10. Multiple regression

Davide Del Prete

Data Analysis 2: Regression analysis

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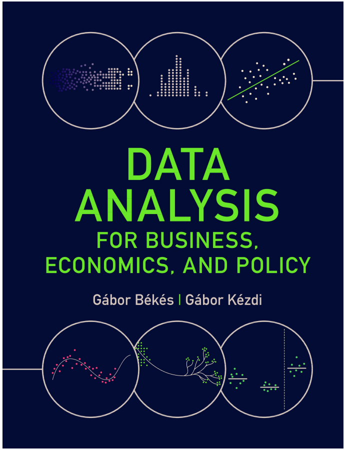
Slideshow for the Békés-Kézdi Data Analysis textbook



- Cambridge University Press, 2021
- gabors-data-analysis.com
  - Download all data and code: gabors-data-analysis.com/data-and-code/
- This slideshow is for Chapter 10

Motivation

- You want to find out how running time, distance and altitude are associated with each other to evaluate your local running time.
- Interested in finding evidence for or against labor market discrimination of women. Compare wages for men and women who share similarities in wage relevant factors such as experience.

Multiple regression analysis

- Multiple regression analysis uncovers average y as a function of more than one x variable: $y^E = f(x_1, x_2, ...)$.
- It can lead to better predictions $\hat{y}$ by considering more explanatory variables.
- It may improve the interpretation of slope coefficients by comparing observations that are different in terms of one of the x_i variable but similar in terms of other x_{-i} variables ($-i$ means all other variable except i).
- Multiple linear regression specifies a linear function of the explanatory variables for the average y.

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_k x_k$$

Multiple regression - case of two regressors

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

- β_1: the slope coefficient on x_1 shows difference in average y across observations with unit difference in x_1, *but the same value of* x_2.
  - β_2 shows difference in average y across observations with with unit difference in x_2, *but the same value of* x_1.
- Can compare observations that are similar in one explanatory variable to see the differences related to the other explanatory variable.

Multiple regression - visual representation

With two explanatory variables visually it means to fit linear plane:

- We are still minimizing the sum of squared errors:

$$\arg\min_{\beta_0,\beta_1,\beta_2} \sum_{i=1}^{N} (y - \beta_0 - \beta_1 x_1 - \beta_2 x_2)^2$$

- For K variables you fit a K dimensional linear plane!
- It is tricky how to visualize multiple regression...
- We cover some of those possibilities.

Multiple regression vs single regression

Compare slope coefficient in simple (β) and in multiple (β_1) linear regression:

$$\text{Simple: } y^E = \alpha + \beta x_1$$
$$\text{Multiple: } y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

To connect β and β_1 you need to regress x_2 on x_1 (called: "$x - x$ regression"):

$$x_2^E = \gamma + \delta x_1$$

Multiple regression vs single regression

Compare slope coefficient in simple (β) and in multiple (β_1) linear regression:

$$\text{Simple: } y^E = \alpha + \beta x_1$$
$$\text{Multiple: } y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

To connect β and β_1 you need to regress x_2 on x_1 (called: "$x - x$ regression"):

$$x_2^E = \gamma + \delta x_1$$

Plug this into the multiple regression:

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2(\gamma + \delta x_1) = \beta_0 + \beta_2\gamma + (\beta_1 + \beta_2\delta)x_1 .$$

It turns out:

$$\beta - \beta_1 = \delta\beta_2$$

Difference in slopes - in words...

- The slope of x_1 in a simple regression is different from its slope in the multiple regression, the difference being the product of its slope in the regression of x_2 on x_1 and the slope of x_2 in the multiple regression.
- The slope coefficient on x_1 in the two regressions is different
  - unless x_1 and x_2 are uncorrelated ($\delta = 0$) OR
  - the coefficient on x_2 is zero in the multiple regression ($\beta_2 = 0$).
- The slope in the simple regression is larger if x_2 and x_1 are positively correlated and β_2 is positive
  - or x_2 and x_1 are negatively correlated and β_2 is negative

Multiple regression - why different?

- If x_1 and x_2 are correlated, comparing observations with or without the same x_2 value makes a difference.
- If they are positively correlated, observations with higher x_2 tend to have higher x_1.
- In the simple regression we ignore differences in x_2 and compare observations with different values of x_1.
- But higher x_1 values mean higher x_2 values, too.
- Corresponding differences in y may be due to differences in x_1 but also differences in x_2.
  - Neglecting x_2, when it is important leads to 'omitted variable bias'.

Multiple regression - omitted variable

- Omitted variables are important, if you are interested in a coefficient value:
  - If you have a measure/variable on x_2 use it and you are done.
  - If you do not have a measure/variable on x_2:
    - similar to measurement errors: think and argue!
    - Is your 'true' parameter smaller or larger than what you estimated?
- Language: The slope on x_1 in the sample is confounded by omitting the x_2 variable, and thus x_2 is a confounder.
  - When you see/report coefficient values with adding more and more other variables to the model:
    - Want to show parameter stability - there is no other important confounder.
    - If your coefficient value changes by adding other variable(s), then you most likely have omitted variable bias problem.

Multiple regression - some language

- Multiple regression with two explanatory variables (x_1 and x_2),
- We measure differences in expected y across observations that differ in x_1 but are similar in terms of x_2.
- Difference in y by x_1, *conditional on* x_2. OR *controlling for* x_2.
- We condition on x_2, or control for x_2, when we include it in a multiple regression that focuses on average differences in y by x_1.

OLS estimator - to see such formulation

For multiple regression usually we use matrix notation:

$$y = \boldsymbol{x}'\boldsymbol{\beta}$$

where, $\boldsymbol{x} = [1, x_1, x_2, \ldots, x_k]$ and $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \ldots, \beta_k]'$.

OLS has a closed form solution in matrix form:

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{x}'\boldsymbol{x}\right)^{-1} \boldsymbol{x}'y$$

Standard Error of Beta

- Inference, confidence intervals in multiple regressions is analogous to those in simple regressions.

$$SE(\hat{\beta}_1) = \frac{Std[e]}{\sqrt{n}Std(x_1)\sqrt{1 - R_1^2}}$$

- Behaviour is the same, the SE is small IF: small Std of the residuals (the better the fit of the regression); large sample, large the Std of x_1.
- New element: $\sqrt{1 - R_1^2}$ term in the denominator - the R-squared of the regression of x_1 on x_2 - refers to the correlation between x_1 and x_2.
- The stronger the correlation between x_1 and x_2 the larger the SE of $\hat{\beta}_1$.
- Note the symmetry: the same applies to the SE of $\hat{\beta}_2$.
- As usual, in practice, use robust SE.

Collinearity of explanatory variables

- Perfectly collinearity is when x_1 is a linear function of x_2.
- Consequence: cannot calculate coefficients (reason: linearly dependent matrix: inverse does not exists...)
  - One will be dropped by software

- Strong but imperfect correlation between explanatory is called *multicollinearity*.
  - Consequence: We can still get the slope coefficients and their standard errors, but:
    - Standard errors may be large.
    - Does not affect the value of β

Multicollinearity and SE of beta

- As a consequence of multicollinearity the standard errors may be large.
  - Concept: Few variables that are different in x_1 but not in x_2. Not enough observations for comparing average y when x_1 is different but x_2 remains the same.
  - Math: R_1^2 is high (x_2 is a good predictor of x_1), thus $\sqrt{1 - R_1^2}$ is (really) small, which makes $SE(\beta_1)$ (very) large.

- This is a small sample problem.
  - May look at pair-wise correlations when start working with data
  - Drop one or the other, or combine them (use z-score/average/PCA).

F-test: joint significance

- *Testing joint hypotheses*: null hypotheses that contain statements about more than one regression coefficient.
- We aim at testing whether a subset of the coefficients (such as all geographical variables) are all zero.
- F-test answers this.
  - Individually they are not all statistically different from zero, but together they may be.
  - Everything is similar to t-tests, but the sampling distribution here is a 'F-distribution'

- We may ask if *all slope coefficients are zero* in the regression.
  - "Global F-test", and its results are often shown by statistical software by default.

Many explanatory variables

- Having more explanatory variables is straightforward extension:

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ...$$

- Interpreting the slope of x_1: on average, y is β_1 units larger in the data for observations with one unit larger x_1 but the same value for all other x variables.
- SE formula - small when R_k^2 is small - R^2 of regression of x_k on all *other* x variables.

$$SE(\hat{\beta}_k) = \frac{Std[e]}{\sqrt{n}Std[x_k]\sqrt{1 - R_k^2}}$$

Non-linear patterns with multiple regression

- Uses splines, polynomials - actually like multiple regression - we have multiple coefficient estimates.
- Multicollinearity - not (perfect) *linear* combinations, but keep in mind...
  - Remember the 'poly()' function? $\rightarrow$ it handles this issue!
- Non-linear function of various x_i variables may be combined.

Understanding the gender difference in earnings

- In the USA (2014), women tend to earn about 20% less than men
- Aim 1: Find patterns to better understand the gender gap.
  - Our focus is the interaction with age.
- Later - Aim 2: Think about if there is a causal link from being female to getting paid less.

Gender gap in earnings - data

- 2014 census data
  - Age between 15 to 65
  - Exclude self-employed (earnings is difficult to measure)
  - Include those who reported 20 hours more as their usual weekly time worked
- Employees with a graduate degree (higher than 4-year college)
- Use log hourly earnings ($\ln(w)$) as dependent variable
- Use gender and add age as explanatory variables

Basic models for gender gap

We are quite familiar with the relation between earnings and gender:

$$\ln w^E = \alpha + \beta female, \qquad \beta < 0$$

Let's extend the model with age:

$$\ln w^E = \beta_0 + \beta_1 female + \beta_2 age$$

We can calculate the correlation between female and age, which is in fact negative.

What do you expect about β, β_1, δ?
Reminder:

$$age^E = \gamma + \delta female$$

Gender gap regression - baseline

Variables	(1) ln wage	(2) ln wage	(3) age
female	-0.195**	-0.185**	-1.484**
	(0.008)	(0.008)	(0.159)
age		0.007**	
		(0.000)	
Constant	3.514**	3.198**	44.630**
	(0.006)	(0.018)	(0.116)
Observations	18,241	18,241	18,241
R-squared	0.028	0.046	0.005

Note: *All employees with a graduate degree. Robust standard errors in parentheses*
Source: `cps-earnings` dataset. 2014 CPS Morg.

Age is a confounder variable

Remember: the omitted variable bias is given by:

$$\beta - \beta_1 = \delta\beta_2$$

which can be calculated easily:

- $\beta - \beta_1 = -0.195 - (-0.185) = -0.01$
- $\delta\beta_2 = -1.48 \times 0.007 \approx -0.01$

Interpretation:

- Age is a confounder, it is different from zero and the value of beta coefficient changes.
- But a weak one: the magnitude of the change is not really large.

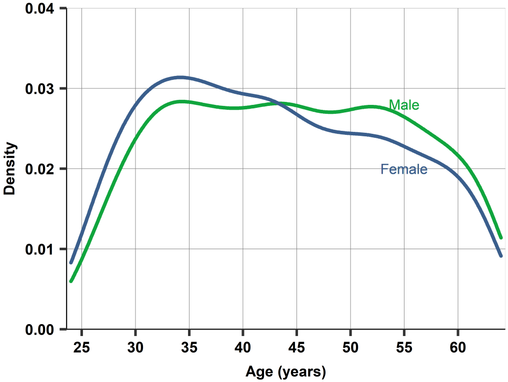
Interpretations and connections of the basic model

Interpretation of model coefficients:

- Women of the same age have a slightly smaller earnings disadvantage in this data because they are somewhat younger, on average
- employees that are younger tend to earn less
- part of the earnings disadvantage of women is thus due to the fact that they are younger.
  - This is a small part: around 1 percentage points of the 20% difference,
  - Overall this is only a 5% share of the entire difference.
    - This is the difference if we control for age or not.
- A single linear variable for age may not be enough.
  - Investigate the impact of age.

Conditional distribution of age based on gender

Age distribution of male and female employees with degrees higher than college



- Relatively few below age 30
- Above 30
  - close to uniform for men
  - for women, the proportion of female employees with graduate degrees drops above age 45, and again, above age 55
- Two possible things
  - fewer women with graduate degrees among the 45+ old than among the younger ones
  - fewer of them are employed

Non-linearity in age, but same effect on gender

Variables	(1) ln wage	(2) ln wage	(3) ln wage	(4) ln wage
female	-0.195**	-0.185**	-0.183**	-0.183**
	(0.008)	(0.008)	(0.008)	(0.008)
age		0.007**	0.063**	0.572**
		(0.000)	(0.003)	(0.116)
age^2			-0.001**	-0.017**
			(0.000)	(0.004)
age^3				0.000**
				(0.000)
age^4				-0.000**
				(0.000)
Constant	3.514**	3.198**	2.027**	-3.606**
	(0.006)	(0.018)	(0.073)	(1.178)
Observations	18,241	18,241	18,241	18,241
R-squared	0.028	0.046	0.060	0.062

Note: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

Using qualitative variables

- Can have binary variables as well as other qualitative variables (factors) .
- Consider a qualitative variable like income categories or continents. How to add it to the regression model?
  - Create binary variables (dummy variables) for all options. Add them - all but one. (Why? $\rightarrow$ linear dependence with the intercept!)
  - Left out one will be the base/reference!

Qualitative variables - example I.

- x is a categorical variable with three values *low*, *medium* and *high*
- binary variable x_m denote if $x = medium$, x_h variable denote if $x = high$.
- for $x = low$ is not included. It is called the *reference category* or left-out category.

$$y^E = \beta_0 + \beta_1 x_m + \beta_2 x_h$$

Qualitative variables - example II.

$$y^E = \beta_0 + \beta_1 x_m + \beta_2 x_h$$

- Pick $x = low$ as the reference category. Other values compared to this.
  - This is the left out variable
- β_0 shows average y in the reference category. Here, β_0 is average y when both $x_m = 0$ and $x_h = 0$: this is the case of $x = low$.
- β_1 shows the difference of average y between observations with $x = medium$ and $x = low$
- β_2 shows the difference of average y between observations with $x = high$ and $x = low$.

Interactions

- Many cases, data is made up of important groups: male and female workers or countries in different continents.
- Some of the patterns we are after may vary across these groups.
- The strength of a relation may also be altered by a special variable.
- In medicine, a *moderator variable* can reduce / amplify the effect of a drug on people.
- In business, financial strength can affect how firms/countries may weather a recession.
- All of these mean different patterns for subsets of observations.

Interactions - when to use?

- Regression with two explanatory variables: x_1 is continuous, D is binary denoting two groups in the data (e.g., male or female employees).
- We wonder if the relationship between average y and x_1 is different for observations with $D = 1$ than for $D = 0$. How to test?

Interaction - parallel lines

- Option 1: Two *parallel lines* for the y - x_1 pattern: one for those with $D = 0$ and one for those with $D = 1$.
- Similar to qualitative variables plus a continuous variable x_1

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 D$$

- The predicted/expected values for the two groups ($y_0^E = E[y^E|D = 0], y_1^E = E[y^E|D = 1]$) can be written as,

$$y_0^E = \beta_0 + \beta_2 \times 0 + \beta_1 x_1$$

$$y_1^E = \beta_0 + \beta_2 \times 1 + \beta_1 x_1$$

Interaction - different slopes

- Option 2: *Allow for different slopes* in the two D groups we have to add an interaction term directly to x_1 as well:

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 D + \beta_3 (x_1 \times D)$$

- Intercepts are kept different by β_2 AND slopes different by β_3. The two slopes are given by,

$$y_0^E = \beta_0 + \beta_1 x_1$$

$$y_1^E = \beta_0 + \beta_2 + (\beta_1 + \beta_3) x_1$$

Interactions vs separate regressions

- Separate regressions in the two groups and the regression that pools observations but includes an interaction term, yield *exactly the same* coefficient estimates.
  - The coefficients of the separate regressions are easier to interpret.
  - The pooled regression with interaction allows for a direct test of whether the slopes are the same.

Interaction with many groups

- You can generalize to three groups
  - Let: D_1, D_2 are binaries and x is continuous:

$$y^E = \beta_0 + \beta_1 x + \beta_2 D_1 + \beta_3 D_2 + \beta_4 (D_1 \times x) + \beta_5 (D_2 \times x)$$

- In general, if you have K groups

$$y^E = \beta_0 + \beta_1 x + \sum_{k=2}^{K} \beta_k D_{k-1} + \beta_{K+k} (D_{k-1} \times x)$$

Interaction with two continuous variable

- Same model used for two continuous variables, x_1 and x_2:

$$y^E = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$$

- Example: Firm level data, 100 industries.
  - y is change in revenue, x_1 is change in global demand, x_2 is firm's financial health
  - The interaction can capture that drop in demand can cause financial problems in firms, but less so for firms with better balance sheet.
- Note: interpretation is tricky! Use the derivative to see why!

Interaction between gender and age

- Why we assume that age has the same slope regardless of gender? We might want to check, whether they are different!
- Are the slopes significantly different?
- Can one get the slope for age for female only from the regression with the interaction?
- How the gender dummy's coefficient changed?

Interaction between gender and age

- Earning for men rises faster with age.
- Pooled EQ with interaction: interaction + age coefficient is the SAME as women's age coefficient.
- β_3 is significant: earning growth by age is different for male and female.
- Constant dummy is close to zero and seems insignificant
  - at birth there would be no difference,
  - but at 25, there is already a significant difference $\rightarrow$ interaction term

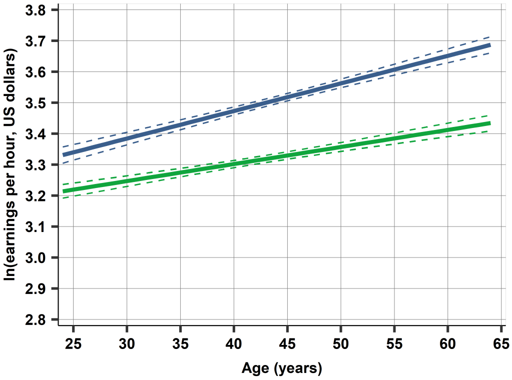
Variables	(1) Women ln wage	(2) Men ln wage	(3) All ln wage
female			-0.036
			(0.035)
age	0.006**	0.009**	0.009**
	(0.001)	(0.001)	(0.001)
female $\times$ age			-0.003**
			(0.001)
Constant	3.081**	3.117**	3.117**
	(0.023)	(0.026)	(0.026)
Observations	9,685	8,556	18,241
R-squared	0.011	0.028	0.047

Nonlinearities and interactions

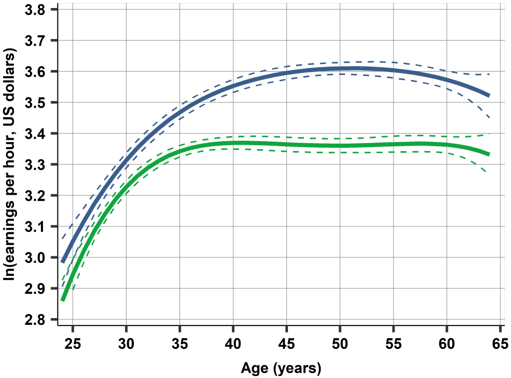
We can estimate interactions with non-linear terms as well:

$$
\begin{aligned}
lnw^E = \beta_0 &+ \beta_1 age + \beta_2 age^2 + \beta_3 age^3 + \beta_4 age^4 \\
&+ \beta_5 female + \beta_6 female \times age + \beta_7 female \times age^2 \\
&+ \beta_8 female \times age^3 + \beta_9 female \times age^4
\end{aligned}
$$

Nonlinearities and interactions



Log earnings per hour and age by gender: predicted values and confidence intervals from a linear regression interacted with gender.



Log earnings per hour and age by gender: predicted values and confidence intervals from a regression with 4th-order polynomial interacted with gender.

Visual inspection in the regression lines

- The average earnings difference is around 10% between ages 25 and 30
- increases to around 15% by age 40, and reaches 22% by age 50,
- from where it decreases slightly to age 60 and more by age 65.
- confidence intervals around the regression curves are rather narrow, except at the two ends.
- Conclusion?

Causal analysis with multiple regression

- One main reason to estimate multiple regressions is to get closer to a causal interpretation.
- Called: Causal analysis or causal inference
- By conditioning on other observable variables, we can get closer to comparing similar objects – "apples to apples" – even in observational data.
- But getting closer is not the same as getting there.
- In principle, one may help that by conditioning on *every* potential confounder: variables that would affect y and the causal variable x_1 at the same time.
  - Ceteris paribus = conditioning on *every* such relevant variable.

Causal analysis - ceteris paribus

- Ceteris paribus = conditioning on every such relevant variable.
- *Ceteris paribus* prescribes what we want to condition on.
  - A multiple regression can condition on what's in the data the way it is measured.
- Importantly, conditioning on everything is impossible in general.
- Multiple regression is never (hardly ever) ceteris paribus.

Causal analysis

- A multiple regression on observational data is rarely capable of uncovering a causal relationship.
  - Cannot capture all potential confounder. (No ceteris paribus comparison)
  - We can never really know. BUT
- multiple regression can get us closer to uncovering a causal relationship
  - Compare units that are the same in many respects - controls
- More on causal inference in Chapters 19-24

Gender difference in earnings - causality?

What may cause the difference in wages?

- Labor discrimination - one group earns less even if they have the same *marginal product*
- Try control for marginal product (or for variables which matters to marginal product)
  - Eg.: occupation (as an indicator for inequality in gender roles), or industry, union status, hours worked and other socio-economic characteristics
- Use variables as controls - does comparing apples to apple change coefficient of female variable?
  - Practice: add more variables if coefficient is the same you are good. Otherwise need to think about OVB...

Causal analysis - results

- More and more confounders added
- Female coefficient reduced from 22% to 14%
- Compare two people, with same age, hours, industry, occupation, geography, background (=confounders) - women earn 14% less, on average.

Variables	(1) ln wage	(2) ln wage	(3) ln wage	(4) ln wage
female	-0.224**	-0.212**	-0.151**	-0.141**
	(0.012)	(0.012)	(0.012)	(0.012)
Age and education		YES	YES	YES
Family circumstances			YES	YES
Demographic background			YES	YES
Job characteristics			YES	YES
Union member			YES	YES
Age in polynomial				YES
Hours in polynomial				YES
Observations	9,816	9,816	9,816	9,816
R-squared	0.036	0.043	0.182	0.195

Restricted sample: employees of age 40 to 60 with a graduate degree

Discussion

- Could not safely pin down the role of labor market discrimination and broader gender inequality

Prediction with multiple regression

- Reason to estimate a multiple regression is to make a *prediction*.
  - find the best guess for the dependent variable y_j for a particular *target observation j*

$$\hat{y}_j = \hat{\beta}_0 + \hat{\beta}_1 x_{1j} + \hat{\beta}_2 x_{2j} + \ldots$$

- When the goal is prediction we want the regression to produce as good a fit as possible.
  - 'good fit' in the general pattern that is representative of the target observation j.
- A common danger is *overfitting* the data: finding patterns in the data that are not true in the general pattern, only for your sample.
- More on prediction in Chapters 13-18

Visualization of fit for multiple regression

- The $\hat{y} - y$ *plot* has $\hat{y}$ on the horizontal axis and y on the vertical axis.
  - The plot features the 45 degree line and the scatterplot around it = the regression line of y regressed on $\hat{y}$.
- The scatterplot around this line shows how actual values of y differ from their predicted value $\hat{y}$.
- Review case study in Chapter 10

Summary take-away

- Multiple regression are linear models with several x variables.
- May include binary variables and interactions
- Multiple regression can take us closer to a causal interpretation and help make better predictions.