UNIVERSITÀ DEGLI STUDI DI NAPOLI
PARTHENOPE

Artificial Intelligence

Knowledge Representation and Inference

LESSON 12

prof. Antonino Staiano

M.Sc. In ''Machine Learning e Big Data'' - University Parthenope of Naples

Propositional Logic: Inference Rules

Inference: General concepts

- Two sentences α and β are logically equivalent ($\alpha \equiv \beta$), if they are *true* under the same models, i.e., *if and only if*
  - $\alpha \vDash \beta$ and $\beta \vDash \alpha$
  - For instance $(P \land Q) \equiv (Q \land P)$
- A sentence is valid if it is *true* in all models
  - It is also called a tautology
    - $P \lor \neg P$
- A sentence is satisfiable if it is *true* only in some model
  - $P \land Q$

Inference: General concepts

- Two useful properties related to the above concepts
  - Deduction theorem
    - For any α and β, $\alpha \vDash \beta$ if and only if $\alpha \Rightarrow \beta$ is valid
      - for instance, given a set KB of premises and a possible conclusion model-checking inference algorithm works by checking whether $KB \Rightarrow \alpha$ is valid

  - satisfiability is related to the standard mathematical proof technique of *reductio ad absurdum* (proof by refutation or by contradiction):

$$\alpha \vDash \beta \text{ if and only if } (\alpha \wedge \neg \beta) \text{ is unsatisfiable}$$

Inference Rules

- Practical inference algorithms are based on inference rules to avoid the exponential computational complexity of model checking
- An inference rule represents a standard pattern of inference:
  - it implements a simple reasoning step whose soundness can be easily proven and applied to a set of premises with a specific structure to derive a conclusion
- Inference rules are represented as follows:

$$\frac{\text{premises}}{\text{conclusion}}$$

Inference Rules

- Modus Ponens

If it is raining, then Harry is inside
It is raining

Harry is inside

Modus Ponens

$$\alpha \rightarrow \beta$$

$$\alpha$$

$$\overline{}$$

$$\beta$$

Inference Rules

- And Elimination

Harry is friends with Ron and Hermione

Harry is friends with Hermione

And Elimination

$$\frac{\alpha \wedge \beta}{\alpha}$$

Inference Rules

- Double Negation Elimination

It is not true that Harry did not pass the test

Harry passed the test

Double Negation Elimination

$$\frac{\neg(\neg\alpha)}{\alpha}$$

Inference Rules

- Implication Elimination

If it is raining, then Harry is inside

It is not raining or Harry is inside

Implication Elimination

$$\frac{\alpha \rightarrow \beta}{\neg\alpha \vee \beta}$$

Inference Rules

- Biconditional elimination

It is raining if and only if Harry is inside

If it is raining, then Harry is inside, and if Harry is inside, then it is raining

Biconditional Elimination

$$\frac{\alpha \leftrightarrow \beta}{(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)}$$

De Morgan’s Law

It is not true that both Harry and Ron passed the test

Harry did not pass the test or Ron did not pass the test

De Morgan’s Law

$$\frac{\neg(\alpha \wedge \beta)}{\neg\alpha \vee \neg\beta}$$

De Morgan's Law

It is not true that Harry or Ron passed the test

Harry did not pass the test and Ron did not pass the test

De Morgan's Law

$$\frac{\neg(\alpha \vee \beta)}{\neg\alpha \wedge \neg\beta}$$

Distributive Property

$$\frac{(\alpha \wedge (\beta \vee \gamma))}{(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}$$

Distributive Property

$$\frac{(\alpha \lor (\beta \land \gamma))}{(\alpha \lor \beta) \land (\alpha \lor \gamma)}$$

Search Problems

- Initial state
- Actions
- Transition model
- Goal test
- Path cost function

Theorem Proving

- Initial state: starting knowledge base
- Actions: inference rules
- Transition model: new knowledge base after inference
- Goal test: check statement we're trying to prove
- Path cost function: number of steps in proof

Proof by Resolution

- What about the completeness of our inference algorithm?
  - If the search algorithm that uses the inference rule is complete and the rules are adequate the inference algorithm is complete
  - However, if the inference rule is not adequate, for instance, the goal is unreachable
  - Therefore, we turn on a single inference rule, the resolution, that yields a complete inference algorithm when coupled with any complete search algorithm

Resolution

- Resolution is based on another inference rule that let us prove anything that can be proven about a KB

(Ron is in the Great Hall) v (Hermione is in the library)

Ron is not in the Great Hall

Hermione is in the library

Resolution: Unit Resolution Rule

$$\frac{P \vee Q \qquad \neg P}{Q}$$

Resolution

$$\frac{P \lor Q_1 \lor Q_2 \lor \ldots \lor Q_n \qquad \neg P}{Q_1 \lor Q_2 \lor \ldots \lor Q_n}$$

Resolution

(Ron is in the Great Hall) v (Hermione is in the library)

(Ron is not in the Great Hall) v (Harry is sleeping)

(Hermione is in the library) v (Harry is sleeping)

Resolution

$$\frac{P \vee Q \qquad \neg P \vee R}{Q \vee R}$$

Resolution

$$\frac{P \vee Q_1 \vee Q_2 \vee \ldots \vee Q_n \qquad \neg P \vee R_1 \vee R_2 \vee \ldots \vee R_m}{Q_1 \vee Q_2 \vee \ldots \vee Q_n \vee R_1 \vee R_2 \vee \ldots \vee R_m}$$

Clause

- A disjunction of literals
  - e.g. P v Q v R
- Disjunction means literals connected with *or*
- Conjunction means literals connected with *and*
- Literal is either a propositional symbol or the opposite of a propositional symbol
- Any logical sentence can be turned into a conjunctive normal form

Conjunctive Normal Form

- Logical sentence that is a conjunction of clauses

$$(A \lor B \lor C) \land (D \lor \neg E) \land (F \lor G)$$

Conversion to CNF

- Eliminate biconditionals
  - turn $(\alpha \leftrightarrow \beta)$ into $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$
- Eliminate implications
  - turn $(\alpha \rightarrow \beta)$ into $\neg\alpha \lor \beta$
- Move ¬ inwards using De Morgan's Laws
  - e.g. turn $\neg(\alpha \land \beta)$ into $\neg\alpha \lor \neg\beta$
- Use distributive law to distribute ∨ wherever possible

Conversion to CNF

$(P \vee Q) \rightarrow R$	
$\neg(P \vee Q) \vee R$	eliminate implication
$(\neg P \wedge \neg Q) \vee R$	De Morgan's Law
$(\neg P \vee R) \wedge (\neg Q \vee R)$	distributive law

- Converting into a CNF is useful in order to apply the resolution
  - Inference by resolution

Inference by Resolution

$$\frac{\begin{array}{c} P \vee Q \\ \neg P \vee R \end{array}}{(Q \vee R)}$$

Inference by Resolution

$$P \lor Q \lor S$$

$$\neg P \lor R \lor S$$

$$(Q \lor S \lor R \lor S)$$

Factoring -> eliminates all redundant variables

Inference by Resolution

$$P$$

$$\neg P$$

$$(\,)$$

Empty clause

- The empty clause is always false
- This is the base of the inference by resolution algorithm

Inference by Resolution

- To determine if KB $\models \alpha$:
  - Check if (KB $\wedge \neg\alpha$) is a contradiction?
    - If so, then KB $\models \alpha$
    - Otherwise, no entailment
- In practice
  - Convert if (KB $\wedge \neg\alpha$) to Conjunctive Normal Form
  - Keep checking to see if we can use the resolution to produce a new clause
    - If ever we produce the empty clause (equivalent to False), we have a contradiction, and KB $\models \alpha$
    - Otherwise, if we can't add new clauses, no entailment

Inference by Resolution

Does $(A \lor B) \land (\neg B \lor C) \land (\neg C)$ entail A?

$$(A \lor B) \land (\neg B \lor C) \land (\neg C) \land (\neg A)$$

$(A \lor B)$ $(\neg B \lor C)$ $(\neg C)$ $(\neg A)$

Inference by Resolution

Does $(A \lor B) \land (\neg B \lor C) \land (\neg C)$ entail A?

$$(A \lor B) \land (\neg B \lor C) \land (\neg C) \land (\neg A)$$

$(A \lor B)$ $\quad$ $\underline{(\neg B \lor C)}$ $\quad$ $\underline{(\neg C)}$ $\quad$ $(\neg A)$

Inference by Resolution

Does $(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C)$ entail A?

$$(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C) \wedge (\neg A)$$

$(A \vee B)$ $\quad$ $\underline{(\neg B \vee C)}$ $\quad$ $\underline{(\neg C)}$ $\quad$ $(\neg A)$ $\quad$ $(\neg B)$

Inference by Resolution

Does $(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C)$ entail A?

$$(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C) \wedge (\neg A)$$

$(A \vee B)$ $(\neg B \vee C)$ $(\neg C)$ $(\neg A)$ $(\neg B)$

Inference by Resolution

Does $(A \lor B) \land (\neg B \lor C) \land (\neg C)$ entail A?

$$(A \lor B) \land (\neg B \lor C) \land (\neg C) \land (\neg A)$$

$\underline{(A \lor B)}$ $(\neg B \lor C)$ $(\neg C)$ $(\neg A)$ $\underline{(\neg B)}$

Inference by Resolution

Does $(A \lor B) \land (\neg B \lor C) \land (\neg C)$ entail A?

$$(A \lor B) \land (\neg B \lor C) \land (\neg C) \land (\neg A)$$

$(A \lor B)$ $(\neg B \lor C)$ $(\neg C)$ $(\neg A)$ $(\neg B)$ (A)

Inference by Resolution

Does $(A \lor B) \land (\neg B \lor C) \land (\neg C)$ entail A?

$$(A \lor B) \land (\neg B \lor C) \land (\neg C) \land (\neg A)$$

$(A \lor B)$ $(\neg B \lor C)$ $(\neg C)$ $(\neg A)$ $(\neg B)$ (A)

Inference by Resolution

Does $(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C)$ entail A?

$$(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C) \wedge (\neg A)$$

$(A \vee B)$ $(\neg B \vee C)$ $(\neg C)$ $\underline{(\neg A)}$ $(\neg B)$ $\underline{(A)}$

Inference by Resolution

Does $(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C)$ entail A?

$$(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C) \wedge (\neg A)$$

$(A \vee B)$ $(\neg B \vee C)$ $(\neg C)$ $\underline{(\neg A)}$ $(\neg B)$ $\underline{(A)}$ $(\,)$

Inference by Resolution

Does $(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C)$ entail A?

$$(A \vee B) \wedge (\neg B \vee C) \wedge (\neg C) \wedge (\neg A)$$

$(A \vee B)$ $(\neg B \vee C)$ $(\neg C)$ $(\neg A)$ $(\neg B)$ (A) $(\,)$

Limitations of propositional logic

- Main problems
  - Limited expressive power
    - Inferences involving the structure of atomic sentences (e.g., All men are mortal, ...) cannot be made
  - Lack of conciseness
    - Even small KBs (in natural language) require many propositional symbols and sentences

From Propositional to Predicate Logic

- The description of many domains of interest for real-world applications (e.g., mathematics, philosophy, AI) involves the following elements in natural language:
  - nouns denoting objects (or persons), e.g.: Wumpus and pits; Socrates and Plato; the numbers one, two, etc.
  - predicates denoting properties of individual objects and relations between them, e.g.: Socrates is a man, five is prime, four is lower than five; the sum of two and two equals four
  - some relations between objects can be represented as functions, e.g.: "father of", "two plus two"
  - facts involving some or all objects, e.g.: all squares neighboring the Wumpus are smelly; some numbers are prime
- These elements cannot be represented in propositional logic, and require the more expressive predicate logic
  - The predicate logic version of the Resolution algorithm is used in automatic theorem provers, to assist mathematicians to develop complex proofs