11. Modelling probabilities

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Data Analysis 2: Regression analysis

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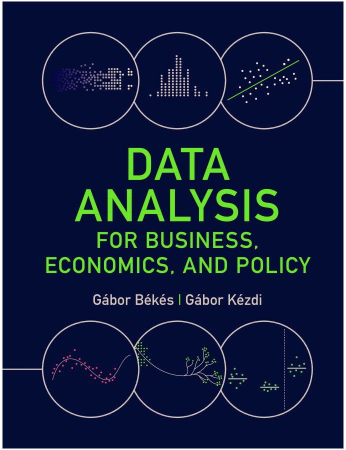
Slideshow for the Békés-Kézdi Data Analysis textbook



- Cambridge University Press, 2021
- gabors-data-analysis.com
  - Download all data and code: gabors-data-analysis.com/data-and-code/
- This slideshow is for Chapter 11

Motivation

- *What are the health benefits of not smoking? Considering the 50+ population, we can investigate if differences in smoking habits are correlated with differences in health status.*

Binary events

- Start with binary events: things that either happen or don't happen captured by binary variable
- How can we model these events?
  - We do not observe 'on average' larger values for y in this case.
- Solution - model instead the probabilities!

$$E[y] = P[y = 1]$$

- The average of a 0–1 binary variable is also the probability that it is one.
  - Frequency (25% of cases) — probability (25% chance)
- Expected value = average probability of event happening
  - Use the same tools, but interpretation is changing!

Linear probability model - LPM

- Modelling probability – regression with *binary dependent variable*.
- *Linear Probability Model (LPM)* is a linear regression with a binary dependent variable

- Differences in average y are also differences in the probability that $y = 1$
- Linear regressions with binary dependent variables show
  - differences in expected y by x, is also differences in the probability of $y = 1$ by x.
- Introduce notation for probability:

$$y^P = P[y = 1 | x_1, x_2, \dots]$$

- Linear probability model (LPM) regression is

$$y^P = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

Linear probability model - interpretation

$$y^P = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

- y^P denotes the probability that the dependent variable is one, conditional on the right-hand-side variables of the model.
- β_0 shows the probability of y if all x are zero.
- β_1 shows the difference in the probability that $y = 1$ for observations that are different in x_1 but are the same in terms of x_2.
- Still true: average difference in y corresponding to differences in x_1 with x_2 being the same.

Linear probability model - modelling

- Linear probability model (LPM) using OLS.
- We can use all transformations in x, that we used before:
  - Log, Polinomials, Splines, dummies, interactions, ect.
- All formulae and interpretations for standard errors, confidence intervals, hypotheses and p-values of tests are the same.
- Heteroskedasticity robust error are essential in this case!

Predicted values in LPM

- Predicted values - $\hat{y}^P$ - may be problematic, calculated the same way, but to be interpreted as probabilities.

$$\hat{y}^P = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$$

- Predicted values need to be between 0 and 1 because they are probabilities
- But in LPM, they may be below 0 and above 1. No formal bounds in the model.
  - With continuous variables that can take any value (GDP, Population, sales, etc), this could be a serious issue
  - With binary variables, no problem ('saturated models')
- Problem if goal is prediction!
- Not a big issue for inference $\rightarrow$ uncover patterns of association.
  - But note in theory it may give biased estimates...

Does smoking pose a health risk?

The question of the case study is whether, and by how much less likely smokers are to stay healthy than non-smokers.

- focus on people of age 50 to 60 who consider themselves healthy
- ask them four years later as well

Research question: Does smoking lead to deteriorating health?

Data

- $y = 1$ if person stayed healthy
- $y = 0$ if person became unhealthy
- Data comes from SHARE (Survey for Health, Aging and Retirement in Europe)
  - 14 European countries
  - Demographic information on all individual
  - 2011 and 2015 participants are used
  - Being healthy means to report "feeling excellent" or "very good"
  - $N = 3,109$

LPM

Start with a simple univariate model with being a smoker.

$$\textit{stays healthy}^{P} = \alpha + \beta \textit{smoker}$$

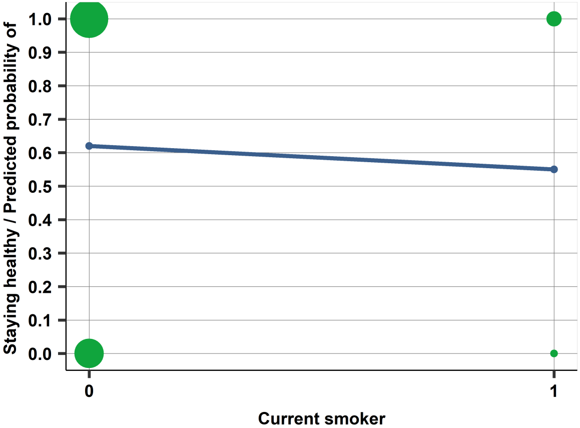
Both dependent and independent models are using only dummy variables.

Estimated β is -0.072

Can we draw a scatterplot?

Scatterplot

Figure: Staying healthy - scatterplot and regression line



LPM Interpretation

- The coefficient on smokes shows the difference in the probability of staying healthy comparing current smokers and current nonsmokers.
- Current smokers are 7 *percentage points* less likely to stay healthy than those that did not smoke.
- Can add additional controls to capture if quitting matters.

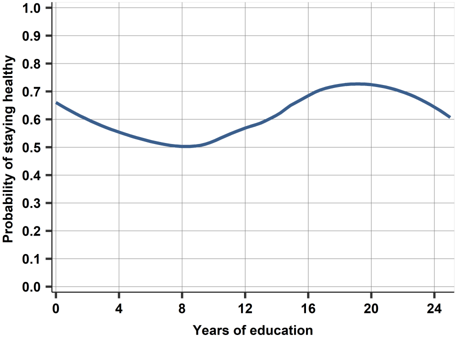
LPM with many regressors I.

- Multiple regression – closer to causality
  - compare people who are very similar in many respects but are different in smoking habits
  - find many confounders that could be correlated with smoking habits and health outcomes
- Smokers / non-smokers – different in many other behaviors and conditions:
  - personal traits
  - behavior such as eating, exercise
  - socio-economic conditions
  - background - e.g. country they live in

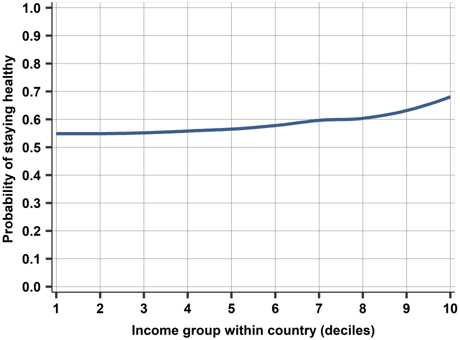
LPM with many regressors II.

- Pick variables:
  - gender dummy, age, years of education,
  - income (measured as in which of the 10 income groups individuals belong within their country),
  - body mass index (a measure of weight relative to height),
  - whether the person exercises regularly, the country in which they live.
  - country - set of binary indicators.
- Think functional form:
  - Continuous control variables might have nonlinear relationship with staying healthy
  - Explore the relationship with nonparametric tools

Functional form selection



Staying healthy and years of education



Staying healthy and income group

Decisions: (1) Include education as a piecewise linear spline with knots at 8 and 18 years; (2) include income in a linear way.

LPM results

Probability of staying healthy - extended model

VARIABLES	Staying healthy	VARIABLES (cnt.)	
Current smoker (Y/N)	-0.061*	Income group	0.008*
	(0.024)		(0.003)
Ever smoked (Y/N)	0.015	BMI (for < 35)	-0.012**
	(0.020)		(0.003)
Female (Y/N)	0.033	BMI (for $>= 35$)	0.006
	(0.018)		(0.017)
Age	-0.003	Exercises regularly (Y/N)	0.053**
	(0.003)		(0.017)
Years of education (for < 8)	-0.001	Years of education (for $>= 18$)	-0.010
	(0.007)		(0.012)
Years of education (for $>= 8$ and < 18)	0.017**	Country indicators	YES
	(0.003)		
Observations	3,109		

Robust standard errors in parentheses. ** p<0.01, * p<0.05

Y/N denotes binary vars. BMI and education entered as spline. Age in years. Income in deciles.

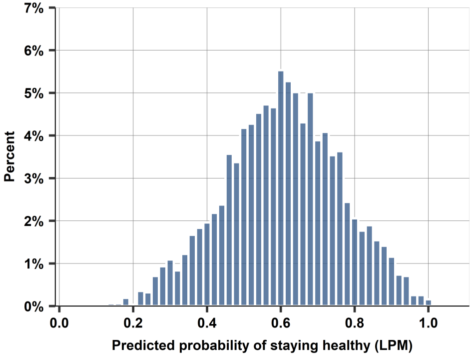
LPM result's interpretation

- Coefficient on currently smoking is -0.06
  - The 95% confidence interval is relatively wide $[-0.11, -0.01]$, but it does not contain zero
- No significant differences in staying healthy when comparing never smokers to those who used to smoke but quit
- Women are 3 percentage points more likely to stay in good health
- Age does not seem to matter in this relatively narrow age range of 50 to 60 years
- Differences in years of education
- Income matters somewhat less, maybe non-linear?
- Regular exercise matters.

LPM's predicted probabilities

- Predicted probabilities are calculated from the extended linear probability model.
- Predicted probability of staying healthy from this linear probability model ranges between 0.036 and 1.011
  - LPM means it can be below 0 or above 1...
  - Here, only marginally above 1

Histogram of the predicted probabilities



Source: `share-health` dataset.

Compare predicted probability distribution

- Drill down in distribution:
  - Looking at the composition of people: top vs bottom part of probability distribution
  - Look at average values of covariates for top and bottom 1% of predicted probabilities!

Top 1% predicted probability:

- no current smokers, women,
- avg 17.3ys of education, higher income
- BMI of 20.7, and 90% of them exercise.

Bottom 1% predicted probability:

- 37.5% current smokers, 63% men
- 7.6 years of education, lower income
- BMI of 30.5, 19% exercise

Probability models: logit and probit

- Prediction: predicted probability need to be between 0 and 1

- For prediction, we use non-linear models
- Relate the probability of the $y = 1$ event to a nonlinear function of the linear combination of the explanatory variables -> 'Link function'
  - Link function is some $F(\cdot)$, s.t. $F(y)$ may be used in linear models.

- Two options: Logit and probit – different link function
  - Resulting probability is always strictly between zero and one.

Link functions I.

The logit model has the following form:

$$y^P = \Lambda(\beta_0 + \beta_1 x_1, \beta_2 x_2 + ...) = \frac{exp(\beta_0 + \beta_1 x_1, \beta_2 x_2 + ...)}{1 + exp(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + ...)}$$

where the link function $\Lambda(z) = \frac{exp(z)}{1+exp(z)}$ is called the *logistic function*.

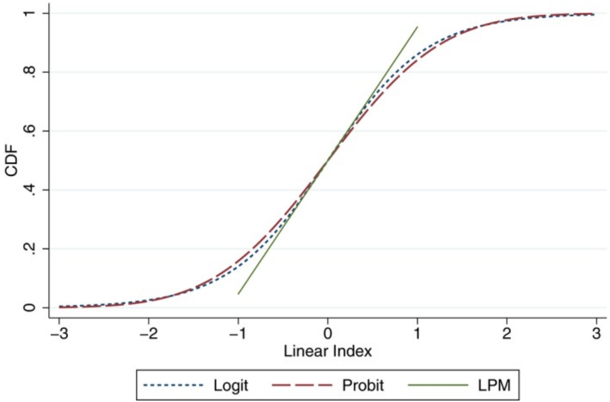
The probit model has the following form:

$$y^P = \Phi(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + ...)$$

where the link function $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{z^2}{2}\right) dz,$ is the cumulative distribution function (CDF) of the standard normal distribution.

Link functions II.

- Both Λ and Φ are increasing S-shape curves, bounded between 0 and 1. (Y here is $\Lambda(z)$ and $\Phi(z)$)
- Plotted against their respective "z" values. (Here -3 to 3)
- Small difference (indistinguishable) - logit less steep close to zero and one = thicker tails than the probit.
- In our models, 'z' is a linear combination of β coefficients and x-s. The parameter estimates are typically different in probit vs logit.



Logit and probit interpretation

- Both the probit and the logit transform the $\beta_0 + \beta_1 x_1 + ...$ linear combination using a link function that shows an S-shaped curve.
- The slope of this curve keeps changing as we change whatever is inside.
  - The slope is steepest when $y^P = 0.5$;
  - it is flatter further away; and it becomes very flat if y^P is close to zero or one.
- The difference in y^P that corresponds to a unit difference in any explanatory variable is not the same.
  - You need to take the partial derivatives. It depends on the value of x
- Important consequence: no direct interpretation of the raw coefficient values!

Marginal differences

- Link functions makes variation in association between x and y^P – for logit and probit models, we do not interpret raw coefficients!
- Instead, transform them into 'marginal differences' for interpretation purposes
- The marginal difference for x is the average difference in the probability of $y = 1$, that corresponds to a one unit difference in x.
  - Software may call them 'marginal effects' or 'average marginal effects' or 'average partial effects'.
- Marginal differences have the exact same interpretation as the coefficients of linear probability models.

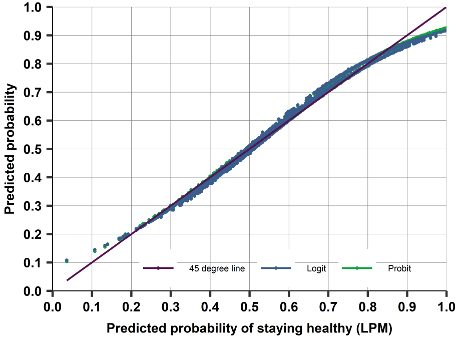
Maximum likelihood estimation

- When estimating a logit or probit model, we use ‘maximum likelihood’ estimation.
  - You specify a (conditional) distribution, that you will use during the estimation.
    - This is logistic for logit and normal for probit model.
  - You maximize this function w.r.t. your β parameters $\rightarrow$ gives the maximum likelihood for this model.
    - No closed form solution $\rightarrow$ need to use search algorithms.
- The maximum value for this function ℓ is then used for model comparisons (e.g. for Pseudo R^2)

Predictions for LMP, Logit and Probit I.

- Compare the three model results
- Baseline is LPM - extended model.
- 45 degree line is LPM
- Predicted probabilities from the logit and the probit shown vs LPM

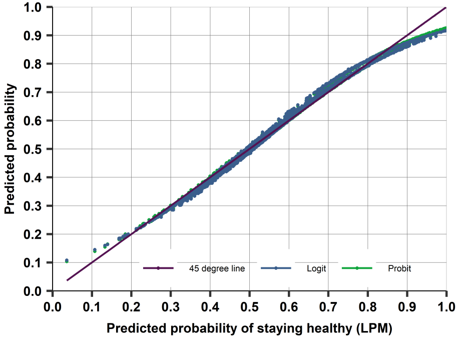
Comparing probabilities from models



Predictions for LMP, Logit and Probit II.

- Predicted probabilities from the logit and the probit are practically the same
  - range is between 0.10 and 0.92, which is narrower than the LPM, which ranges from 0.036 to 0.101
- LPM, logit and probit models produce almost exactly the same predicted probabilities
- except for the lowest and highest probabilities

Comparing probabilities from models



Coefficient results for logit and probit

Dep.var.: stays healthy	(1) LPM	(2) logit coeffs	(3) logit marginals	(4) probit coeffs	(5) probit marginals
Current smoker	-0.061*	-0.284**	-0.061**	-0.171*	-0.060*
	(0.024)	(0.109)	(0.023)	(0.066)	(0.023)
Ever smoked	0.015	0.078	0.017	0.044	0.016
	(0.020)	(0.092)	(0.020)	(0.056)	(0.020)
Female	0.033	0.161*	0.034*	0.097	0.034
	(0.018)	(0.082)	(0.018)	(0.050)	(0.018)
Years of education (if < 8)	-0.001	-0.003	-0.001	-0.002	-0.001
	(0.007)	(0.033)	(0.007)	(0.020)	(0.007)
Years of education (if $>= 8$ and < 18)	0.017**	0.079**	0.017**	0.048**	0.017**
	(0.003)	(0.016)	(0.003)	(0.010)	(0.003)
Years of education (if $>= 18$)	-0.010	-0.046	-0.010	-0.029	-0.010
	(0.012)	(0.055)	(0.012)	(0.033)	(0.012)
Income group	0.008*	0.036*	0.008*	0.022*	0.008*
	(0.003)	(0.015)	(0.003)	(0.009)	(0.003)
Exercises regularly	0.053**	0.255**	0.055**	0.151**	0.053**
	(0.017)	(0.079)	(0.017)	(0.048)	(0.017)
Age, BMI, Country	YES	YES	YES	YES	YES
Observations	3,109	3,109	3,109	3,109	3,109

Does smoking pose a health risk?– logit and probit

- LPM – interpret the coefficients.
- Logit, probit - Interpret the *marginal differences*. Basically the same.
  - Marginal differences are essentially the same across the logit and the probit.
  - Essentially the same as the corresponding LPM coefficients.
- Happens often:
  - We could not know which is the "right model" for inference
  - Often LPM is good enough for interpretation.
  - Check if logit/probit very different.
    - Investigate functional forms if yes.

Goodness of fit measures

- There is no comprehensively accepted goodness of fit measure...
  - This is because we do not observe probabilities only 1 and 0...
- R-squared is not the same meaning as before
  - Evaluating fit for probability models, we compare predictions that are between zero and one to values that are zero or one.
  - But predicted probabilities would not fit the zero-one variables, so we'd never get it right.
- R-squared less natural measure of fit, but we can calculate it as usual.
  - *But*: R-squared can not be interpreted the same way we did for linear models.

Brier score

- Brier score
$$Brier = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i^P - y_i)^2$$
- The Brier score is the average distance (mean squared difference) between predicted probabilities and the actual value of y.
- Smaller the Brier score, the better.
  - When comparing two predictions, the one with the smaller Brier score is the better prediction because it produces less (squared) error on average.
- Related to a main concept in prediction: mean squared error (MSE)

Pseudo R2

- Pseudo R-squared
  - Similar to the R-squared – measures the goodness of fit, tailored to binary outcomes.
  - Many versions of this measure. Most widely used: McFadden's R-squared
    - Computes the ratio of log-likelihood of the model vs intercept only.
  - Can be computed for the logit and the probit but not for the linear probability model. (No likelihood function there...)

- Another alternative is 'Log-loss' measure
  - Negative number. Better prediction comes with a smaller log-loss in absolute values.

Practical use

- There are several measured of model fit, they often give the same ranking of models.
- Do not use: R-squared could be computed for any model, but it no longer has the interpretation we had for linear models with quantitative dependent variable.
- Only probit vs logit: pseudo R-squared may be used to rank logit and probit models.
- Use, especially for prediction: Brier score is a metric that can be computed for all models and is used in prediction.

Does smoking pose a health risk?– Goodness of fit

Table: Statistics of goodness of fit for probability predictions models

Statistic	Linear probability	Logit	Probit
R-squared	0.103	0.104	0.104
Brier score	0.215	0.214	0.214
Pseudo R-squared	n.a.	0.080	0.080
Log-loss	-0.621	-0.617	-0.617

Source: `share-health` data. People of age 50 to 60 from 14 European countries who reported to be healthy in 2011. N=3109.

Does smoking pose a health risk?– Goodness of fit

- Stable ranking – better predictions have a
  - higher R-squared and pseudo R-squared
  - and a lower Brier score
  - a smaller log-loss in absolute values.
- Logit and the probit are of the same quality.
- Logit/probit better than the predictions from linear probability model. The differences are small.

Bias of the predictions

- Post-prediction: we may be interested to study some features of our model
- One specific goal: evaluating the *bias of the prediction*.
  - Probability predictions are *unbiased* if they are right on average = the average of predicted probabilities is equal to the actual probability of the outcome.
  - If the prediction is unbiased, the bias is zero.
- If, in our data, 20% of observations have $y = 0$ and 80% have $y = 1$, and the average of our prediction is $N^{-1}\sum_{i=1}^{N}\hat{y}_i = 0.8$, then our prediction is unbiased.
- A large value of bias indicates a greater tendency to underestimate or overestimate the chance of an event.

Calibration

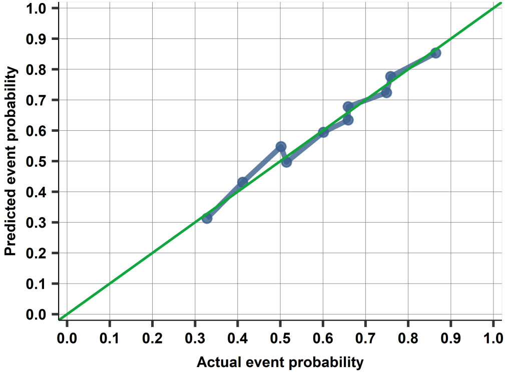
- Unbiasedness refers to the whole distribution of probability predictions is
- A finer and stricter concept is *calibration*
  - A prediction is *well calibrated* if the actual probability of the outcome is equal to the predicted probability for each and every value of the predicted probability.
- You take predicted probabilities which are around 10% and check the average for the realized outcome. If it is 10%, then the prediction is well calibrated.
- 'Calibration curve' is used to show this.
- A model may be unbiased (right on average) but not well calibrated
  - underestimate high probability events and overestimate low probability ones

Calibration curve

- A *calibration curve*
  - Horizontal axis shows the values of all predicted probabilities ($\hat{y}^P$).
  - Vertical axis shows the fraction of $y = 1$ observations for all observations with the corresponding predicted probability.
- A well-calibrated case, the calibration curve is close to the 45 degree line.
- In practice we create bins for predicted probabilities and make comparisons of the actual event's probability.
  - Use percentiles in general. Some cases equal widths are used (this is a more noisy estimate)

Calibration curve

- A calibration curve for the logit model
- 10 bins
- Not only unbiased, but well calibrated!



Probability models summary

- Find patterns with ease when y is binary - model probability with regressions
- Linear probability model is mostly good enough, easy inference.
  - Predicted values could be below 0, above 1
- Logit (and probit) - better when aim is prediction, predicted values strictly between 0-1
- Most often, LPM, logit, probit - similar inference
  - Use marginal (average) differences
- No trivial goodness of fit. Brier score or pseudo-R-Squared.
- Calibration is useful diagnostics tool: well-calibrated models will predict a 20% chance for events that tend to happen one out of five cases.