementary Linear Maps

$$t_A \;:\; x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

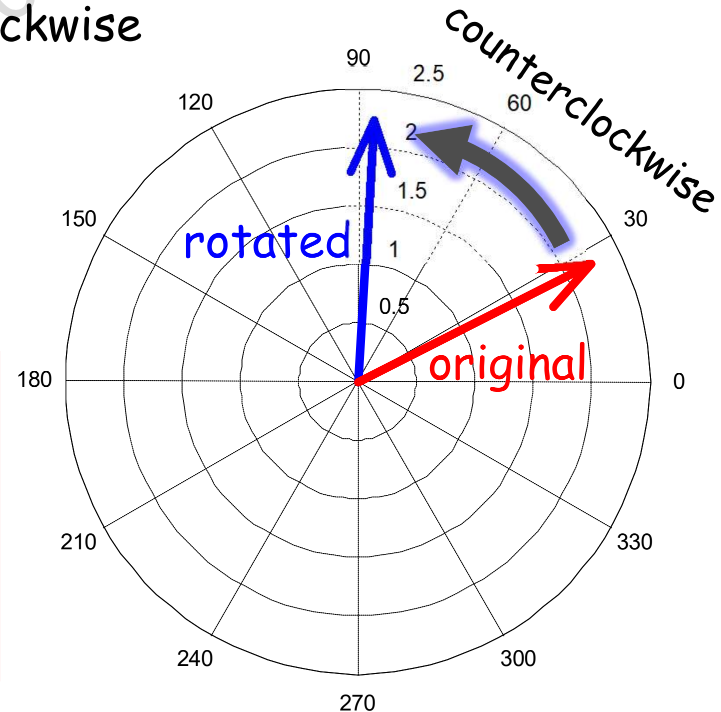
Rotation around O by an angle $\theta$

($t_A$ automorphism)

**Example:** Rotation in $\mathbb{R}^2$ by $+\pi/3$

$$A = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$$

$\theta \geq 0$: counterclockwise
$\theta < 0$: clockwise



```
theta=pi/3; c=cos(theta); s=sin(theta);
A=[c -s;s c];
x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b'); set(h,'LineWidth',3)
hold on; axis('tight')
h=compass(x(1),x(2),'r'); set(h,'LineWidth',3)
```

# Properties of 2D rotation matrices

rotation

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\det(A) = \cos^2 + \sin^2 = 1$$

$A^{-1} = A^{\mathsf{T}}$ $A$: orthogonal matrix

$$A = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

$A^{-1} = A^{\mathsf{T}}$ $A$: orthogonal matrix

$\det(A) = -1$

It is not only a 2D rotation

orthogonal matrix

$A^{-1} = A^{\mathsf{T}}$

$\det(A) = \pm 1$

... why? prove it.

$\det(A) = +1$ → proper rotation

$\det(A) = -1$ → improper rotation (rotation + reflection)

# 2D Elementary Linear Maps

$$t_A \quad : \quad x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad r \in \mathbb{R}$$

Horizontal shear (or x-shear map or shear parallel to the x-axis)

***r*** shear factor

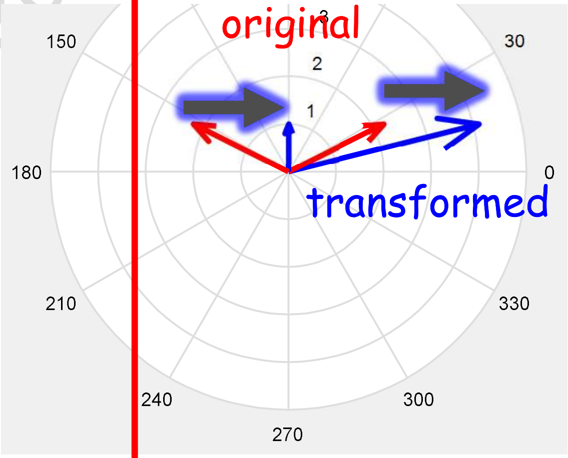
($t_A$ automorphism)

## **Example:** Horizontal shear in $\mathbb{R}^2$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

1

```
A=[1 2;0 1]; x=[2 1]';  y=A*x;
h=compass(y(1),y(2),'b'); set(h, …
h=compass(x(1),x(2),'r'); set(h, …
x=[-2 1]';  y=A*x; …
```



2

```
A=[1 2;0 1]; x=[5 0]'; y=A*x
y =
      5
      0
```

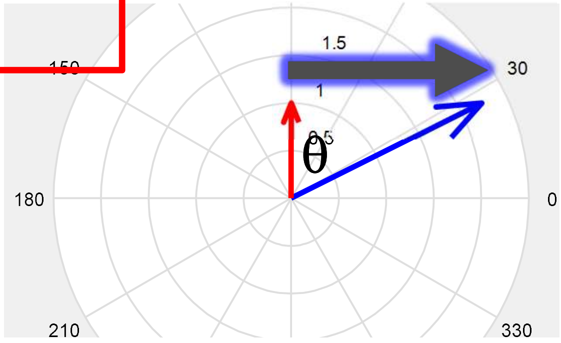
a horizontal vector remains the same

3

```
A=[1 2;0 1]; x=[0 1]';  y=A*x
y =
      2
      1
```

the arrowhead of a vertical vector is displaced horizontally

$$A = \begin{pmatrix} 1 & \tan(\theta) \\ 0 & 1 \end{pmatrix}$$



4

```
th=acos(dot(x,y)/(norm(x)*norm(y))); disp(tan(th))
```

② geometric interpretation of $r$: $r = \tan(\theta)$

# 2D Elementary Linear Maps

$$t_A \quad : \quad x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

$$A = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad r \in \mathbb{R}$$

$r$ shear factor

($t_A$ automorphism)

Vertical shear (or y-shear map or shear parallel to the y axis)

## **Example:** Vertical shear in $\mathbb{R}^2$

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

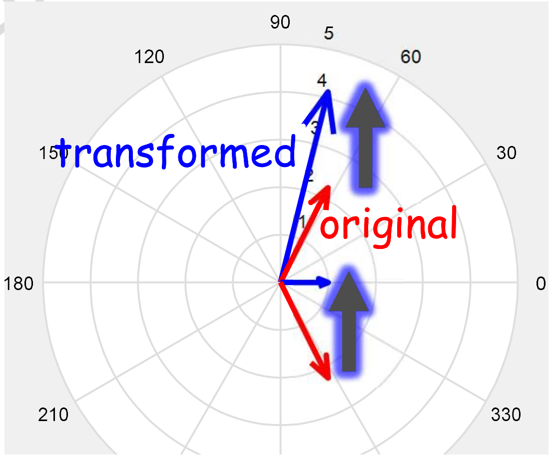
1

```
A=[1 0;2 1]; x=[1 2]';  y=A*x;
h=compass(y(1),y(2),'b'); set(h, …
h=compass(x(1),x(2),'r'); set(h, …
x=[1 -2]';  y=A*x; …
```

2

```
A=[1 0;2 1]; x=[0 5]'; y=A*x
y =
     0
     5
```

a vertical vector remains the same

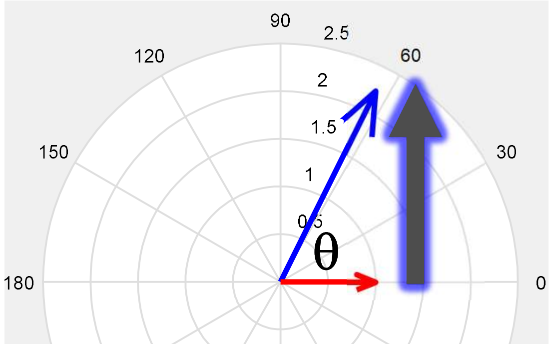


3

```
A=[1 0;2 1]; x=[1 0]';  y=A*x
y =
     1
     2
```

the arrowhead of a horizontal vector is displaced vertically

$$A = \begin{pmatrix} 1 & 0 \\ \tan(\theta) & 1 \end{pmatrix}$$



4

```
th=acos(dot(x,y)/(norm(x)*norm(y))); disp(tan(th))
     2
```

geometric interpretation of $r$: $r = \tan(\theta)$

# A translation (or shift) is not a linear map

## Example

$$T \quad : \quad x \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} \in \mathbb{R}^2$$

There is no matrix $A(2\times2)$ such that $y = T(x) = Ax$

But $T$ can be written as: $y = T(x) = \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} = Ix + \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$$y = Ax + v$$

$$\boxed{F \quad : \quad x \in \mathbb{R}^n \longrightarrow y = Ax + v \in \mathbb{R}^m}$$

Matrix form of an Affine Map (see later)

# A translation is an affine map

# Homogeneous coordinates

A Translation in $\mathbb{R}^n$ becomes a linear map (in $\mathbb{R}^{n+1}$) if we use the homogeneous coordinates.

**Example:** homogeneous coordinates in $\mathbb{R}^2$

$$P \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

Cartesian coordinates

$$\begin{cases} x_1 = \dfrac{X_1}{X_3} \\ x_2 = \dfrac{X_2}{X_3} \end{cases}$$

$$P \equiv \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathbb{R}^3$$

Homogeneous coordinates

The point at ∞ is represented as $\infty \equiv \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$

# **Example:** translation in homogeneous coordinates

translation $T \; : \; x \in \mathbb{R}^2 \longrightarrow y = \begin{pmatrix} x_1 \boxed{-2} \\ x_2 \boxed{-1} \end{pmatrix} \in \mathbb{R}^2$

puts the new Origin at $\mathbf{P}_0(x_0,y_0)=(2,1)$
and the old Origin is transformed into $(-2,-1)$

$T$ becomes in homogeneous coordinates:

$$y = T(x) = \begin{pmatrix} \dfrac{X_1}{X_3} - 2 \\[2ex] \dfrac{X_2}{X_3} - 1 \end{pmatrix} = \begin{pmatrix} \dfrac{X_1 - 2X_3}{X_3} \\[2ex] \dfrac{X_2 - X_3}{X_3} \end{pmatrix} \equiv \begin{pmatrix} X_1 - 2X_3 \\ X_2 - X_3 \\ X_3 \end{pmatrix} = Y$$

$$Y = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Then $T = \begin{pmatrix} I & \boxed{\begin{pmatrix} -x_0 \\ -y_0 \end{pmatrix}} \\ 0 & 1 \end{pmatrix}$ ← displacement

$$T = t_A \; : \; X \in \mathbb{R}^3 \longrightarrow Y = AX = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} -2 \\ -1 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathbb{R}^3$$

displacement

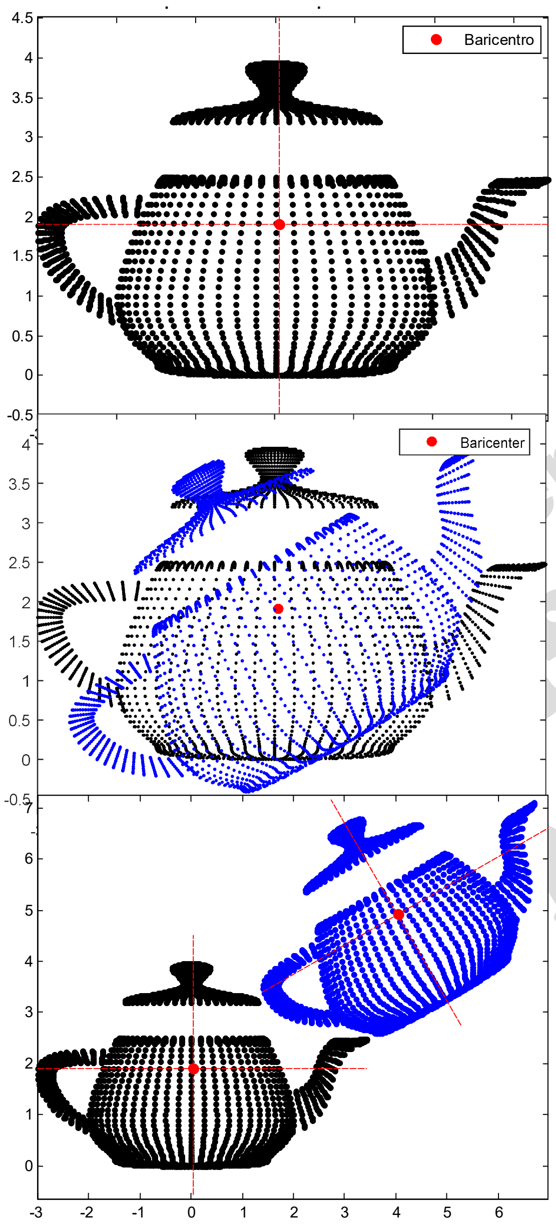
identity matrix

Linear Mappings

(prof. M. Rizzardi)

# **Example**: roto-translation of N points

Rotation of a angle $\theta=30°$ around the barycenter and then translation in $Q=(2,3)$



## in cartesian coordinates

computational complexity

**1) rotation around barycenter**

1.1) compute the barycenter B $O(N)$

1.2) translate the origin in B $O(N)$

1.3) rotate around the new origin (B) $O(N)$

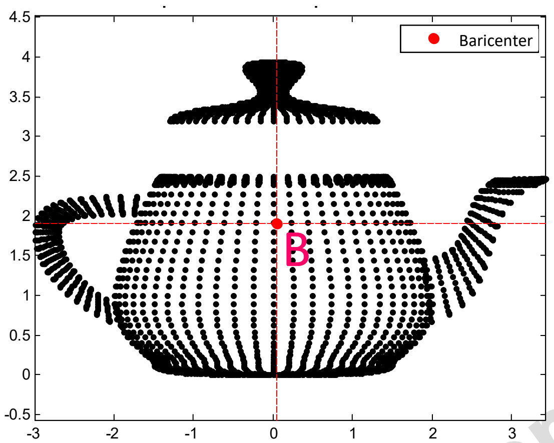
1.4) apply inverse translation to reset the origin $O(N)$

**2) translation in Q** $O(N)$

total computational complexity: $O(5N)$

# Example: (contd.)

Rotation of a angle $\theta=30°$ around the barycenter and then translation in $Q=(2,3)$



## in homogeneous coordinates

**1) Rotation around barycenter**

computational complexity

1.1) compute the barycenter B, $O(N)$

$$T=\begin{pmatrix}1 & 0 & -B(1)\\ 0 & 1 & -B(2)\\ 0 & 0 & 1\end{pmatrix}$$

1.2) the matrix of translation $\boldsymbol{T}$

1.3) compute the rotation matrix $\boldsymbol{R}$

$$R=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

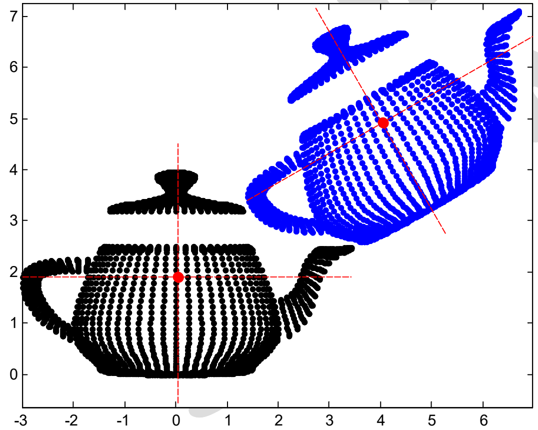
1.4) compute the translation matrix $\boldsymbol{T}^{-1}$

$$\boldsymbol{T}^{-1}=\begin{pmatrix}1 & 0 & +B(1)\\ 0 & 1 & +B(2)\\ 0 & 0 & 1\end{pmatrix}$$

($\boldsymbol{T}^{-1}$ is not really computed)

**2) Matrix $T_2$ of translation in Q**

$$T_2=\begin{pmatrix}1 & 0 & Q(1)\\ 0 & 1 & Q(2)\\ 0 & 0 & 1\end{pmatrix}$$



**3) Compute matrix $M=T_2\times T^{-1}\times R\times T$ and the images of the points P: M*P** $O(N)$

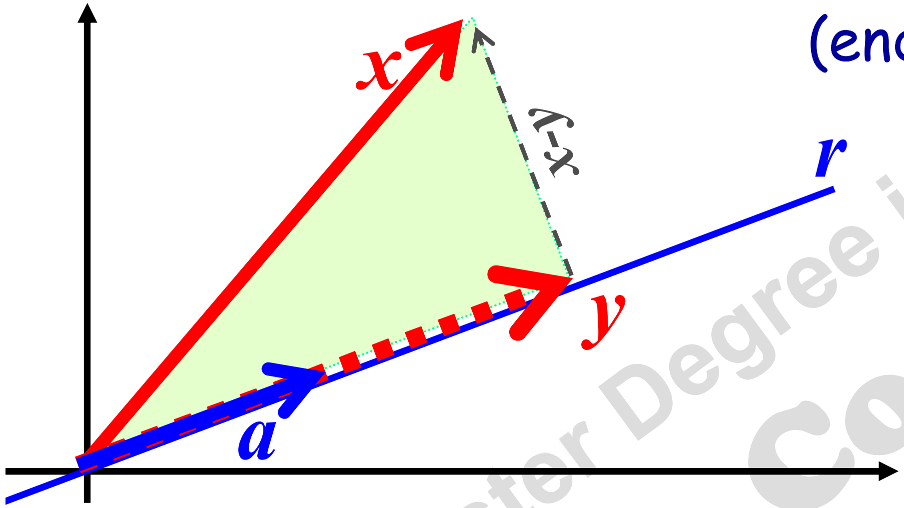
total computational complexity: $O(2N)$

# 2D Elementary Linear Maps

$$t_A \;:\; x\in\mathbb{R}^2 \longrightarrow y=t_A(x)=Ax\in\mathbb{R}^2 \qquad A\ (2\times 2)$$

## Orthogonal projection onto a line $r=\text{span}\{a\},\ a\in\mathbb{R}^n$

(endomorphism)



such that $\begin{cases} y\in r \\ x-y\perp r \end{cases}$

$\langle,\rangle$ standard dot product

$$\begin{cases} y\in r \\ x-y\perp r \end{cases} \Leftrightarrow \begin{cases} y=\lambda a \\ \langle x,a\rangle-\lambda\langle a,a\rangle=0 \end{cases} \Rightarrow \lambda=\frac{\langle a,x\rangle}{\langle a,a\rangle}=\frac{a^{\mathsf{T}}x}{a^{\mathsf{T}}a}$$

$a$ is a vector, then $a^{\mathsf{T}}a=\|a\|^2$ is a scalar

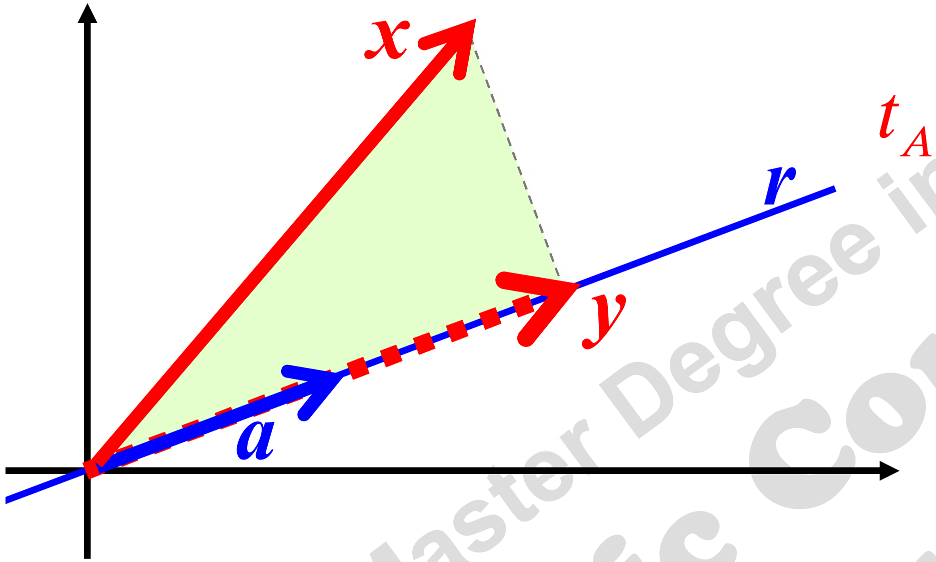
$$t_A \;:\; x\in\mathbb{R}^2 \longrightarrow y=\frac{1}{\|a\|^2}aa^{\mathsf{T}}x\in\mathbb{R}^2 \Rightarrow \boxed{A=\frac{1}{\|a\|^2}aa^{\mathsf{T}}}$$

**Exercise:** Compute $\mathcal{N}(A)$ and $\mathcal{N}(I-A)$: what do they represent?

# Example

## Orthogonal projection onto the line

$$r=\text{span}\{\underbrace{(2,1)^{\mathsf T}}_{a}\}$$

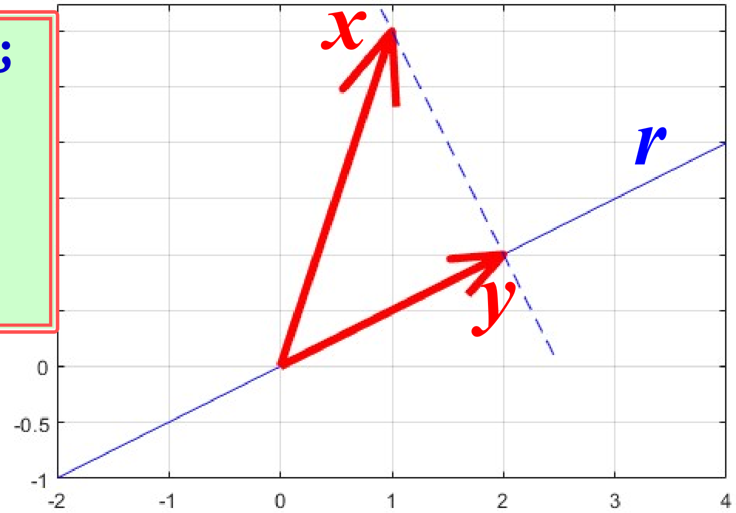


$$t_A : x \in \mathbb{R}^2 \longrightarrow y = \frac{1}{\|a\|^2}\, aa^{\mathsf T} x \in \mathbb{R}^2$$

$$y = t_A(x) = \left[\frac{1}{\|a\|^2}\, aa^T\right] x =$$

$$= \left[\frac{1}{10}\begin{pmatrix}3\\1\end{pmatrix}\begin{pmatrix}3 & 1\end{pmatrix}\right] x = 0.1\begin{pmatrix}9 & 3\\ 3 & 1\end{pmatrix} x$$

```
a=[2 1]'; syms b real; r=b*a; A=[a*a'/norm(a)^2];
x=[1 3]'; y=A*x; n=null(a'); % normal
fplot(r(1),r(2),[-2 2],'Color','b'); hold on
h=compass([x(1) y(1)],[x(2) y(2)] ,'r'); set(h,…
axis equal; grid on; fplot(a(1)+n(1),a(2)+n(2),[-1 2.5],
                              'Color','b','LineStyle','--')
```



What are the Null Space and Image Space of this transformation?

# **Application:** distance between a point and a line

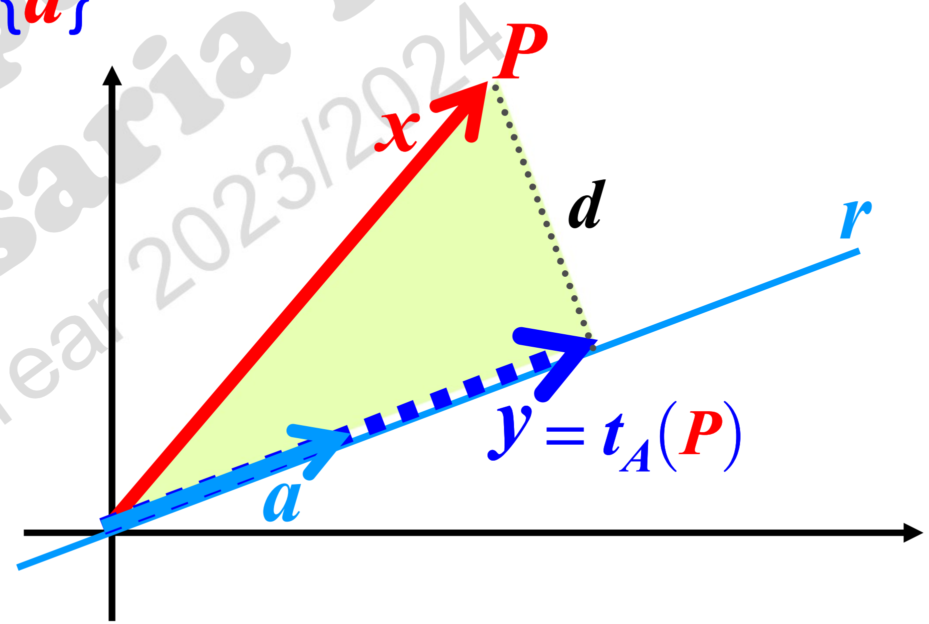
In the Euclidean Space $\mathbb{R}^2$, the distance between a point $\boldsymbol{P}$ and a line $\boldsymbol{r}$ is defined as

$$d(P,r) = \|x-y\|_2 = \min\{d(P,Q)\ \forall Q \in r\}$$

where $\boldsymbol{x}$ is the vector given by $\boldsymbol{P}$ and $\boldsymbol{y}$ is the orthogonal projection of $\boldsymbol{x}$ onto $\boldsymbol{r} = \mathrm{span}\{\boldsymbol{a}\}$

$$y = \underbrace{\left[\frac{1}{\|a\|^2}aa^T\right]}_{A} x$$

```
a=[2 1]'; x=[1 3]';
A=[a*a' / norm(a)^2];     y=A*x;
[norm(x-y)  sqrt(sum((x-y).^2))]
ans =
          2.5298       2.5298
```



**Exercise**

Compute the distance between two parallel lines

# MATLAB Lab: factorize $t_A$ into elementary linear maps

$$t_A \ : \ x \in \mathbb{R}^2 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^2$$

1 $A = QR$

```
A=rand(2);  disp(rank(A))
        2
[Q,R] = qr(A)
Q =
     -0.66874         -0.74349
     -0.74349          0.66874
R =
      -1.2183         -0.76401
            0           0.5164
```

$$A = \begin{pmatrix} 0.81472 & 0.12699 \\ 0.90579 & 0.91338 \end{pmatrix}$$

Q is orthogonal

Is Q a rotation? $R_\alpha \overset{?}{=} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

```
disp(det(Q))
  -1
```

no!

We need to permute its rows $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

what is $P$ from a geometrical point of view?

$Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q$, $Q_1 = \underbrace{\begin{pmatrix} -0.74349 & 0.66874 \\ -0.66874 & -0.74349 \end{pmatrix}}$

cos(α)<0, sin(α)<0 ⇒ α∈3rd quadrant

```
disp(det(Q1))
  1
```

ok!

rotation

by angle $\alpha = -138.03°$

atan(sine/cosine)  atan2(sine,cosine)

```
P=[0 1;1 0]; Q1=P*Q;
disp(atan2(Q1(2,1),Q1(1,1))*180/pi)
   -138.03                3rd quadrant (OK!)
disp(atan(Q1(2,1)/Q1(1,1))*180/pi)
     41.97                1st quadrant (NO!)
disp(asin(Q1(2,1))*180/pi)
        -41.97            4th quadrant (NO!)
disp(acos(Q1(1,1))*180/pi)
        138.03            2nd quadrant (NO!)
```

$$A = QR$$

2 $A = P\,Q_1\,R$

what is $P$ ?

what is $R$ ?

# MATLAB Lab: factorize $t_A$ into elementary linear maps (contd.)

$$A=\begin{pmatrix}0.81472 & 0.12699\\ 0.90579 & 0.91338\end{pmatrix} \qquad A = P\,Q_1 R$$

What is $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ ?

## $P$ is a particular permutation matrix

An elementary permutation matrix $P$ is a square matrix obtained from the same size identity matrix by a permutation of 2 rows (or of 2 cols). A general permutation matrix is obtained by multiplying two or more elementary permutation matrices.

or

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \Rightarrow P=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \qquad B=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$$

```
P=[0 1 0;1 0 0;0 0 1];
syms a b c d e f g h i real
B=[a b c;d e f; g h i];
```

$$PB=\begin{pmatrix}d&e&f\\a&b&c\\g&h&i\end{pmatrix}$$

```
disp(P*B)
[d, e, f]
[a, b, c]
[g, h, i]
```

left mult. ⇔ permute rows

right mult. ⇔ permute cols

$$BP=\begin{pmatrix}b&a&c\\e&d&f\\h&g&i\end{pmatrix}$$

```
disp(B*P)
[b, a, c]
[e, d, f]
[h, g, i]
```

$P=P^{\mathsf{T}}$: symmetric

```
all(all(P==P'))
ans =
  logical
   1
```

$PP=I$: idempotent

```
all(all(P*P==eye(3)))
ans =
  logical
   1
```

```
det(P)
ans =
    -1
```

⇒ all elementary permutations $P$ are reflections

$P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $P$ is a reflection ⇒ Axis? Which vectors : $Pv=v$ ?

$Pv-v=0$ ⇒

```
N=null(sym(P - eye(2)))
N =
  1
  1
```

$(P-\mathrm{I})\,v=\underline{0}$

$$\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

$P$: reflection across the bisector of 1st and 3rd q.

# MATLAB Lab: factorize $t_A$ into elementary linear maps (contd.)

$$A = \begin{pmatrix} 0.81472 & 0.12699 \\ 0.90579 & 0.91338 \end{pmatrix}$$

$\boxed{2}$ $A = \underset{\text{reflection}}{P}\,\underset{\text{rotation}}{Q_1}\,R$

$\boxed{3}$ … and $R$? $R = \begin{pmatrix} -1.2183 & -0.76401 \\ 0 & 0.5164 \end{pmatrix}$ $R$: upper triangular matrix

## Can $R$ contain a horiz. shear?

$$H = \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$$

We need to extract pivots

```
S=diag(diag(R))
S =
     -1.2183            0
           0       0.5164
H=S\R   the same as H=inv(S)*R    R = SH
H =
   shear   1        0.62712   tan(θ)
           0              1
theta=atan(H(1,2))*180/pi
theta =
     32.093     degrees
```

$$R = SH$$

$$R = \underbrace{\begin{pmatrix} -1.2183 & 0 \\ 0 & 0.5164 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 1 & \boxed{0.62712} \\ 0 & 1 \end{pmatrix}}_{H}$$

$\boxed{\tan(\theta)}$

$S$: reflection + non-uniform scaling

$H$: horiz. shear

$S = S_1 S_2$

```
S1=diag(sign(diag(S)))
S1 =
    -1       0   y-reflection
     0       1
```

```
S2=S1\S
S2 =  non-uniform scal.
   1.2183            0
        0       0.5164
```

$$A = \underset{\text{reflection}}{P}\,\underset{\text{rotation}}{Q_1}\,\underset{\text{reflection}}{S_1}\,\underset{\text{non-u. scal.}}{S_2}\,\underset{\text{horiz. shear}}{H}$$

# Exercise

Given as input a square matrix A, computed as

```
A=rand(2);
```

explain which elementary linear maps come from the following factorizations of A:

- `[L,U,P]=lu(A);`
- `[U,S,V]=svd(A);`

# Contents

- **Examples of 3D Linear Maps:** uniform and non-uniform scaling, (proper and improper) rotations, reflections and their properties, orthogonal projection onto a plane and onto a line.

- **Summary of properties for an orthogonal matrix.**

# 3D Elementary Linear Maps

$$t_A \;:\; x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$$

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

**Radial homothety** centered at O of factor $\rho$

(or uniform scaling or isotropic scaling)

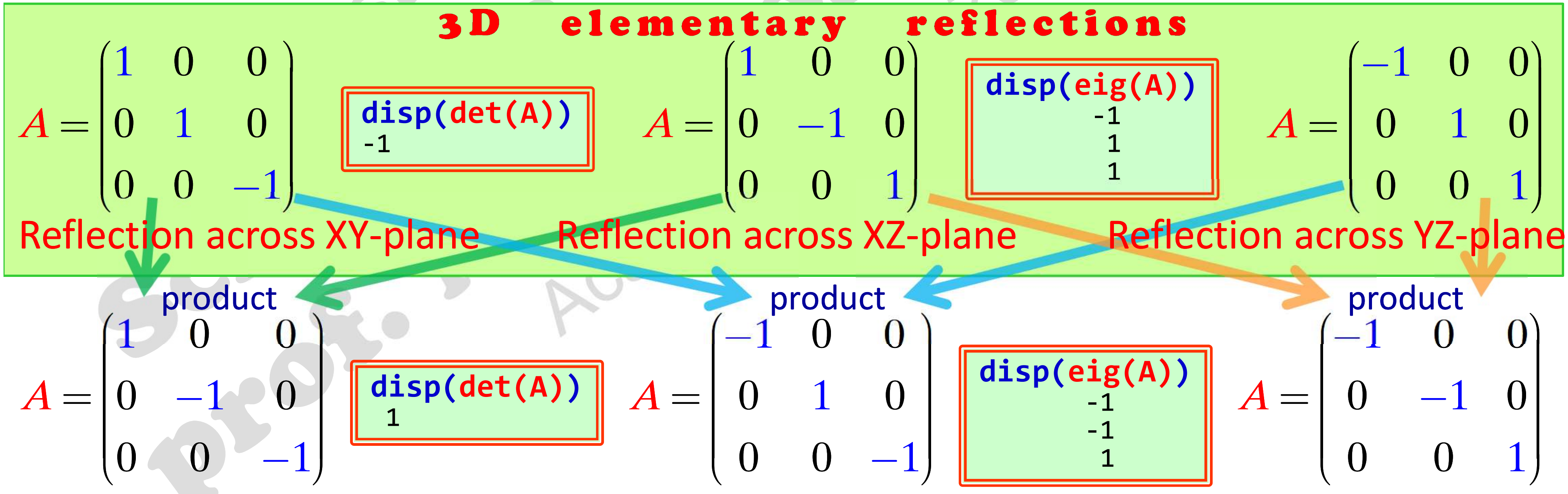
$0 < \rho < 1$ contraction

$1 < \rho$ dilation

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

**Non-uniform scaling** (or anisotropic scaling) centered at O

## Particular reflections



**3D elementary reflections**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
disp(det(A))
-1
```

Reflection across XY-plane

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
disp(eig(A))
 -1
  1
  1
```

Reflection across XZ-plane

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection across YZ-plane

product

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
disp(det(A))
 1
```

Reflection across X-axis

product

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
disp(eig(A))
 -1
 -1
  1
```

Reflection across Y-axis

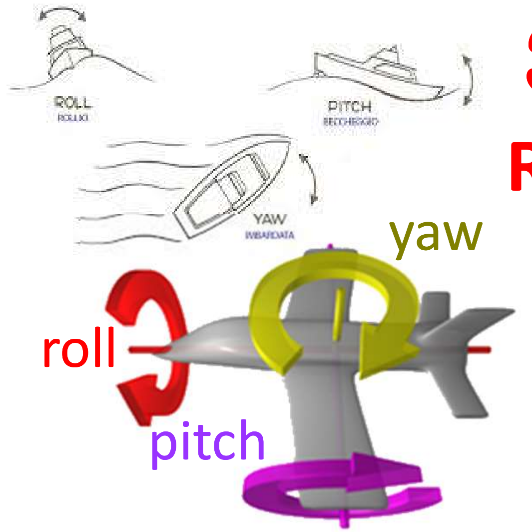
product

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection across Z-axis

# 3D Elementary Linear Maps

## Rotation centered at O around a cartesian axis



$$t_A \quad : \quad x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$$

permutation matrix

```
P=[0 0 1;
   1 0 0;
   0 1 0]
```

1

```
syms a real
A=[1      0        0;
   0 cos(a) -sin(a);
   0 sin(a) cos(a)];
```

2

```
B=P*A*P'; disp(B)
[ cos(a), 0, sin(a)]
[      0, 1,      0]
[-sin(a), 0, cos(a)]
```

3

```
C=P*B*P'; disp(C)
[cos(a), -sin(a), 0]
[sin(a),  cos(a), 0]
[     0,       0, 1]
```

2 and 3 derive from 1 : why?

1 $R_x(\alpha)$=**rotation around X-axis** by an angle $\alpha$

**roll** (it rollio)

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

2 $R_y(\beta)$=**rotation around Y-axis** by an angle $\beta$

**pitch** (it beccheggio)

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

3 $R_z(\theta)$=**rotation around Z-axis** by an angle $\theta$

**yaw** (it imbardata)

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Generic 3D Rotation = $R_x(\alpha)\, R_y(\beta)\, R_z(\theta)$

# Properties of rotations

$$t_A \quad : \quad x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$$

rotation $\Rightarrow$ $A^{-1} = A^{\mathsf{T}}$, $\det(A) = 1$ $\quad$ $A$: orthogonal matrix

orthogonal matrix $\Leftrightarrow$ $A^{-1} = A^{\mathsf{T}}$, $\det(A) = \pm 1$ $\leftarrow$ **Prove it**

remember that: $\det(A) = \det(A^{\mathsf{T}})$

$$A \text{ orthogonal} \begin{cases} \det(A) = +1 \\ \det(A) = -1 \end{cases}$$

$\det(A) = +1$ $\Rightarrow$ proper rotation

$\det(A) = -1$ $\Rightarrow$ improper rotation (rotation + reflection)

# 3D Rotation around an axis $a$ by an angle $\theta$

**Theor.**: In the Linear Space $\mathbb{R}^3$, the matrix $R_a(\theta)$ of a 3D rotation around an axis $r=\text{span}\{a\}$ and by an angle $\theta$ is

$$R_{\boldsymbol{a}}(\theta)=\begin{pmatrix} \boxed{c+(1-c)a_x^2} & (1-c)a_xa_y-sa_z & (1-c)a_xa_z+sa_y \\ (1-c)a_xa_y+sa_z & \boxed{c+(1-c)a_y^2} & (1-c)a_ya_z-sa_x \\ (1-c)a_xa_z-sa_y & (1-c)a_ya_z+sa_x & \boxed{c+(1-c)a_z^2} \end{pmatrix}$$

+ +

where $c=\cos(\theta)$, $s=\sin(\theta)$, $\boldsymbol{a}=(a_x, a_y, a_z)^T$ such that $\|\boldsymbol{a}\|_2=1$.

**Theor.**: Given the matrix $R$ of a proper rotation, then

- Its rotation axis $\boldsymbol{a}$ can be found as the following Null Space*:

$$R_a\boldsymbol{a}=\boldsymbol{a} \quad \Longleftrightarrow \quad \mathcal{N}(R_a-\text{I})=\text{span}\{\boldsymbol{a}\}$$

- and its rotation angle $\theta$ can be found by the following formulas:

$$\text{Tr}(R)=1+2c \qquad \Rightarrow \qquad c=\cos(\theta)=[\text{Tr}(R)-1]/2$$

$$\text{If} \quad a_z\neq 0, \quad R_{2,1}=(1-c)a_xa_y+sa_z \quad \Rightarrow \quad s=\sin(\theta)=[R_{2,1}-(1-c)a_xa_y]/a_z$$

$$\text{If} \quad a_y\neq 0, \quad R_{1,3}=(1-c)a_xa_z+sa_y \quad \Rightarrow \quad s=\sin(\theta)=[R_{1,3}-(1-c)a_xa_z]/a_y$$

$$\text{If} \quad a_x\neq 0, \quad R_{3,2}=(1-c)a_ya_z+sa_x \quad \Rightarrow \quad s=\sin(\theta)=[R_{3,2}-(1-c)a_ya_z]/a_x$$

$$\Rightarrow \quad \boxed{\theta=\text{atan2}(s,c)}$$

where Tr() denotes the trace of a matrix.

trace = sum of elements of main diagonal

* Alternatively, the rotation axis $\boldsymbol{a}$ is given by: $R_a\boldsymbol{a}=\boldsymbol{a}$, that is the (unitary) *eigenvector* of $R_a$ related to the *eigenvalue* 1.

# 3D rotation: example 1

```
R=[-1 0 0;0 0 -1;0 1 0]
R =
    -1     0     0
     0     0    -1
     0     1     0
disp([R'*R  R*R'])
     1     0     0     1     0     0
     0     1     0     0     1     0
     0     0     1     0     0     1
disp(det(R))
  -1
```

orthogonal matrix

improper rotation

```
P=[1 0 0;0 0 1;0 1 0]
P =
     1     0     0
     0     0     1
     0     1     0
R1=R*P
R1 =
cos(θ) -1     0     0
sin(θ)  0    -1     0
        0     0     1
```

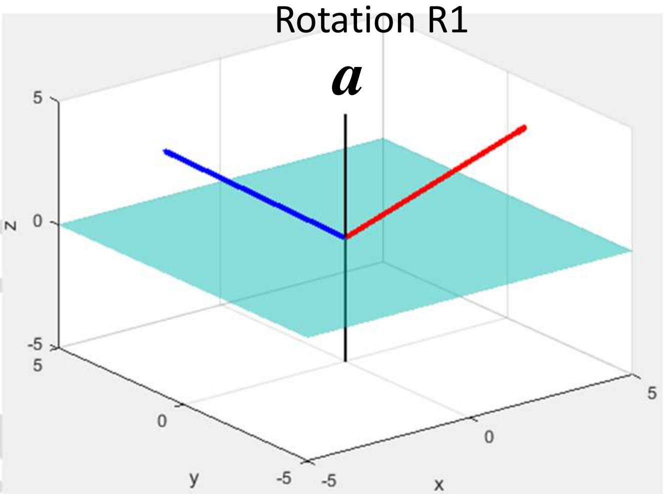
axis of rotation

P: elementary permutation matrix

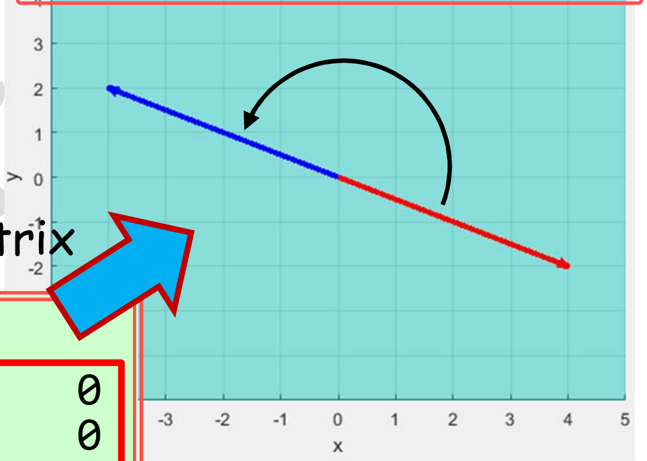
```
disp([R1'*R1  R1*R1'])
     1     0     0     1     0     0
     0     1     0     0     1     0
     0     0     1     0     0     1
disp(det(R1))
  1
...
```

R1: proper rotation

θ=180°, axis = span{$(0,0,1)^T$}



```
view([a(1),a(2),a(3)])
```



R=R1*P

## Exercise

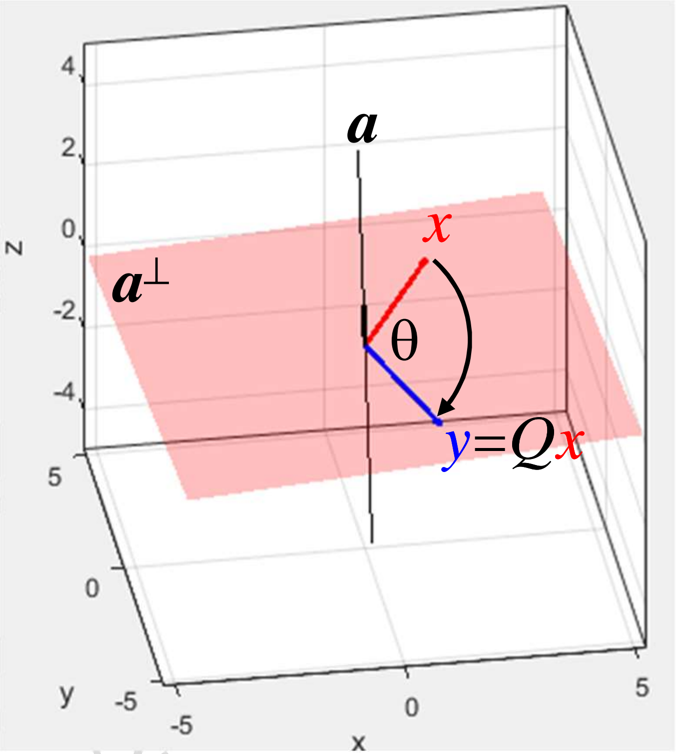
What kind of linear transformation does the permutation matrix P induce? And $P^{-1}$? Display their effects.

# 3D rotation: example 2

```
rng('default');  A=rand(3);  [Q,R]=qr(A);
disp([Q'*Q  Q*Q'])
     1      0      0      1      0      0
     0      1      0      0      1      0
     0      0      1      0      0      1
disp(det(Q))
   1
```

orthogonal matrix

proper rotation



$a$: rotation axis

$$x\in a^\perp \;\Rightarrow\; y=Qx\in a^\perp$$

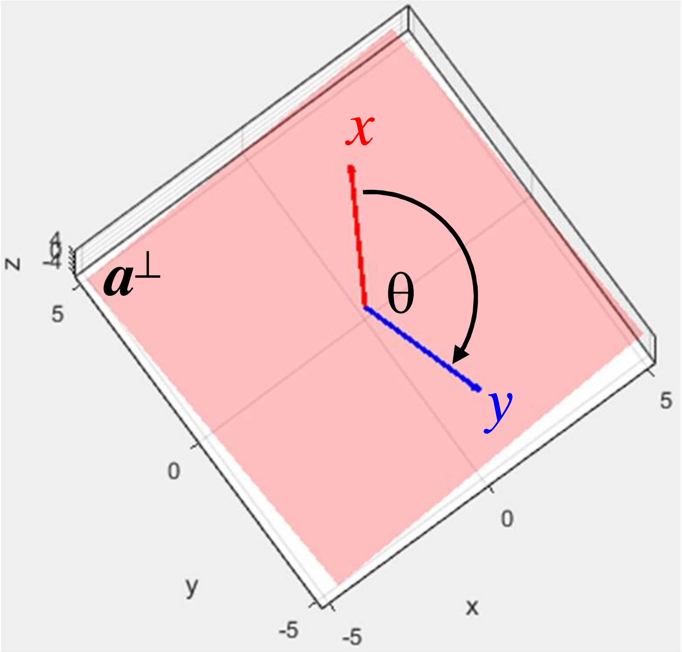
```
a=null(Q-eye(3))
a =
    -0.038844
     -0.05243
      0.99787
cosTh=(trace(Q)-1)/2
cosTh =
    -0.66766 < 0
sinTh=(Q(1,3)-(1-cosTh)*a(1)*a(3))/a(2)
    -0.74447 < 0
Th=rad2deg(atan2(sinTh,cosTh))
Th =
    -131.89
```

rotation axis

normalized vector

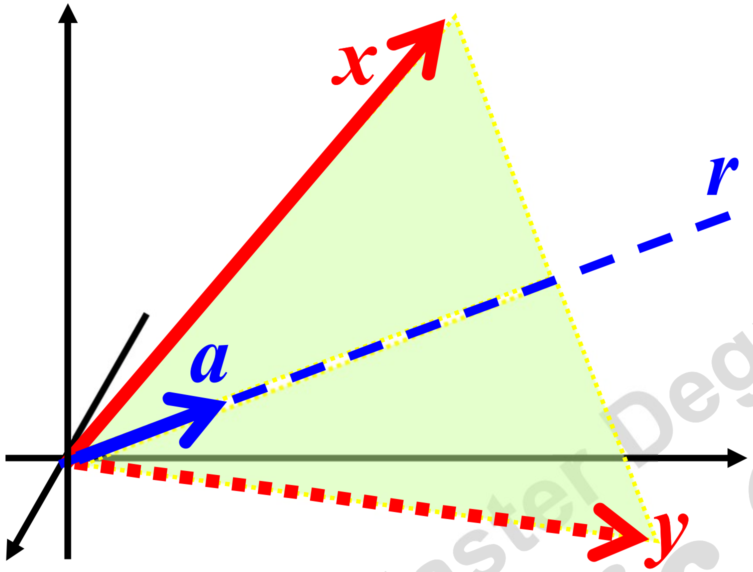
$$s=\sin(\theta)=\left[R_{1,3}-(1-c)a_x a_z\right]/a_y$$

rotation angle



# Orthogonal reflection across a line $r$=span{$a$}: how can we get the transformation matrix?

$$t_A \ : \ x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3$$



$t_A$ : such that $\begin{cases} x + y \in r \\ x - y \perp r \end{cases} \Rightarrow \begin{cases} x + y = \lambda a \\ \langle x - y, a \rangle = 0 \end{cases}$

$$\Downarrow$$

$$\begin{cases} y = \lambda a - x \\ \langle 2x - \lambda a, a \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} y = \lambda a - x \\ 2\langle a, x \rangle - \lambda \langle a, a \rangle = 0 \end{cases}$$

$$\Leftrightarrow \lambda = 2\frac{\langle a, x \rangle}{\langle a, a \rangle} = 2\frac{\langle a, x \rangle}{\|a\|_2^2} \Rightarrow y = 2a\frac{\langle a, x \rangle}{\|a\|_2^2} - x = \frac{2}{\|a\|_2^2} aa^\top x - x = \left( \frac{2}{\|a\|_2^2} aa^\top - I_3 \right) x$$

$$\Rightarrow \boxed{A = \left( \frac{2}{\|a\|_2^2} aa^\top - I \right)}$$

**Exercise**

What are $\mathcal{N}(A)$, $\mathcal{R}(A^\top)$, $\mathcal{N}(A^\top)$ and $\mathcal{R}(A)$?
And $\mathcal{N}(A - \mathrm{I})$? Is $t_A$ an automophism?

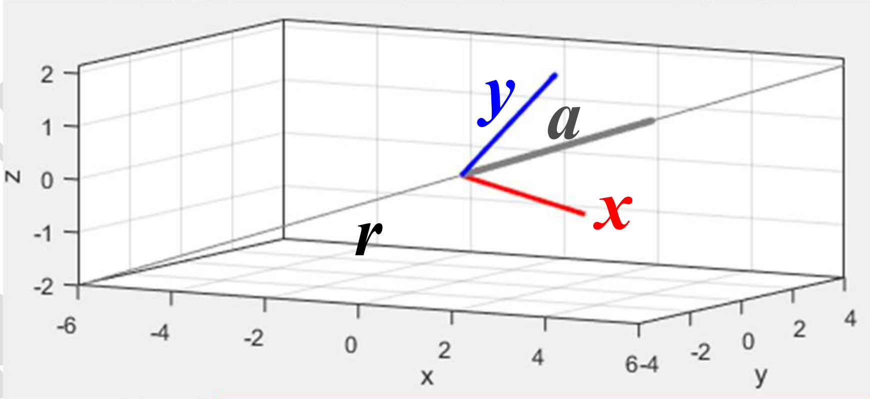
# **MATLAB Lab:** 3D reflection across a generic line

$r=\mathrm{span}\{a\}:\ a=(3,2,1)^T$

## What are the properties of a 3D reflection matrix?

```
a=[3 2 1]';  syms t real
ezplot3(t*a(1),t*a(2),t*a(3),[-2 2]) r
axis equal; hold on; box on
h=quiver3(0,0,0,a(1),a(2),a(3),1); set(…
A = 2/norm(a)^2*a*a'-eye(size(a,1))
A =
    0.28571   0.85714   0.42857
    0.85714  -0.42857   0.28571
    0.42857   0.28571  -0.85714
x=[1 3 -1]'; y=A*x;
h=quiver3(0,0,0,x(1),x(2),x(3),1); set(…
h=quiver3(0,0,0,y(1),y(2),y(3),1); set(…
```

symmetric



```
ax=null(A-eye(3)); % rotation axis
disp(rank([a ax]))
     1
cosTH=(trace(A)-1)/2;
sinTH=(A(1,3) - ...
      (1-cosTH)*ax(1)*ax(3))/ax(2);
TH=atan2(sinTH,cosTH)*180/pi
TH =
    -180
```

```
disp(det(A))
1
```

the determinant of A equals 1*

* a reflection with det(A)=1 is a rotation around its axis by an angle = ±π

```
disp(eig(A))
            -1
            -1
             1
```

its eigenvalues are -1, 1

```
disp(A*A)
            1   -4.4409e-16            0
  -4.4409e-16             1            0
            0             0            1
```

the matrix equals its inverse

```
disp(A'*A)
            1   -1.5266e-16  -1.1102e-16
  -1.5266e-16             1  -2.7756e-17
  -1.1102e-16   -2.7756e-17            1
```

```
disp(A*A')
            1   -1.5266e-16  -1.1102e-16
  -1.5266e-16             1  -2.7756e-17
  -1.1102e-16   -2.7756e-17            1
```

orthogonal matrix

# Reflections across a line

$$A=\left(\frac{2}{\|a\|_2^2}aa^{\mathrm{T}}-I\right)$$

## 3D

```
N=3; syms a [N 1] real
A=simplify(2/norm(a)^2*a*a'-eye(N));
disp(det(A))
1
```

proper rotation

```
disp(eig(A))
-1
-1
 1
```

eigenvalues

```
all(all(A == A.'))
ans =
  logical
   1
```

symmetric matrix

```
all(all(simplify(A*A) == eye(N)))
ans =
  logical
   1
```

$A = A^{-1}$

```
all(all(simplify(A'*A) == eye(N)))
ans =
  logical
   1
```

$A = A^{\mathrm{T}} = A^{-1}$

```
all(all(simplify(A*A' == eye(N))))
ans =
  logical
   1
```

orthogonal matrix

```
ax=null(A-eye(N))
ax =
a1/a3
a2/a3
    1
```

**a** is the 3D rotation axis

```
cosTH=simplify((trace(A)-1)/2)
cosTH =
-1
```

the rotation angle is 180°

only for 3D rotation

## 4D

```
N=4; syms a [N 1] real
A=simplify(2/norm(a)^2*a*a'-eye(N));
disp(det(A))
-1
```

improper rotation

```
disp(eig(A))
-1
-1
-1
 1
```

eigenvalues

```
all(all(A == A.'))
ans =
  logical
   1
```

symmetric matrix

```
all(all(simplify(A*A) == eye(N)))
ans =
  logical
   1
```

$A = A^{-1}$

```
all(all(simplify(A'*A) == eye(N)))
ans =
  logical
   1
```

$A = A^{\mathrm{T}} = A^{-1}$

```
all(all(simplify(A*A' == eye(N))))
ans =
  logical
   1
```

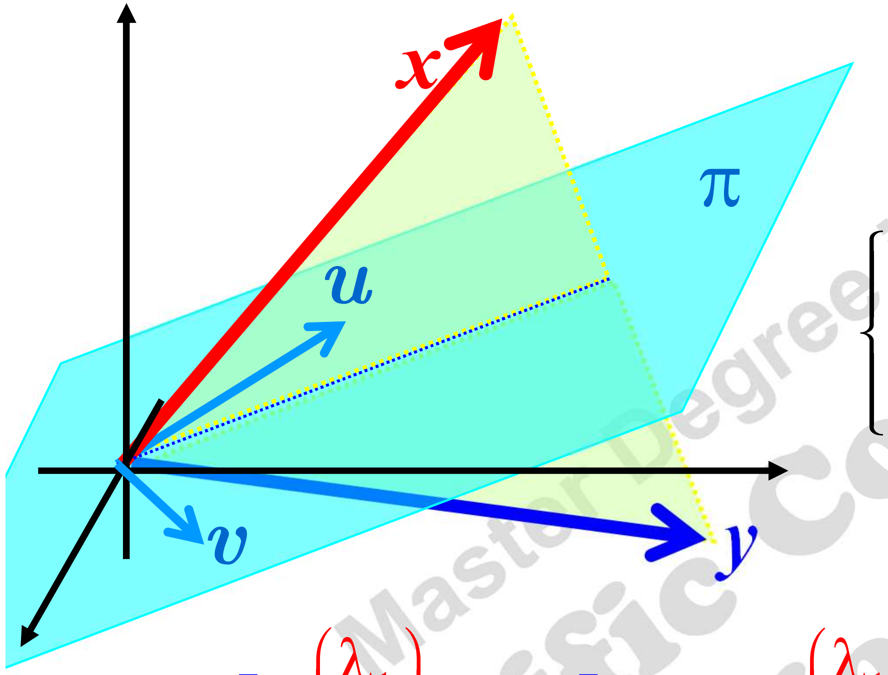
orthogonal matrix

```
ax=null(A-eye(N))
ax =
a1/a4
a2/a4
a3/a4
    1
```

**a** is reflection axis

# Elementary 3D reflection across a plane $\pi=\text{span}\{u,v\}$: how can we get the transformation matrix?

$$t_A \;:\; x\in\mathbb{R}^3 \longrightarrow y=t_A(x)=Ax\in\mathbb{R}^3$$



$$\boxed{\begin{array}{c} U=[u,v] \\ \pi=\mathcal{R}(U) \\ x-y\in\pi^{\perp} \end{array}}$$

$t_A$ such that $\begin{cases} x+y\in\pi \\ x-y\perp\pi \end{cases}$

$$\begin{cases} x+y\in\mathcal{R}(U) \\ x-y\in\mathcal{N}(U^{\mathsf{T}}) \end{cases} \Leftrightarrow \begin{cases} x+y=U\lambda \\ U^{\mathsf{T}}(x-y)=0 \end{cases}$$

$$y=U\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}-x \Rightarrow U^{\mathsf{T}}\left[2x-U\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\right]=0$$

$$\Leftrightarrow U^{\mathsf{T}}U\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}=2U^{\mathsf{T}}x \Leftrightarrow \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}=2\left(U^{\mathsf{T}}U\right)^{-1}U^{\mathsf{T}}x \qquad \boxed{y=U\lambda-x \Rightarrow}$$

$$\Rightarrow y=U\left[2\left(U^{\mathsf{T}}U\right)^{-1}U^{\mathsf{T}}x\right]-x=\underbrace{\left[2U\left(U^{\mathsf{T}}U\right)^{-1}U^{\mathsf{T}}-I_3\right]}x$$

matrix of an orthogonal reflection across $\pi$

$$F=t_A : x\in\mathbb{R}^3 \longrightarrow y=Ax\in\mathbb{R}^3$$

# Properties of 3D reflection matrices

$$A = 2U\left(U^{\top}U\right)^{-1}U^{\top} - I_3$$

across a line : $U=[u]$ (vector)
across a plane: $U=[u,v]$ (matrix)

1. The matrix of a reflection is symmetric.

2. The inverse of a reflection is the reflection itself.

Proof:

$$A\cdot A = \left[2U\left(U^{\top}U\right)^{-1}U^{\top} - I_3\right]\left[2U\left(U^{\top}U\right)^{-1}U^{\top} - I_3\right] =$$

$$= 4U\left(U^{\top}U\right)^{-1}\cancel{U^{\top}U\left(U^{\top}U\right)^{-1}}U^{\top} - 4U\left(U^{\top}U\right)^{-1}U^{\top} + I_3 =$$

$$= \cancel{4U\left(U^{\top}U\right)^{-1}U^{\top} - 4U\left(U^{\top}U\right)^{-1}U^{\top}} + I_3 = I_3$$

3. The matrix of a reflection is orthogonal.

4. $U=[u,v]$ ⇨ its eigenvalues are $-1, +1, +1$ and its determinant is $-1$.
   $U=[u]$ ⇨ its eigenvalues are $-1, -1, +1$ and its determinant is $+1$.

## Orthonormal basis

across a line : $U=[u]$ $\|u\|=1$
across a plane: $U=[u,v]$ $U^{\top}U=\mathrm{I}$

⇨

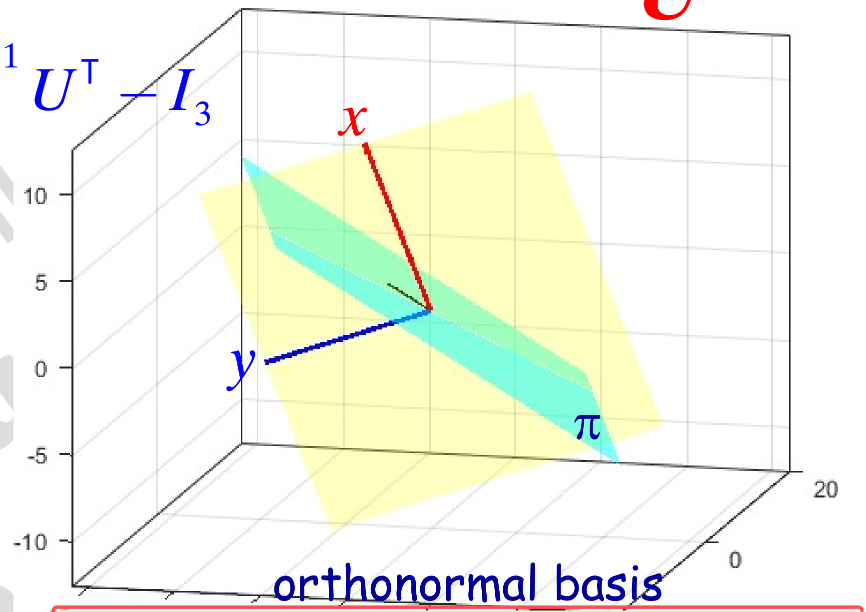
$$A = 2u\,u^{\top} - I_3$$
$$A = 2UU^{\top} - I_3$$

the inverse matrix is no longer needed: simpler formula

# Example: 3D reflection across $\pi=\mathrm{span}\{\underbrace{u,v}_{U}\}$

```
u=[2 3 1]'; v=[3 1 0]'; U=[u v];
syms a b real; p=U*[a b]';
A=2*U*inv(U'*U)*U' - eye(3);
all(all(A == A'))
ans =
  logical                 symmetric matrix
   1
all(all(abs(A*A - eye(3)) < 1e-10))
ans =
  logical                           A = A⁻¹
   1
x=10*rand(3,1); y=A*x;
disp(rank([U x+y])) x+y∈ℛ(U)
     2
disp((x-y)'*U)
  -5.3291e-15   4.4409e-16 (x-y)⊥π
disp(det(A))
                    -1
disp(eig(A))
                    -1
                     1
                     1
```

$$A = 2U\left(U^{\mathsf{T}}U\right)^{-1}U^{\mathsf{T}} - I_3$$



orthonormal basis

symbolic

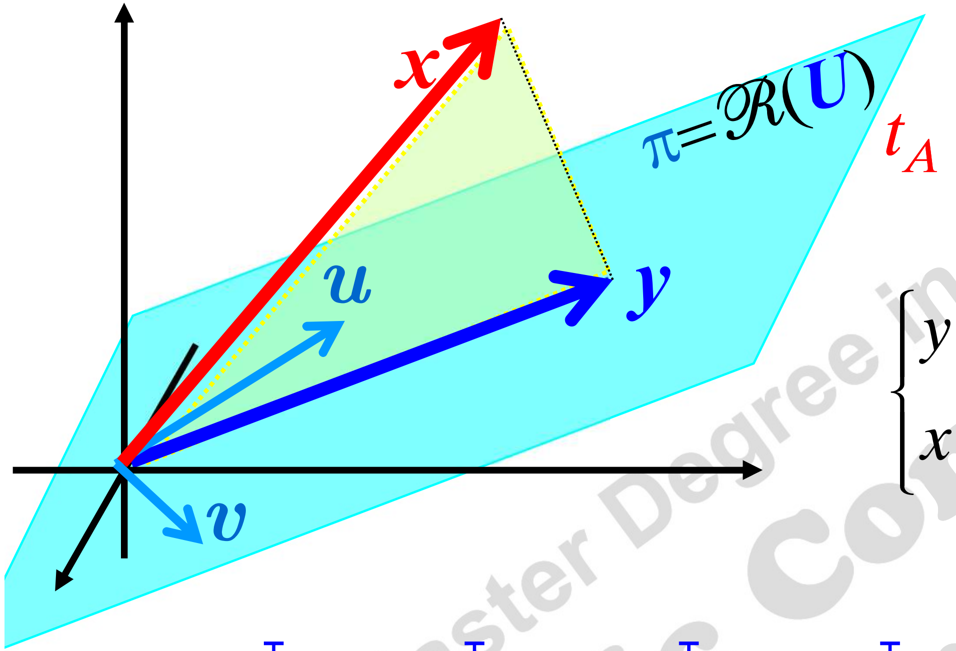
```
syms u v [3 1] real;  U=[u v];
A=simplify(2*U*inv(U'*U)*U'-eye(3));
disp(simplify(det(A)))
-1
disp(simplify(eig(A)))
-1
 1
 1
all(all(A == A'))
ans =
  logical          symmetric matrix
   1
all(all(simplify(A*A) == eye(3)))
ans =
  logical                    A = A⁻¹
   1
disp(rank([U x+y]))  x+y∈ℛ(U)
2
disp(simplify((x-y)'*U))
[0, 0]
                           (x-y)⊥π
```

```
u=[2 3 1]'; v=[3 1 0]';
U=orth([u v]);
disp(U'*U)
           1   2.0817e-17
  2.0817e-17            1
A=2*U*U'-eye(3);
x=10*rand(3,1); y=A*x;
disp(rank([U x+y]))          x+y∈ℛ(U)
    2
disp((x-y)'*U)
  -5.3291e-15   4.4409e-16  (x-y)⊥π
all(all(A == A'))
ans =
  logical
   1
disp(det(A))
  -1
disp(eig(A))
          -1 the product of eigenvalues gives
           1 the value of determinant
           1
```

$$A = 2UU^{\mathsf{T}} - I_3$$

# Orthogonal projection onto a plane $\pi=\mathrm{span}\{u,v\}$

$$t_A \;:\; x\in\mathbb{R}^3 \longrightarrow y=t_A(x)=Ax\in\mathbb{R}^3$$



$$U=[u,v]$$
$$\pi=\mathcal{R}(U)$$
$$x-y\in\pi^{\perp}$$

$t_A$ such that $\begin{cases} y\in\pi \\ x-y\perp\pi \end{cases}$

$$\begin{cases} y\in\mathcal{R}(U) \\ x-y\in\mathcal{N}(U^\top) \end{cases} \Leftrightarrow \begin{cases} y=U\lambda \\ U^\top(x-y)=0 \end{cases}$$

Gram matrix

$$\Leftrightarrow U^\top x-U^\top y=U^\top x-U^\top U\lambda=0 \Leftrightarrow \boxed{U^\top U}\lambda=U^\top x$$

$$\Leftrightarrow \lambda=\left(U^\top U\right)^{-1}U^\top x \qquad y=U\lambda \;\Rightarrow\; y=U\left(U^\top U\right)^{-1}U^\top x$$

$A$: matrix for the orthogonal projection onto $\mathcal{R}(U)$

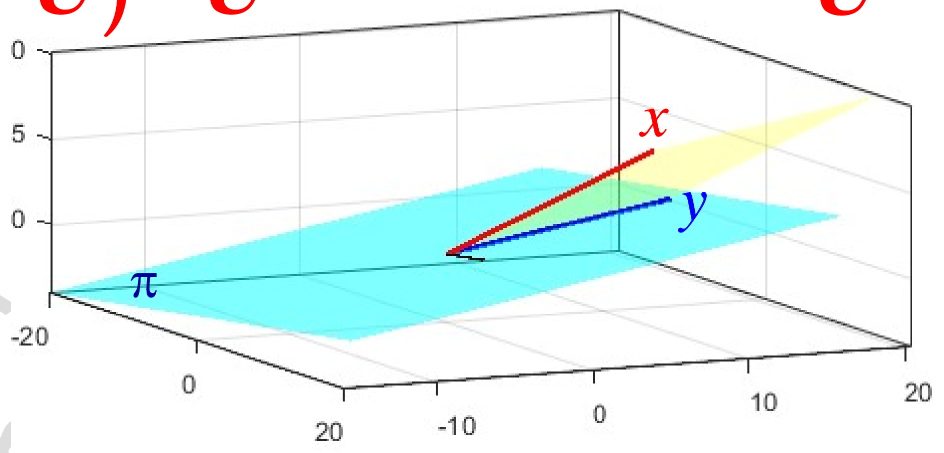
$$y=\boxed{\left[U\left(U^TU\right)^{-1}U^T\right]}x \qquad A=U(U^\top U)^{-1}U^\top$$

$U$ has orthonormal columns ⇒ $U^\top U=I$ ⇒ simpler $A=UU^\top$

# Example: 3D orthogonal projection onto $\pi=\text{span}\underbrace{\{u,v\}}_{U}$

$$A = U(U^\top U)^{-1}U^\top$$

```
u=[2 3 1]'; v=[3 1 0]'; U=[u v];
syms a b real; p=U*[a b]';
A=U*inv(U'*U)*U';
all(all(abs(A - A') < 1e-10))
ans =
  logical                  symmetric matrix
   1
all(all(abs(A*A - A) < 1e-10))
ans =
  logical                 A*A=A  idempotent
   1
x=10*rand(3,1); y=A*x;
disp(y'*(x-y))    y⊥(x-y)
    6.728e-14
disp(rank([U y]))
        2
disp(rank(A))
        2
disp(det(A))
  -2.3051e-17
disp(eig(A))
  -4.1973e-17
            1
            1
```



symbolic

```
syms u v [3 1] real;  U=[u v];
A=simplify(U*inv(U'*U)*U');
all(all(A == A'))
ans =
  logical           symmetric matrix
   1
syms x [3 1] real; y=A*x;
disp(simplify(y'*(x-y)))
0                          y⊥(x-y)
disp(rank([U y]))
      2                  y∈ℛ(U)
disp(rank(A))
      2
disp(simplify(det(A)))
0
disp(simplify(eig(A)))
0
1
1
```

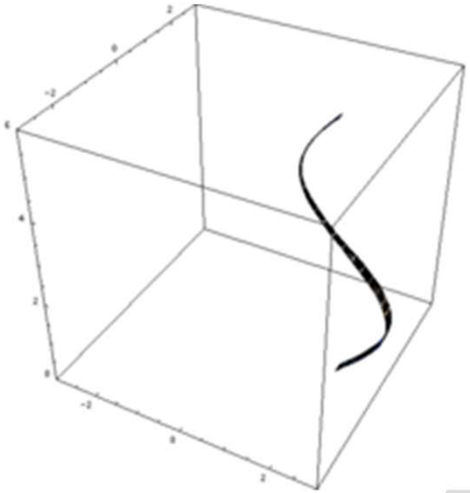
orthonormal basis

```
u=[2 3 1]'; v=[3 1 0]';
U=orth([u v]);
disp(U'*U)
               1   2.0817e-17
   2.0817e-17               1
A=U*U'; x=10*rand(3,1); y=A*x;
disp(rank([U y]))
    2                 y∈ℛ(U)
disp(y'*(x-y))
   2.4869e-14          y⊥(x-y)
all(all(A == A'))
ans =
  logical
   1
disp(det(A))
   0
disp(eig(A))
   8.3267e-17 the product of eigenvalues gives
            1
            1 the value of determinant
```

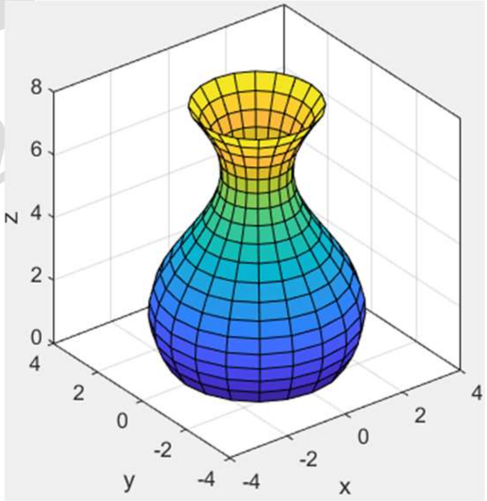
$$A = UU^\top$$

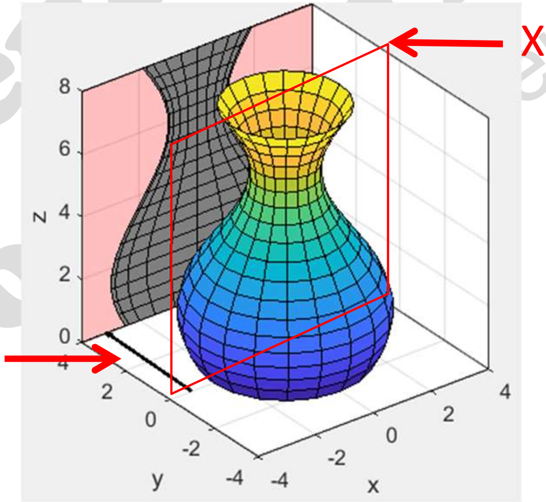
# **Exercise** on a solid of revolution

Display the shadow parallel to XZ-plane (or to YZ-plane) of a solid of revolution. This shadow can be computed as the orthogonal projection of the solid onto that plane, then moved to a side of the graphic figure by means of a translation.



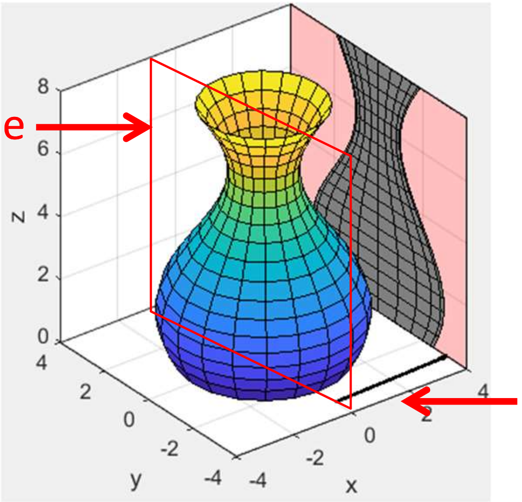
```
t=(-pi/3:pi/10:2*pi-pi/2)';
y=2+cos(t);
[X,Y,Z]=cylinder(y); Z=8*Z;
figure(1); surf(X,Y,Z); axis equal
box on; hold on
set(gca,'FontSize',12)
AX=[-4 4 -4 4 0 8]; axis(AX)
xlabel('x'); ylabel('y'); zlabel('z')
```





XZ-plane

XZ-plane displacement

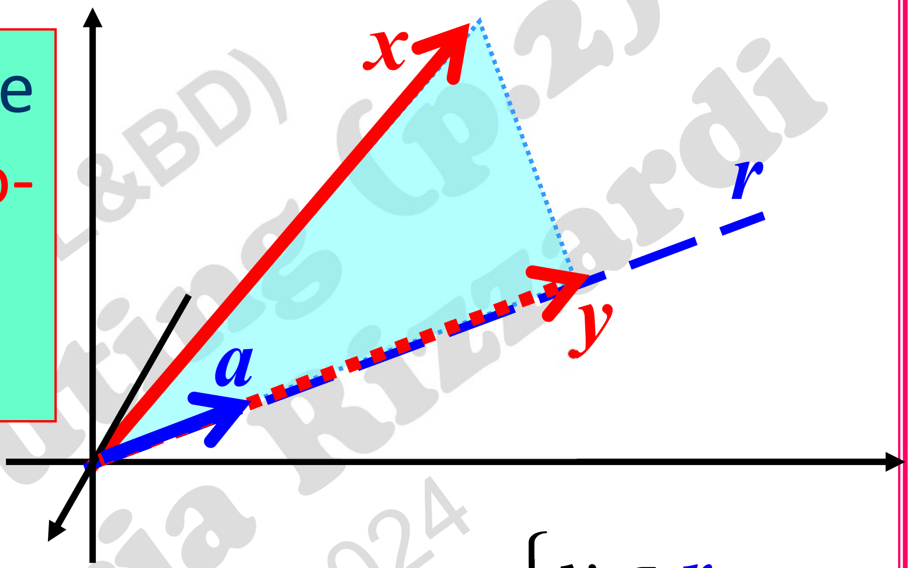


YZ-plane

YZ-plane displacement

# Exercises

**1** Find the matrix form of the endomorphism for the orthogonal projection onto a line
$r = \text{span}\{a\}$, assigned $a \in \mathbb{R}^3$.



Hints:

$$t_A \; : \; x \in \mathbb{R}^3 \longrightarrow y = t_A(x) = Ax \in \mathbb{R}^3 \text{ such that } \begin{cases} y \in r \\ x - y \perp r \end{cases}$$

$$\begin{cases} y \in r \\ x - y \perp r \end{cases} \Leftrightarrow \begin{cases} y = \lambda a \\ \langle x - y, a \rangle = 0 \end{cases}$$

**2** Compute the distance between a point and
- a line $r$
- a plane $\pi$.

# Contents

- Generalized inverse, ABCD Theor., pseudoinverse and one sided inverse.
- Solutions of an underdetermined linear system.
- Least-norm solution of underdetermined linear systems.

# Generalized inverses of a matrix

Let A be a matrix (m×n) of rank r:

G (n×m) is called a *generalized inverse* of A $\overset{def}{\Longleftrightarrow}$ AGA = A

G always exists, but might not be unique in general.

A: square and invertible ⇒ $G=IGI=A^{-1}AGAA^{-1}=A^{-1}A\,A^{-1}=A^{-1}$

this leads to the name generalized inverse

## Example

$$A=\begin{pmatrix}1 & 2\\ 2 & 4\\ 3 & 6\end{pmatrix}$$

```
A=[1 2;2 4;3 6]; G=[1 0 0;0 0 0]; AGA=A*G*A
AGA =
      1      2
      2      4
      3      6
```

**G** is a *generalized inverse* of **A**

```
AG=A*G
AG =
   1      0      0
   2      0      0
   3      0      0
```

```
GA=G*A
GA =
      1      2
      0      0
```

Is G unique?

```
NA=null(sym(A));
syms a b c real
G1=G + NA*[a b c]
G1 =
[1 - 2*a, -2*b, -2*c]
[      a,    b,    c]
AG1A=A*G1*A
AG1A =
     1      2
     2      4
     3      6
```

∀a,b,c

**G1** generalized inverse

infinitely many matrices **G1**

# Generalized inverses of a matrix
## (g-inverses)

### ABCD Theorem

Let $A$ be an m×n matrix with rank($A$)=r≤min{m,n}. One can show that, **after a suitable reordering of its rows and columns**, $A$ can be written in partitioned form as:

**permuted** $A$ $\quad A = \begin{pmatrix} A_r & B \\ C & D \end{pmatrix}$ where

$A_r$ is r×r and invertible,
$B$ is r×(n−r),
$C$ is (m−r)×r,
$D$ is (m−r)×(n−r)

Then $D = C\,A_r^{-1}B$, so that

$$A = \begin{pmatrix} A_r & B \\ C & CA_r^{-1}B \end{pmatrix}$$

### Construction of a generalized inverse

Let $A$ be as in ABCD Theor., then a generalized inverse of $A$ is $G$ (n×m):

$$G_{n\times m} = \begin{pmatrix} A_r^{-1} & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{pmatrix}$$

# Generalized inverses of a matrix: example 1

The submatrix $A_r$ does not need any rearrangement of rows and columns in $A$

ABCD Theor.

1

```
A=sym([1 2 3;4 5 6;7 8 9]);
[m,n]=size(A);
r=rank(A)
r =
     2
S=rref(A)
S =
[1, 0, -1]
[0, 1,  2]
[0, 0,  0]
```

$$\begin{array}{ccc} & A_r & B \\ A = & \boxed{\begin{matrix}1 & 2\\ 4 & 5\end{matrix}} & \boxed{\begin{matrix}3\\ 6\end{matrix}} \\ & \boxed{\begin{matrix}7 & 8\end{matrix}} & \boxed{9} \\ & C & D \end{array}$$

2

```
Ar=A(1:2,1:2)
Ar =
[1, 2]
[4, 5]
rank(Ar)
ans =
     2
```

3

```
B=A(1:2,3)
B =
3
6
C=A(3,1:2)
C =
[7, 8]
D=C*inv(Ar)*B
D =
9
```

G: g-inverse

```
G=[inv(Ar) zeros(r,m-r); zeros(n-r,m)]
G =
[-5/3,  2/3, 0]
[ 4/3, -1/3, 0]
[   0,    0, 0]
all(all(isAlways(A*G*A == A)))
ans =
  logical
   1
```

check

uniqueness of G?

```
NA=null(A);   syms a b c real
G1=G + NA*[a b c];
all(all(isAlways(A*G1*A == A)))
ans =
  logical
   1
```

infinitely many g-inverses

# Generalized inverses of a matrix: example 2

symbolic or numeric matrix

```
A=sym([1 2;1 2;1 1]);
[m,n]=size(A);
r=rank(A)
r =
     2
S=rref(A)
S =
[1, 0]
[0, 1]
[0, 0]
[~,~,Prow]=lu(A)
Prow =
     1     0     0
     0     0     1
     0     1     0
PA=Prow*A
PA =
[1, 2]
[1, 1]
[1, 2]
```

$r = n$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Columns 1 and 2 in A are lin. ind., but not rows 1 and 2

permutation matrix

Now, rows 1 and 2 in PA are linearly independent

**Teor. ABCD**

… "suitable rearrangement of rows and columns in $A$" …

$A_r$

$$PA = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$C$

```
Ar=PA(1:r,1:r)
Ar =
[1, 2]
[1, 1]
rank(Ar)
ans =
     2
```

```
G=[inv(Ar) zeros(r,m-r); zeros(n-r,m)]
G =
[-1,  2, 0]
[ 1, -1, 0]
all(all(isAlways(A*G*A == A)))
ans =
  logical
   0
all(all(isAlways(PA*G*PA == PA)))
ans =
  logical
   1
```

G is not a generalized inverse of A

G is a generalized inverse of PA

G is a generalized inverse of PA

$$\cancel{P_{row}}AGP_{row}A = \cancel{P_{row}}A$$

$$\boxed{A(GP_{row})A = A}$$

```
GP=G*Prow
GP =
[-1, 0,  2]
[ 1, 0, -1]
all(all(isAlways(A*GP*A == A)))
ans =
  logical
   1
```

permute columns in G

GP is a generalized inverse of A

# Generalized inverses of a matrix: example 2 (cont)

## Algorithm 1

symbolic or numeric matrix

**ABCD Theor.**

… "suitable rearrangement of rows and columns in A" …

1

```
A=sym([1 2;1 2;1 1]);
[m,n]=size(A); r=rank(A)
r =
     2
[~,~,Prow]=lu(A)
Prow =
   1      0      0
   0      0      1
   0      1      0
[~,~,Pcol]=lu(A')
Pcol =
     1      0
     0      1
```

row-permutation matrix

col-permutation matrix

actually, , it's not needed here

$$A=\begin{pmatrix}1 & 2\\ 1 & 2\\ 1 & 1\end{pmatrix}$$

permuted matrix

2

```
PrAPc=Prow*A*Pcol
PrAPc =
[1, 2]
[1, 1]
[1, 2]
Ar=PrAPc(1:r,1:r)
Ar =
[1, 2]
[1, 1]
rank(Ar)
ans =
     2
```

sottomatrice Ar di rango r=2

G: g-inverse

```
G=[inv(Ar) zeros(r,m-r); zeros(n-r,m)]
G =
[-1,  2, 0]
[ 1, -1, 0]
all(all(isAlways(A*G*A == A)))
ans =
  logical
   0
all(all(isAlways(PrAPc*G*PrAPc == PrAPc)))
ans =
  logical
   1
```

check

G is not a g-inverse of A

G is a g-inverse of permuted matrix

g-inverse of **A** matrix

$$\boxed{A*\boxed{(P_{col}*G*P_{row})}*A = A}$$

G is a generalized inverse of $P_{row}*A*P_{col}$

$$\cancel{P_{row}}*A*P_{col}*G*P_{row}*A*\cancel{P_{col}} = \cancel{P_{row}}*A*\cancel{P_{col}}$$

# Generalized inverses of a matrix: example 3

**Theor. ABCD**

… "suitable rearrangement of rows and columns in A" …

## Algorithm 1

```
A=sym([4 8 4 -2;4 8 4 -2;-2 -4 -2 10]);
[m,n]=size(A)
m =
     3
n =
     4
r=rank(A)
r =
     2
[~,~,Prow]=lu(A)
Prow =
     1     0     0
     0     1     0
     0     0     1
[~,~,Pcol]=lu(A')
Pcol =
     1     0     0     0
     0     1     0     0
     0     0     0     1
     0     0     1     0
```

$$A=\begin{pmatrix} 4 & 8 & 4 & -2 \\ 4 & 8 & 4 & -2 \\ -2 & -4 & -2 & 10 \end{pmatrix}$$

← actually, it's not needed here

permuted matrix

$\boxed{2}$

```
PrAPc=Prow*A*Pcol
PrAPc =
[ 8,  4, -2,  4]
[ 8,  4, -2,  4]
[-4, -2, 10, -2]
Ar=PrAPc(1:r,1:r)
Ar =
[8, 4]
[8, 4]
rank(Ar)
ans =
     1
```

the rank of **Ar** submatrix is not r=2

## Algorithm 1 doesn't always work

# Generalized inverses of a matrix: example 3 (cont)

Theor. ABCD

… "suitable rearrangement of rows and columns in $A$" …

$$A=\begin{pmatrix} 4 & 8 & 4 & -2 \\ 4 & 8 & 4 & -2 \\ -2 & -4 & -2 & 10 \end{pmatrix}$$

```
A=[4 8 4 -2;4 8 4 -2;-2 -4 -2 10];
[m,n]=size(A)
m =
     3
n =
     4
r=rank(A)
r =
     2
[~,pCOL]=rref(A)
pCOL =
     1     4
AA=[A(:,pCOL) A(:,setdiff((1:n),pCOL))]
AA =
     4    -2     8     4
     4    -2     8     4
    -2    10    -4    -2
[~,pROW]=rref(AA')
pROW =
     1     3
AAA=[AA(pROW,:);AA(setdiff((1:m),pROW),:)]
AAA =
     4    -2     8     4
    -2    10    -4    -2
     4    -2     8     4
```

this syntax only holds for numeric arrays

the first r=2 columns are linearly independent, but the first r=2 rows are not

the submatrix Ar, of size r×r, has rank=2

## Algorithm 2

```
[m,n]=size(A); r=rank(A);
[~,pCOL]=rref(A)
pCOL =
     1     4
Pcol=eye(n);
Pcol=[Pcol(:,pCOL) Pcol(:,setdiff((1:n),pCOL))]
Pcol =
     1 0 0 0
     0 0 1 0
     0 0 0 1
     0 1 0 0
[~,pROW]=rref(A')
pROW =
     1     3
Prow=eye(m);
Prow=[Prow(pROW,:);Prow(setdiff((1:m),pROW),:)]
Prow =
     1 0 0
     0 0 1
     0 1 0
PAP=Prow*A*Pcol; Ar=PAP(1:r,1:r)
Ar =
     4    -2
    -2    10
G=[inv(Ar) zeros(r,m-r); zeros(n-r,m)];
all(all(A* (Pcol*G*Prow) *A == A))
ans =
  logical
   1
```

only valid for numeric matrices

col-permutation matrix

only valid for numeric matrices

row-permutation matrix

the submatrix Ar, of size r×r, has rank=2

(Pcol*G*Prow) g-inverse of A

# Exercise

Compute a generalized inverse of

$$A = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 10 \end{pmatrix}$$

and of **`diag(diag(A))`**

# Other kinds of generalized inverses:
# Pseudoinverse matrix
## (Moore-Penrose inverse)

* for complex matrices, replace T with H

A real* matrix (m×n) of rank r:

B matrix (n×m) is called the pseudoinverse of A, and denoted by $\boxed{B=A^+}$, if B satisfies all four conditions below:

(1) ABA = A (B is a generalized inverse of A)

(2) BAB = B (A is a generalized inverse of B)

B is a reflexive generalized inverse of A

(3) $(AB)^T = AB$ (AB is symmetric)

(4) $(BA)^T = BA$ (BA is symmetric)

def ⇒ $\boxed{B=A^+ \quad \text{and} \quad A=B^+}$

## Theorem of the Penrose inverse

For each matrix $A \in \mathbb{R}^{m\times n}$, the pseudoinverse $A^+$ of A always exists and is unique.

MATLAB **pinv()** uses **SVD** to form the pseudoinverse $A^+$ of A:

the inverse matrix is not really computed

$$A = U\Sigma V^T = U\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V^T \Rightarrow \Sigma^+ = \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{A^+ = V\Sigma^+U^T}$$

# Pseudoinverse: example 1

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Is $\mathbf{A_p}$, the Moore-Penrose inverse of **A**, computed by means of SVD?

```
A=[1 2 0; 0 1 1];
```

```
[U,S,V]=svd(A);
```

or

```
[U,S,V]=svd(A,'econ');
```

or

```
[U,S,V]=svd(A,0);
Sp=[inv(S(:,1:2));zeros(1,2)]
Ap=V*Sp*U'
Ap =
     0.33333      -0.33333
     0.33333       0.16667
    -0.33333       0.83333
```

pseudo-inverse

```
[U,S,V]=svd(A,0); S
S =
      2.4495          0           0
      0               1           0
```

```
pinv(A)
ans =
      0.33333      -0.33333
      0.33333       0.16667
     -0.33333       0.83333
```

equal

```
[U,S,V]=svd(A);
Sp=[diag(diag(S(:,1:2)).^(-1)); zeros(1,2)]
Ap=V*Sp*U'
Ap =
     0.33333      -0.33333
     0.33333       0.16667
    -0.33333       0.83333
```

no inverse is computed

# Pseudoinverse: example 2

```
A=sym([1 2 3;4 5 6;7 8 9]);
rank(A)
ans =
      2
```

$$A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$$

**Ps** is the Moore-Penrose inverse of **A**

```
Ps=pinv(A);
all(all(isAlways(A*Ps*A == A)))
ans =
  logical      (1)        OK
   1
all(all(isAlways(Ps*A*Ps == Ps)))
ans =
  logical      (2)        OK
   1
all(all(isAlways((A*Ps)' == A*Ps)))
ans =
  logical      (3)        OK
   1
all(all(isAlways((Ps*A)' == Ps*A)))
ans =
  logical      (4)        OK
   1
```

uniqueness of **Ps**: **P1** is not the Moore-Penrose inverse of **A**

```
NA=null(A); syms a b c real
assumeAlso(a>0 & b>0 & c>0)
P1=Ps + NA*[a b c];                      ∀a,b,c
all(all(isAlways(A*P1*A == A)))
ans = logical               (1), (3)   OK
       1
eig(P1*A*P1-P1)'
ans =
[0, 0, 2*b - a - c]         (2)        NO
all(all(isAlways((P1*A)' == P1*A)))
ans = logical
       0                    (4)        NO
```

**G** is a generalized inverse of **A**

```
S=rref(A)... Ar=A(1:2,1:2);
G=[inv(Ar) zeros(r,n-r); zeros(m-r,n)]
G =
[-5/3,  2/3, 0]
[ 4/3, -1/3, 0]
[   0,    0, 0]
```

Is **G** the pseudoinverse of **A?**

```
all(all(isAlways(A*G*A == A)))
ans =
  logical      (1)        OK
   1
all(all(isAlways(G*A*G == G)))
ans =
  logical      (2)        OK
   1
all(all(isAlways((A*G)' == A*G)))
ans =
  logical      (3)        NO
   0
all(all(isAlways((G*A)' == G*A)))
ans =
  logical      (4)        NO
   0
```

# Other kinds of generalized inverses: left and right (one-sided) inverses

Let $A$ be a matrix (m×n) of rank r:

The $C$ matrix (n×m) is called a right inverse of $A$ $\overset{\text{def}}{\Longleftrightarrow}$ $AC = I_m$

The $B$ matrix (n×m) is called a left inverse of $A$ $\overset{\text{def}}{\Longleftrightarrow}$ $BA = I_n$

If $A$ has both a right inverse and a left inverse, then $A$ is square and invertible.

**for maximum rank matrices**

## Theor. (existence ⇔ surjectivity)

The system $Ax=b$ admits at least one solution $x$ for each b if, and only if, $\mathscr{R}(A)=\mathbb{R}^m$, that is if r=m. This is only possible when m≤n. In this case there exists a **right inverse** matrix $C$: $C = A^{\mathsf{T}}\left(AA^{\mathsf{T}}\right)^{-1}$

## Theor. (uniqueness ⇔ injectivity)

If the system $Ax=b$ is compatible, it admits at most one solution $x$ for each $b$ if, and only if, $\mathscr{R}(A^{\mathsf{T}})=\mathbb{R}^n$, that is if r=n. This is only possible if n≤m. In this case there is a single **left inverse** matrix $B$:

$$B = \left(A^{\mathsf{T}}A\right)^{-1}A^{\mathsf{T}}$$

When m=n=r, the two theorems ensure the existence and uniqueness of the inverse $A^{-1}$.

# Right inverses of a maximum rank matrix

$$A=\begin{pmatrix}\textcircled{1}&2&0\\0&\textcircled{1}&1\end{pmatrix}$$

rank($A$) = 2 = m < n $\Longrightarrow$ ∃ a right inverse $C$:

$$\boxed{A}\;\boxed{C}=\boxed{I_2}$$

2 linear systems

$$\left(A\quad I_2\right)=\left(\begin{array}{ccc|cc}1&2&0&1&0\\0&1&1&0&1\end{array}\right)$$

coefficients | constant terms

$\boxed{AC=I_2}$ $\Longleftrightarrow$ multiple underdetermined linear systems

## 2 underdetermined linear systems

$$\boxed{\begin{cases}x_1+2x_2 &= 1\\ \quad x_2+x_3 &= 0\end{cases}}\qquad\qquad \boxed{\begin{cases}x_1+2x_2 &= 0\\ \quad x_2+x_3 &= 1\end{cases}}$$

$$\begin{cases}2x_2 = 1-\textcircled{x_1}\\ x_3 = 0-x_2\end{cases}\quad\text{free variable}\qquad \begin{cases}2x_2 = 0-\textcircled{x_1}\\ x_3 = 1-x_2\end{cases}$$

$$\begin{cases}x_2 = \tfrac{1}{2}-\tfrac{1}{2}\textcircled{x_1}\\ x_3 = -\tfrac{1}{2}+\tfrac{1}{2}\textcircled{x_1}\end{cases}\qquad \begin{cases}x_2 = 0-\tfrac{1}{2}\textcircled{x_1}\\ x_3 = 1+\tfrac{1}{2}\textcircled{x_1}\end{cases}$$

particular solutions
$x_1=0$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\textcircled{0}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}+\alpha\boxed{\begin{pmatrix}2\\-1\\1\end{pmatrix}}\qquad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\textcircled{0}\\0\\1\end{pmatrix}+\beta\boxed{\begin{pmatrix}2\\-1\\1\end{pmatrix}}$$

general solutions

```
A=[1 2 0; 0 1 1];
[m,n]=size(A);
format rational
xp=A\eye(2)
xp =
        0            0
      1/2            0
     -1/2            1
xn=null(A,"rational")
xn =
        2
       -1
        1
syms a b real
C=xp + xn*[a b]
C =
[     2*a,    2*b]
[1/2 - a,     -b]
[a - 1/2, b + 1]
disp(A*C)
[1, 0]
[0, 1]
```

**xp**: particular solutions

xn = solution of homogeneous system

$\mathcal{N}(A)$=span$\{(2,-1,1)^T\}$

**C**: general right inverse

```
disp(A*xp)
          1        0
-1.1102e-16        1
```

**xp**: particular right inverse

# How are the pseudoinverses of full-rank matrices?

$$A=\begin{pmatrix}1&2&0\\0&1&1\end{pmatrix}\qquad AX_p=I_2\qquad X_p=\begin{pmatrix}0&0\\ \frac{1}{2}&0\\ -\frac{1}{2}&1\end{pmatrix}\qquad AC=I_2\qquad C=\left(\begin{pmatrix}0\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\1\end{pmatrix}\ \begin{pmatrix}0\\0\\1\end{pmatrix}+\beta\begin{pmatrix}2\\-1\\1\end{pmatrix}\right)$$

$X_p$: particular right inverse

$C$: general right inverse

**Exercise:** Verify that $X_p$ is not the pseudoinverse of $A$.

Decompose columns of $X_p$ along $\boxed{\mathscr{R}(A^T)}$ and along $\boxed{\mathscr{N}(A)}$; mantain only those in $\mathscr{R}(A^T)$

$\mathscr{R}(A^T)\oplus\mathscr{N}(A)$

$$X_p=\boxed{X_r}+X_n$$

$$\begin{pmatrix}\boxed{\begin{matrix}1&0\\2&1\\0&1\end{matrix}}&\boxed{\begin{matrix}2\\-1\\1\end{matrix}}\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix}=\boxed{\begin{pmatrix}0&0\\ \frac{1}{2}&0\\ -\frac{1}{2}&1\end{pmatrix}}$$

```
Xp=sym(xp);
NA=sym(xn);
coef=[A' NA]\Xp
coef =
[ 1/3, -1/3]
[-1/3,  5/6]
[ 1/3, -5/6]
Xr=A'*coef(1:2,:)
Xr =
[ 1/3, -1/3]
[ 1/3,  1/6]
[-1/3,  5/6]
A'*inv(A*A')
ans =
[ 1/3, -1/3]
[ 1/3,  1/6]
[-1/3,  5/6]
A'/(A*A')
ans =
[ 1/3, -1/3]
[ 1/3,  1/6]
[-1/3,  5/6]
```

or equivalently by orthogonal projection onto $\mathscr{R}(A^T)$

```
RAT=orth(A');
Po=RAT*RAT';
Xr=Po*Xp
Xr =
[ 1/3, -1/3]
[ 1/3,  1/6]
[-1/3,  5/6]
```

ortho-normal basis

If r=m<n a **right pseudoinverse** $C$ is:

$$C=A^T\left(AA^T\right)^{-1}$$

since $AA^T$ is invertible.

If r=n<m a **left pseudoinverse** $B$ is:

$$B=\left(A^TA\right)^{-1}A^T$$

since $A^TA$ is invertible.

```
all(all(isAlways(A*Xr*A == A)))
ans =  logical
   1
all(all(isAlways(Xr*A*Xr == Xr)))
ans =  logical
   1
all(all(isAlways((A*Xr)' == A*Xr)))
ans =  logical
   1
all(all(isAlways((Xr*A)' == Xr*A)))
ans =  logical
   1
```

(1) OK

(2) OK

(3) OK

(4) OK

**Xr** is the right pseudoinverse of **A**

# Left inverses of a maximum rank matrix

$$A = \begin{pmatrix} \textcircled{1} & 1 \\ 2 & \textcircled{0} \\ 0 & 1 \end{pmatrix}$$

rank = 2 = n < m ⟹ ∃ left inverse matrix $B$: $\boxed{B}\,\boxed{A} = \boxed{\mathbf{I}_3}$

$$BA = I_2 \iff (BA)^{\mathsf{T}} = A^{\mathsf{T}}B^{\mathsf{T}} = I_2 \quad \boxed{A^{\mathsf{T}}}\,\boxed{B^{\mathsf{T}}} = \boxed{\mathbf{I}_2}$$

$\iff$ $\boxed{A^{\mathsf{T}}B^{\mathsf{T}} = I_2}$ underdetermined multiple system

2 linear systems

$$\begin{pmatrix} A^{\mathsf{T}} & I_2 \end{pmatrix} = \left(\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

coefficients

known terms

## Exercise

Apply to $A^{\mathsf{T}}$ the algorithm of the previous example to compute a particular matrix $X_p$ and the general form of left inverses of the matrix $A$ (3×2).

Compute also the left pseudoinverse of $A$ by the orthogonal projection of $X_p$ onto $\mathscr{R}(A^{\mathsf{T}})$, and compare it with $B$:

if r=n<m a **left pseudoinverse** $B$ is:

$$B = \left(A^{\mathsf{T}}A\right)^{-1}A^{\mathsf{T}}$$

since $A^{\mathsf{T}}A$ is invertible.

# Underdetermined linear systems

A system $Ax=b$ of linear equations is considered **under-determined** if there are fewer equations than unknowns, in contrast to an **overdetermined system**, where there are more equations than unknowns.

An **underdetermined linear system** has either **no solution** or **infinitely many solutions**.

**Examples:** 3 unknowns, 2 equations

$$\begin{cases} x+y+z=1 \\ x+y+z=0 \end{cases} \Rightarrow$$

**no solution**

$$\begin{cases} x+y+z\phantom{2} = 1 \\ x+y+2z=3 \end{cases} \Rightarrow$$

**infinitely many solutions**

For **system compatibility** check the Rouché–Capelli Theorem:

coefficient matrix

$$\text{rank}(\boxed{A}) = \text{rank}(\boxed{[A\ b]})$$

augmented matrix

# Least-norm solution of underdetermined linear systems

$$Mx = y, \quad M(r \times c)$$

$r$: num of rows
$c$: num of cols

$$\mathscr{R}(M^\top) \oplus \mathscr{N}(M) = \mathbb{R}^c \qquad \mathscr{R}(M^\top) = \mathscr{N}(M)^\perp \qquad \mathscr{N}(M) = \mathscr{R}(M^\top)^\perp$$

The general solution of a compatible underdetermined system $Mx=y$ is given by:

$$X = \{x : Mx = y\} = \{x = x_p + z : Mx_p = y \ \wedge \ z \in \mathscr{N}(M)\}$$

$x_p$: any particular solution

Null Space of $M$

**Theorem**

it is unique

Least-norm solution: $\|x_{LN}\|_2 = \min\limits_{x:Mx=y} \|x\|_2 \Leftrightarrow \boxed{x_{LN}} \in \mathscr{R}(M^\top) \cap X$

If rank$(M) = c < r$ (max rank): ⇒ $\mathscr{N}(M) = \{\underline{0}\}$ ⇒ $\exists! x = x_{LN} : Mx = y$

determined system $\qquad x_{LN} = \left[\left(M^\top M\right)^{-1} M^\top\right] y \qquad$ left pseudoinverse of $M$

If rank$(M) = r < c$ (max rank): ⇒ $\dim[\mathscr{N}(M)] = c - r > 0$

underdetermined system $\qquad x_{LN} = \left[M^\top \left(MM^\top\right)^{-1}\right] y \qquad$ right pseudoinverse of $M$

If rank$(M) = k < \min\{r,c\}$ (non-max rank): ⇒ underdetermined system

$x_{LN}$ is the orthogonal projection of $x_p$ onto $\mathscr{R}(M^\top) = \mathscr{N}(M)^\perp$

# Examples: solve underdetermined systems

$$X=\{x:Mx=y\}=\{x=x_p+z:Mx_p=y\ \wedge\ z\in\mathcal{N}(M)\}$$

general solution

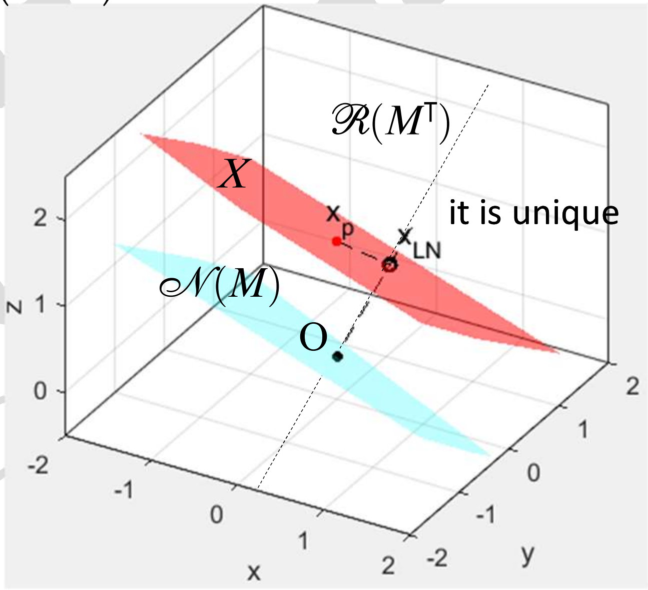
least-norm solution $\|x_{LN}\|_2=\min\limits_{x:Mx=y}\|x\|_2 \Leftrightarrow x_{LN}\in\mathcal{R}(M^{\mathsf{T}})$

$$M=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$$

$$y=\begin{pmatrix}4\\4\end{pmatrix}$$

rank($M$)=1
< r=2 < c=3

```
M=[1 1;2 2;3 3]'; y=[4;4];
xp=M\y;     % particular solution
N=null(M)  % basis for the Null Space
syms a b real; n=N*[a;b]; X=xp + n;
RMT=orth(M'); % orthonormal basis of ℛ(M^T)
P=RMT*RMT'; % orthogonal projection matrix
Pxp=P*xp;     % projection of xp onto ℛ(M^T)
xLN=pinv(M)*y;
disp([norm(xLN) norm(Pxp) norm(xp)])
     1.069    =    1.069    <    1.3333
```

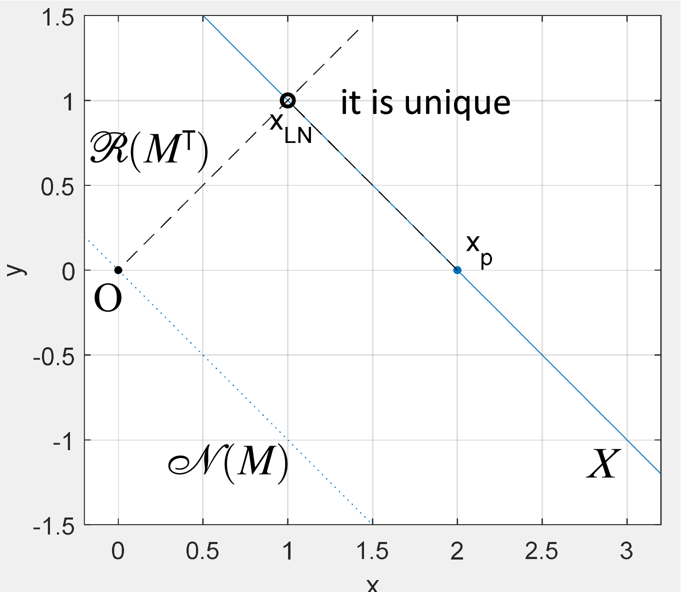


$$M=\begin{pmatrix}1&1\\2&2\\3&3\end{pmatrix}$$

$$y=\begin{pmatrix}2\\4\\6\end{pmatrix}$$

rank($M$)=1
< c=2 < r=3

```
M=[1 1;2 2;3 3]; y=[2;4;6];
xp=M\y;     % particular solution
N=null(M)  % basis for the Null Space
syms a real; n=N*a; X=xp + n;
RMT=orth(M'); % orthonormal basis of ℛ(M^T)
P=RMT*RMT'; % orthogonal projection matrix
Pxp=P*xp;     % projection of xp onto ℛ(M^T)
xLN=pinv(M)*y;
disp([norm(xLN) norm(Pxp) norm(xp)])
     1.4142    =    1.4142    <      2
```



# Examples: solve underdetermined systems

$$X=\{x:Mx=y\}=\{x=x_p+z:Mx_p=y\ \wedge\ z\in\mathcal{N}(M)\}$$

general solution

$$M=\begin{pmatrix}1&4&7\\2&3&9\\2&2&8\end{pmatrix}\quad y=\begin{pmatrix}6\\7\\6\end{pmatrix}$$

rank($M$)=2 < r=c=3

M is a square and singular matrix

Least Norm solution $\|x_{LN}\|_2=\min\limits_{x:Mx=y}\|x\|_2 \Leftrightarrow x_{LN}\in\mathcal{R}(M^\top)$

```
M=[1 4 7;2 3 9;2 2 8]; y=[6;7;6];
xp=M\y     % particular solution
Warning: Matrix is singular to working precision.
xp =
   NaN
   NaN
   NaN
```

???

```
[m,n]=size(M)
m =
     3
n =
     3
rank(M)
ans =
     2
```

```
rank([M y])
ans =
     2
```

compatible and under-determined system

to solve the system the factorization P*M=L*U is used

```
[L,U,P]=lu(M)
L =
           1           0           0
         0.5           1           0
           1        -0.4           1
U =
           2           3           9
           0         2.5         2.5
           0           0           0
P =
     0     1     0
     1     0     0
     0     0     1
```

1) solve L*w=P*y

2) solve U(1:r)*xp=w(1:r)

```
w=L\(P*y)
w =
         7
       2.5
         0
```

1)

```
xp=U(1:r,:)\w(1:r)
xp =
         0
   0.33333
   0.66667
```

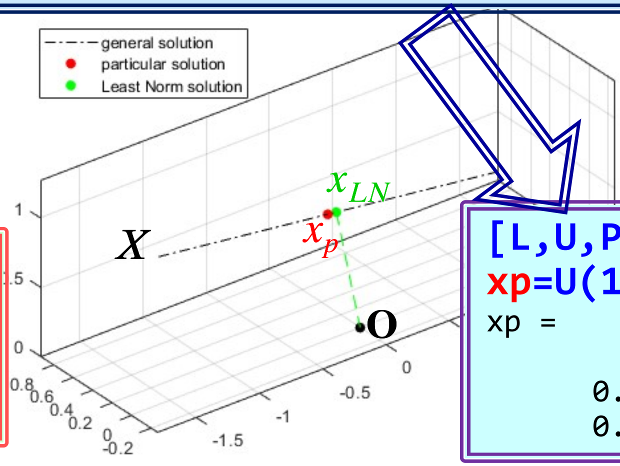
particular solution

2)

general solution of the system

```
N=null(A)
N =
    -0.90453
    -0.30151
     0.30151
syms a [n-r 1] real
Xg=xp + N*a
Xg =
  -(3*11^(1/2)*a1)/11
1/3 - (11^(1/2)*a1)/11
(11^(1/2)*a1)/11 + 2/3
```

general solution



```
[L,U,P]=lu([M y]);
xp=U(1:r,1:n)\U(1:r,n+1)
xp =
         0
   0.33333
   0.66667
```

1) + 2)

Least Norm solution

```
RMT=orth(M'); P=RMT*RMT'; Pxp=P*xp;
xLN=pinv(M)*y;
disp([norm(xLN) norm(Pxp) norm(xp)])
     0.73855   =   0.73855   <   0.74536
```