UNIVERSITÀ PARTHENOPE
DiST
DIPARTIMENTO DI SCIENZE E TECNOLOGIE

Machine Learning (part II)

Hopfield Neural Network

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Introduction

- Hopfield Neural Network
  - 1982 - John Hopfield
  - based on concept of statistical mechanics
    - Energy function
- Extension
    - Boltzmann Machine

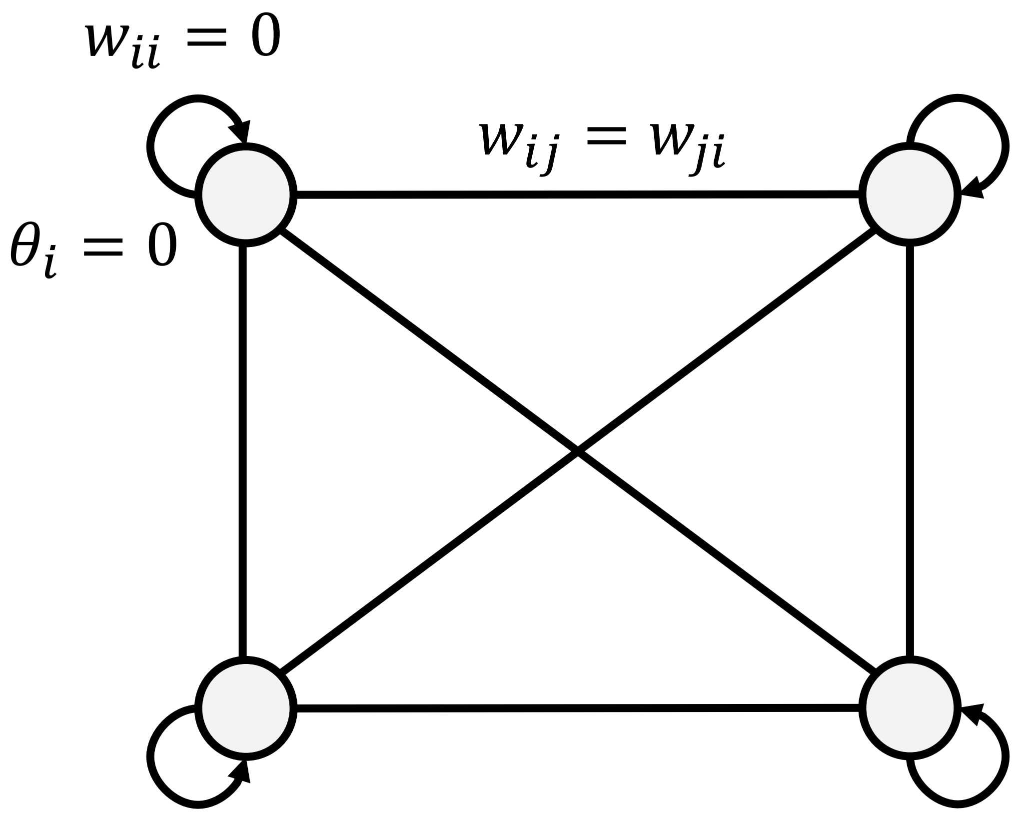
Manuale sulle reti neurali, D. Floreano, C. Mattiussi, Il Mulino, 2002

Learning

- Weights adaption
  - Activations change based on the outputs of other units
  - Finite states sequence up to the stability point
  - The point of stability is the state of equilibrium

- Aim
  - storage and recognition
  - learn a series of patterns
  - after the training phase the network is able to recover an original pattern from one incomplete or corrupted by noise

Hopfield NN



N bipolar units

$$A_i = \sum_{j=1}^{N} w_{ij}\, x_j \qquad \Phi(A_i) = \begin{cases} 1 \; if \; A_i > \theta_i \\ -1 \; otherwise \end{cases}$$

Hebb's rule

- Each input unit performs input and output functions

- Pattern di input

$$\mathrm{p} = [p_1, \dots, p_N]$$

$$x_i = p_i$$

- Hebb's rule

1/N Constant of proportionality

$$\Delta w_{ij}^{\mu} = x_i^{\mu}\ x_j^{\mu}$$

$$w_{ij} = \frac{1}{N}\sum_{\mu=1}^{M} x_i^{\mu}\ x_j^{\mu}$$

For M patterns

Moemorization of patterns

- Stability of a pattern

$$\Phi(A_i^*) = p_i^* \qquad \forall i$$

$$A_i^* = \sum_j w_{ij} x_j^* = \frac{1}{N} \sum_j \sum_\mu x_i^\mu x_j^\mu x_j^*$$

- Without pattern p*

$$A_i^* = x_i^* + \frac{1}{N} \sum_j \sum_{\mu \neq *} x_i^\mu x_j^\mu x_j^*$$

Interference (crosstalk)

Net capability

- NN storage capability

$$\frac{M}{N} \quad \begin{matrix} \textit{patterns} \\ \textit{nodes} \end{matrix}$$

- Storage quantity

$$C_i^* = -\frac{1}{N} x_i^* \sum_j \sum_{\mu \neq *} x_i^\mu x_j^\mu x_j^*$$

It it is *negative* the crosstalk term has the same sign as the desired pattern x_i^*.

If it is *positive* and large then 1, it is change the sign of x_i^* and bit *i* of pattern * is unstable.

Net capability

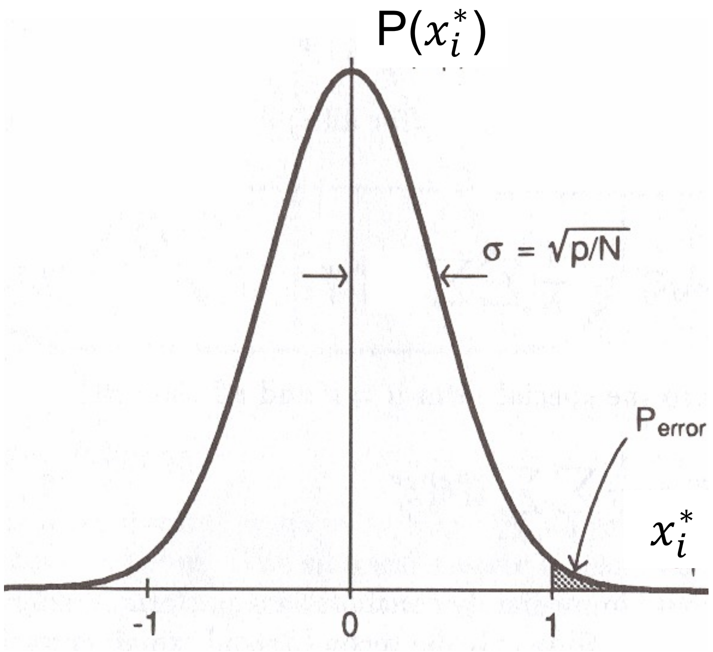
- For random patterns

x_i^* behaves as a binomial distribution

$$P_{error} = \text{Prob}(\, x_i^* > 1)$$

- Binomial distribution

$$\mu = 0 \qquad \sigma^2 = \frac{M}{N}$$



For large $M * N$ approximation (Gaussian)

Net capability

- Gaussian distribution

$$P_{error} = \frac{1}{\sqrt{2\pi\sigma}} \int_1^{\infty} e^{-\frac{x^2}{2\sigma^2}}\, dx =$$

$$\frac{1}{2}\left[1 - erf\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right] = \frac{1}{2}\left[1 - erf\left(\sqrt{\frac{N}{2M}}\right)\right]$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\, du$$

ML – Hopfield NN

Net capability

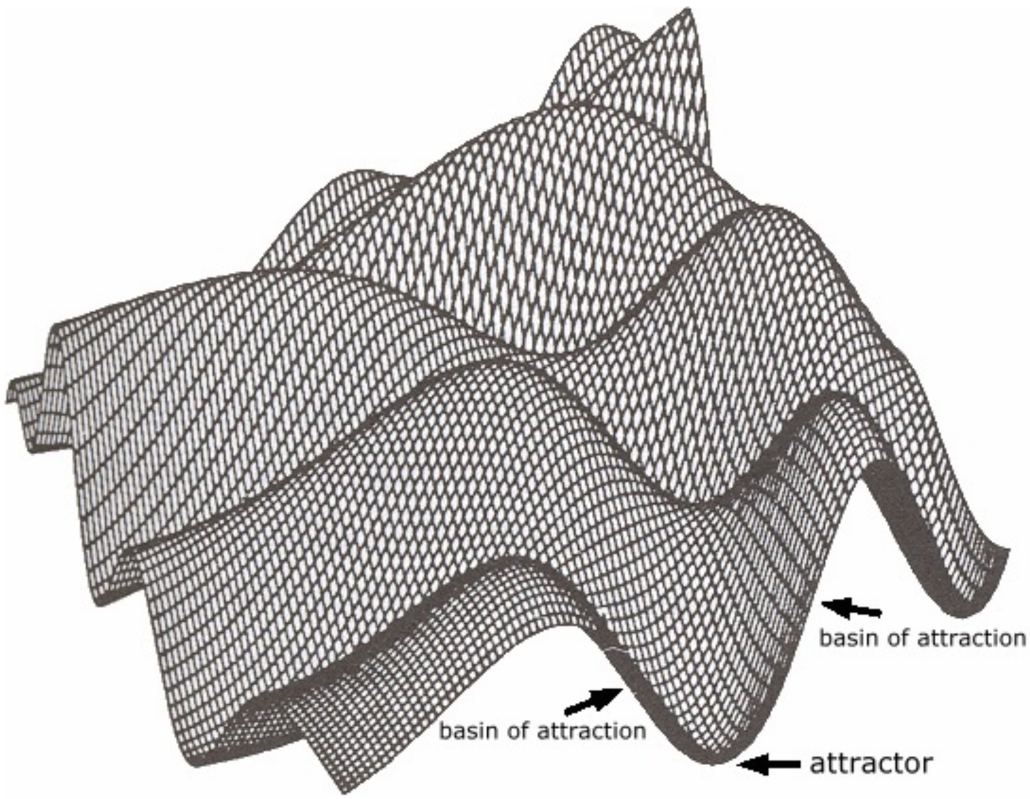
- Error and limitation

P_{error}	M / N
0.001	0.105
0.0036	0.138
0.01	0.185
0.05	0.37
0.1	0.61

Small percentage of error

- For linearment independent patterns
  - Pseudoinversion method
  - Storage capability N-1 patterns

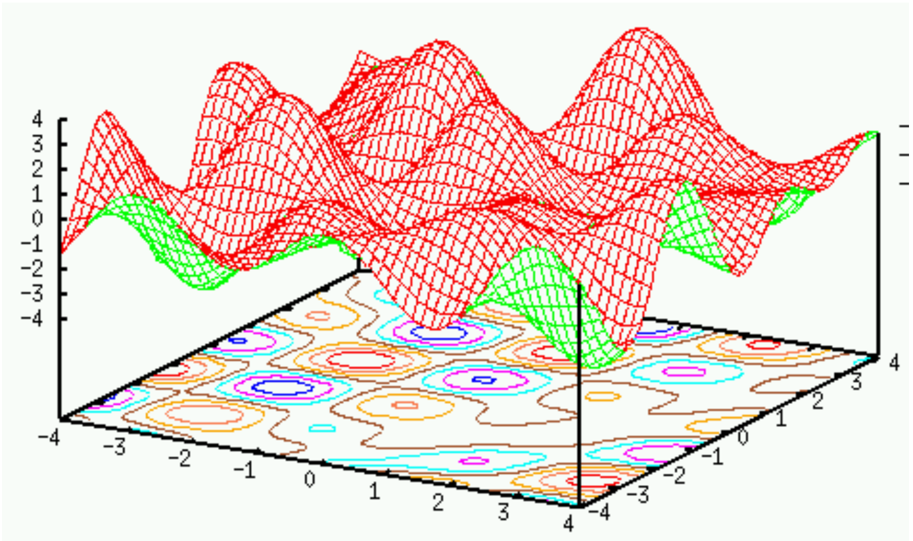
Energy function

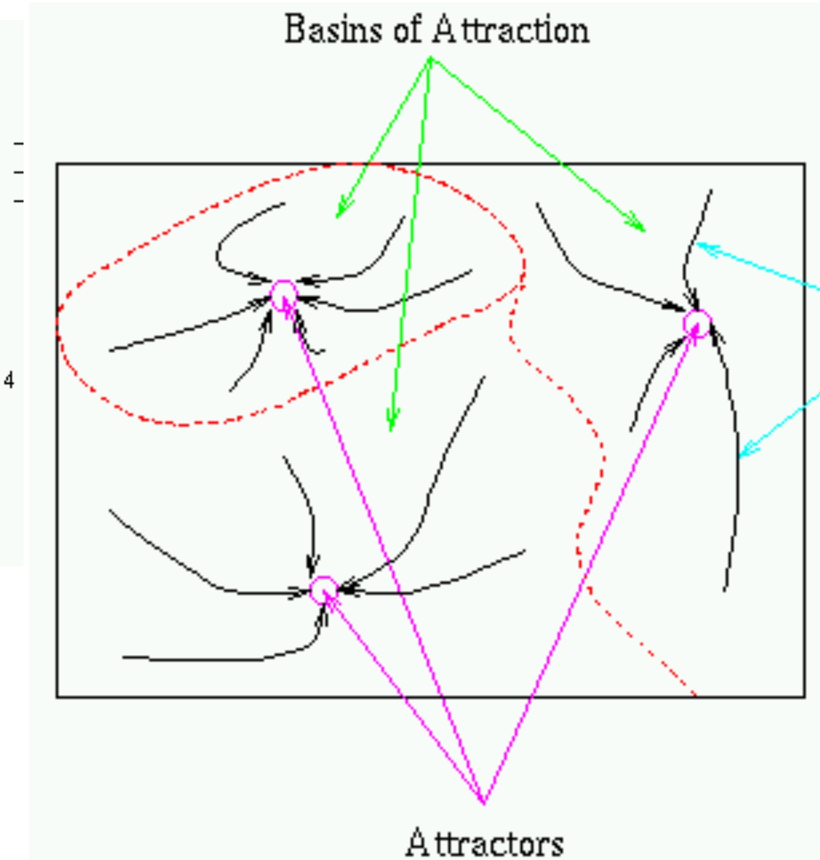


$$H = -\frac{1}{2}\sum_i \sum_j w_{ij} x_i x_j + \sum_i x_i \theta_i$$

Energy function

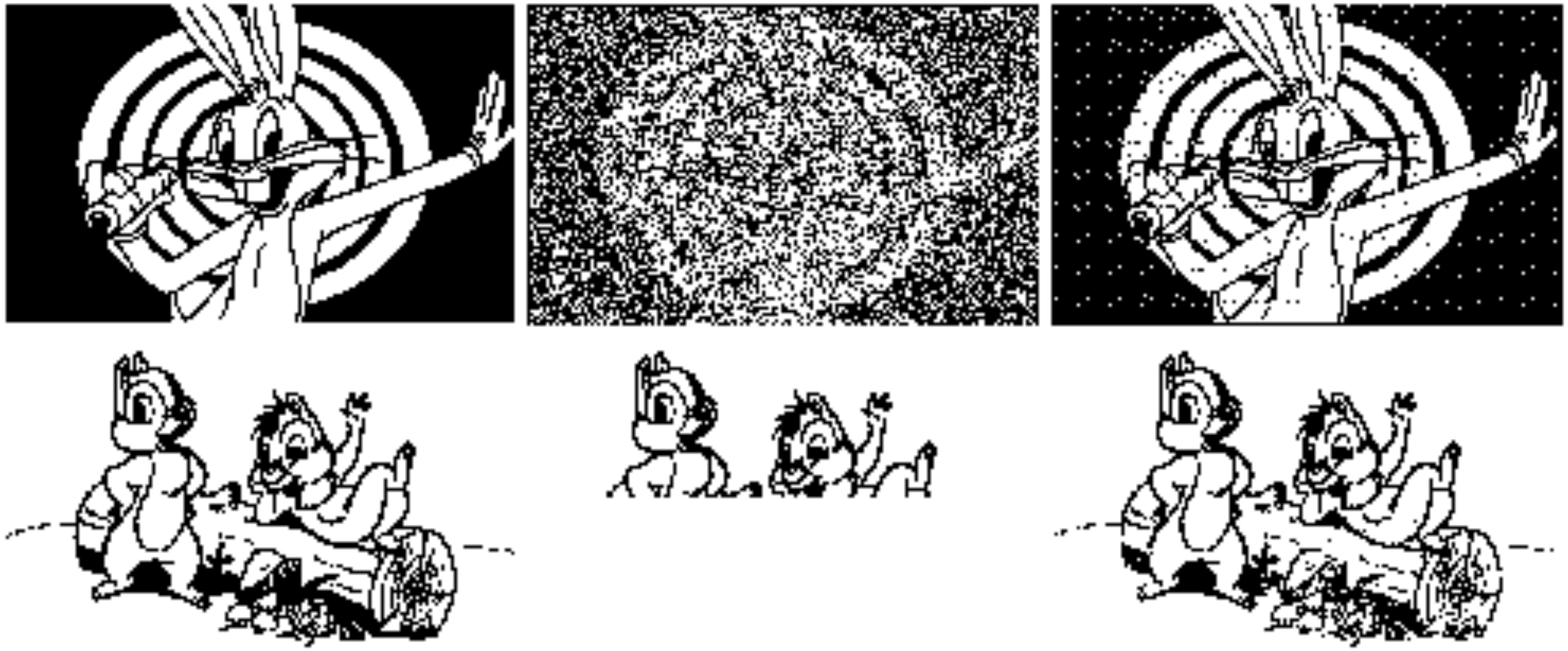
Energy function





Energy function

Storage examples



Images reconstruction examples

Storage and energy reduction

- For random patterns

$$H = -\frac{1}{2}\sum_{i}\sum_{j\neq i} w_{ij}x_i x_j$$

- New pattern

$$H = \left[-\frac{1}{2}\sum_{i}\sum_{j\neq i} w_{ij}^{\mu\neq *} x_i x_j\right] + \left[-\frac{1}{2}\sum_{i}\sum_{j\neq i} w_{ij}^{*} x_i^{*} x_j^{*}\right]$$

Storage and energy reduction

- Optimum of the second term

Positive values

$$\frac{1}{2}\sum_{i}\sum_{j\neq i} w_{ij}^{*}x_i^{*}x_j^{*} = \frac{1}{2}\sum_{i}\sum_{j\neq i} x_i^2 x_j^2$$

- Hebbian learning rule

$$w_{ij}^{*} = x_i^{*}x_j^{*}$$

$$w_{ij} = \sum_{\mu} x_i^{\mu}x_j^{\mu}$$

Pattern recovering and energy reduction

- *k*-th unit

$$H = -\frac{1}{2}\sum_{i\neq k}\sum_{j\neq k} w_{ij}\, x_i x_j - \frac{1}{2} x_k \sum_i w_{ik} x_i - \frac{1}{2} x_k \sum_j w_{kj} x_j$$

- Reduction of the energy

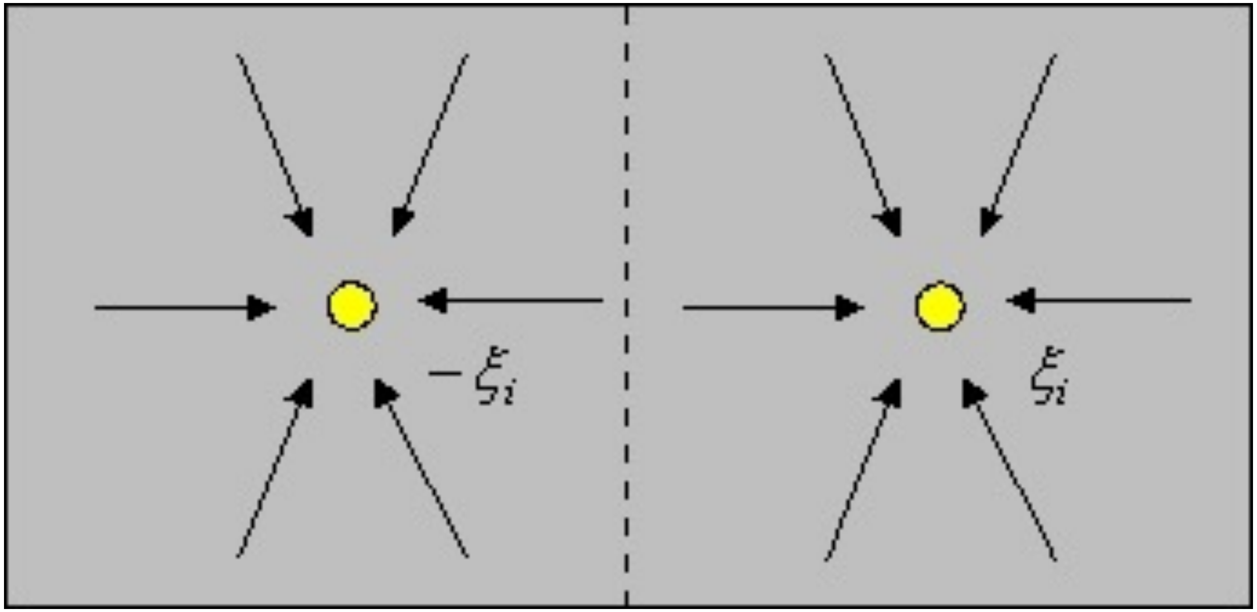
$$\Delta H = -\frac{1}{2}\left[\Delta x_k \sum_i w_{ik} x_i + \Delta x_k \sum_j w_{kj} x_j\right]$$

- For symmetry (Hebbian learning)

$$\Delta H = -\left[\Delta x_k \sum_j w_{kj} x_j\right]$$

$$\Phi(A_i) = \begin{cases} 1 \; if \; A_i > \theta_i \\ -1 \; otherwise \end{cases}$$

Attractors



Configuration space is symmetrically divided into two basins of attraction

Spurious states

- Hebb
  - dynamical system which has attractors (the minima of the energy function)
  - desired patterns which have been stored and are called retrieval states
- Other attractors
  - reversed states
  - mixture states
  - spin glass states

Stochastic units

- Biased random decisions
  - Replace the binary threshold units by binary stochastic units
  - The "temperature" controls the amount of noise
  - Decreasing all the energy gaps between configurations is equivalent to raising the noise level
  - Simulated annealing

$$P(x_i = 1) = \frac{1}{1 + e^{-\beta\left(\sum_j w_{ij} x_j\right)}}$$

$$\beta = \frac{1}{kT}$$

k is the Boltzamnn constant

Stochastic units

- Probability of the states
  - Boltzmann distribution

- Probability of the state

$$P_1 = k\, e^{-\frac{H_1}{T}}$$

$$\frac{P_1}{P_2} = \frac{k\, e^{-\frac{H_1}{T}}}{k\, e^{-\frac{H_2}{T}}} = e^{-\frac{(H_1 - H2)}{T}}$$

$$H_1 < H_2$$

$$\frac{P_1}{P_2} = e^{-\frac{(H_1 - H2)}{T}} > 1$$

$$P_1 > P_2$$