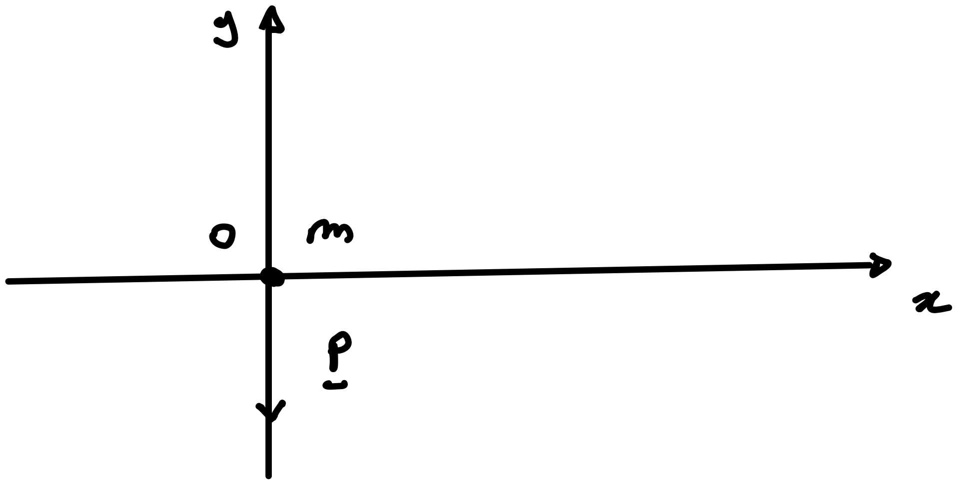
EQUAZIONI DIFFERENZIALI



$$\cancel{m}\,\underline{a} = \underline{P} = \cancel{m}\,\underline{g}$$

$$y''(t) = -g \qquad y'(t) = -gt + c \,, \quad c \in \mathbb{R}$$

$$\int \qquad \int$$

$$y(t) = -\frac{1}{2} g t^2 + ct + d$$

$$c, d \in \mathbb{R}$$

$$\begin{cases} y'' = -g \\ y(0) = 0 \\ \underbrace{y'(0)}\ = 0 \end{cases}$$

$$y' = -gt + c$$

$$0 = y'(0) = c \iff \boxed{c = 0}$$

$$y(t) = -\frac{1}{2} g t^2 + d \longrightarrow y(t) = -\frac{1}{2} g t^2$$

$$0 = y(0) = d \iff \boxed{d = 0}$$

$$m\underline{a} = \underline{F}(t, P, \underline{v})$$

$$\begin{cases} m y''(t) = F(t, y(t), y'(t)) \\ y(0) = y_0 \\ y'(0) = y'_0 \end{cases}$$

PROBLEMA DI CAUCHY

ES. $\underbrace{y'(t) = -\lambda y(t)}$ $\lambda > 0$

EQ. DIFF. I° ORDINE!

ES. $g(x)$ continua in $[a,b]$ $F'(x) = g(x)$

primitive: $y'(x) = g(x)$

$x_0 \in [a,b]$ $$y(x) = \underbrace{\int_{x_0}^{x} g(t)\,dt + c}_{\text{tutte le primitive di } g(x)}, \quad c \in \mathbb{R}$$

$y(x_0) = y_0 \in \mathbb{R}$ CONDIZ. INIZIALE

$$\begin{cases} y'(x) = g(x) \\ y(x_0) = y_0 \end{cases} : \quad y_0 = y(x_0) = \underbrace{\int_{x_0}^{x_0} g(t)\,dt}_{0} + c = c$$

$\Leftrightarrow \boxed{c = y_0}$

$$y(x) = \int_{x_0}^{x} g(t)\,dt + y_0$$

$$y'(x) = \lambda y(x)$$

: VERIFICHIAMO CHE $y(x) = c e^{\lambda x}$, $c \in \mathbb{R}$ RISOLVE L'EQUAZIONE

$$y'(x) = \lambda c e^{\lambda x}$$
$$\lambda y(x) = \lambda c e^{\lambda x}$$

: $y'(x) = \lambda y(x)$

$y(x) = c e^{\lambda x}$ tutte le soluzioni ?
$c \in \mathbb{R}$

SUPPONIAMO CHE $y(x)$ sia soluzione dell'equazione,

CIOÈ : $y'(x) = \lambda y(x) \quad \forall x$

$$z(x) := e^{-\lambda x} y(x)$$

$$z'(x) = -\lambda e^{-\lambda x} y(x) + e^{-\lambda x} y'(x)$$
$$= e^{-\lambda x}\left(-\lambda y(x) + y'(x)\right) = 0$$

$\Rightarrow \quad z(x) = c, \quad c \in \mathbb{R}$

QUINDI $e^{-\lambda x} y(x) = c \implies \boxed{y(x) = c e^{\lambda x}}$

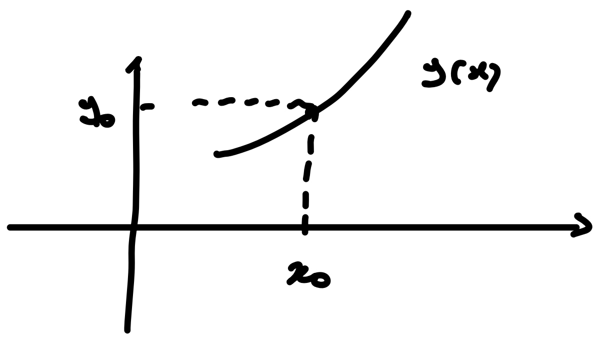
$$\begin{cases} y' = \lambda y \\ y(x_0) = y_0 \end{cases}$$

$$y_0 = y(x_0) = c e^{\lambda x_0}$$

$$c = y_0 e^{-\lambda x_0}$$

$$y(x) = y_0 e^{\lambda (x - x_0)}$$

$\lambda > 0$



$$y' = y^2 \qquad y'(x) = y^2(x)$$

$$y' = x^3 y^2 \qquad = \log x \; y^2$$

$$y' = \log\left(\frac{x}{y}\right) = f(x, y)$$

$$y'(x) = f(x, y(x)) \qquad f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$$

equazione del I° ORDINE IN FORMA NORMALE

$$y''(x) = -\lambda y'(x) + y(x) + x^2$$

$$y''(x) = [y'(x)]^2$$

$$y''(x) = f(x, y(x), y'(x)) \qquad f(x, y, z)$$

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases} \qquad y = y(x) \qquad y'(x) = y^2(x)$$

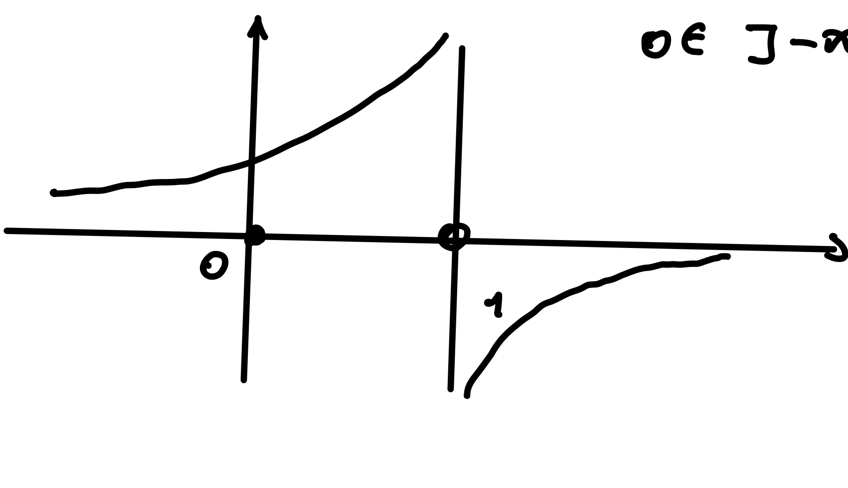
$$y(x) = \frac{1}{1-x} \qquad \boxed{x \neq 1}$$

$$y'(x) = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$= y^2(x)$$

$$y(0) = 1$$

$$0 \in \,]-\infty, 1[$$



Def.

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

PROBLEMA DI CAUCHY
EQUAZIONE I° ORDINE

• RISOLUZIONE LOCALE, O IN PICCOLO: la soluzione che si determina è definita **solo** in un intorno del punto iniziale x_0

(es. $\begin{cases} y'=y^2 \\ y(0)=1 \end{cases} \longrightarrow y=\dfrac{1}{1-x}$, $x\in \underbrace{]-\infty,1[}_{\text{INTORNO } 0}$

• RISOLUZIONE GLOBALE O IN GRANDE:

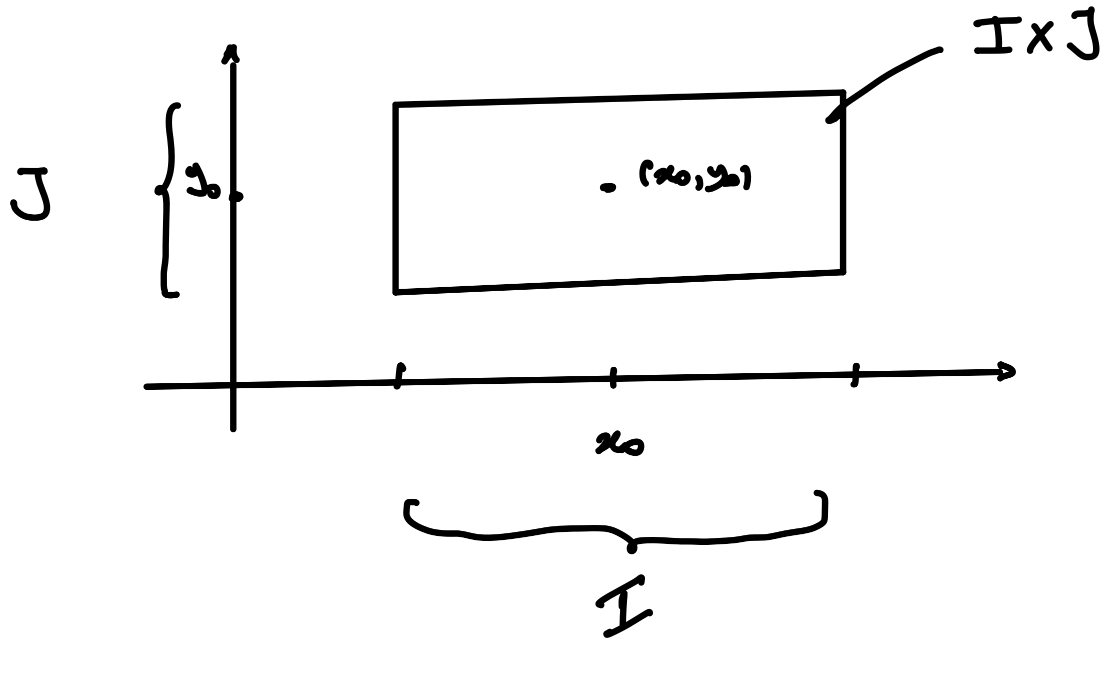
la soluzione che si determina è definita in tutto l'intervallo di definizione dell'equazione

(es. $\begin{cases} y'=\overset{\text{in } \mathbb{R}}{\lambda} y \\ y(x_0)=y_0 \end{cases} \longrightarrow y(x)=y_0 e^{\lambda(x-x_0)}$

$(x_0,y_0)\in\mathbb{R}^2$

$I=[x_0-a,\ x_0+a]$, $a>0$

$J=[y_0-b,\ y_0+b]$, $b>0$



$f = f(x,y)$, $f: I \times J \longrightarrow \mathbb{R}$ <u>CONTINUA</u> $\odot$

$$\left| f(x, y_1) - f(x, y_2) \right| \leq L \, |y_1 - y_2|$$

$$L > 0, \ \forall x \in I, \ \forall y_1, y_2 \in J$$

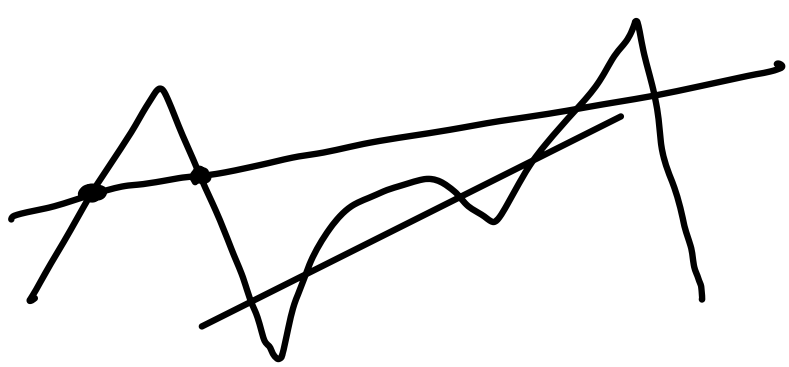
"f LIPSCHITZIANA RISPETTO AD y,

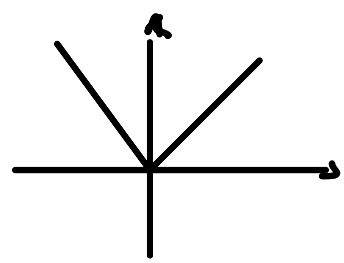
LIPSCHITZ UNIFORMEMENTE RISPETTO AD x"

— $g = g(x)$, $g : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ SI DICE LIPSCHITZIANA

SE $\exists L > 0$: $|g(x_1) - g(x_2)| \leq L \, |x_1 - x_2|$, $\forall x_1, x_2 \in X$

$$\left| \frac{g(x_1) - g(x_2)}{x_1 - x_2} \right| \leq L \qquad x_1 \neq x_2$$



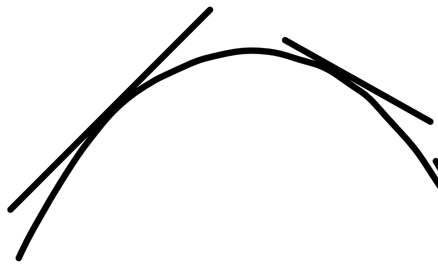


L
"

$g(x) = |x|$: $|g(x_1) - g(x_2)| = \big| |x_1| - |x_2| \big| \leq 1 \cdot |x_1 - x_2|$

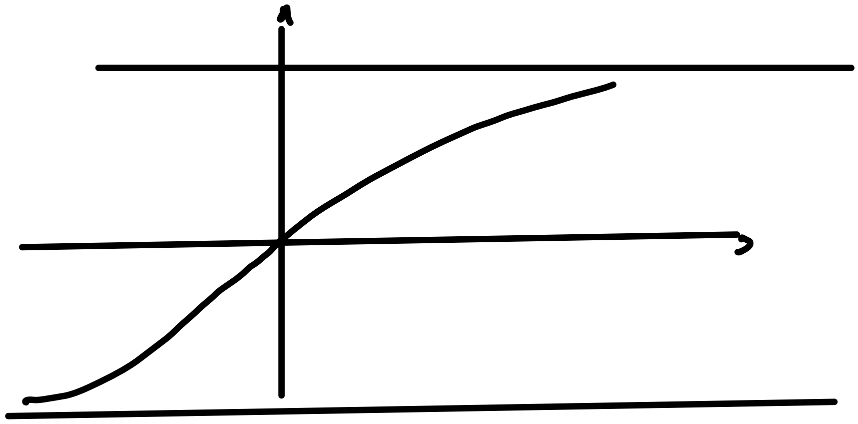
"Se $g(x)$ è LIPSCHITZIANA $\Longrightarrow$ $g(x)$ è CONTINUA"

Prop. Se $g \in C^1$: $|g'(x)| \le L$



$\Longleftrightarrow$ g LIPSCHITZIANA

ES. $g(x) = \operatorname{arctg} x$ $\qquad g'(x) = \dfrac{1}{1+x^2} \le 1$



$g(x) = x^2$; $\qquad g'(x) = 2x$

$x > 0$ $\qquad g'(x) \le \underbrace{2a} = L$

$x \in [0, a]$

$f = f(x,y)$, $f : I \times J \longrightarrow \mathbb{R}$ CONTINUA (•)

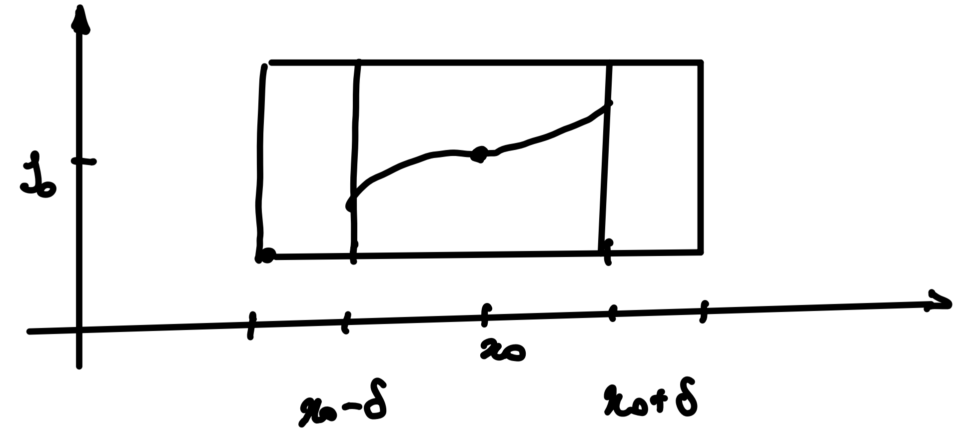
$$|f(x,y_1) - f(x,y_2)| \leq L\,|y_1 - y_2|$$

$L > 0$, $\forall x \in I$, $\forall y_1, y_2 \in J$ (••)

TEOREMA (di ESISTENZA ED UNICITÀ LOCALE DI CAUCHY)

Sotto le condizioni (•) e (••) esiste $0 < \delta < a$ ed esiste un'unica funzione $y = y(x)$, $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ derivabile, che risolve in $[x_0 - \delta, x_0 + \delta]$ il problema di Cauchy

$$\begin{cases} y' = f(x,y) \; : \; y'(x) = f(x, y(x)) \quad \forall x \in [x_0 - \delta, x_0 + \delta] \\ y(x_0) = y_0 \end{cases}$$



COROLLARIO Se f è continua in $I \times J$ e $f \in C^1(I \times J)$

$$\Downarrow$$

vale la tesi del teorema di Cauchy.

EQUAZIONI LINEARI DEL I° ORDINE.

$$y' = \underbrace{a(x)}_{\text{COEFFICIENTE}} y + \underbrace{b(x)}_{\text{TERMINE NOTO}}$$

$$y' = \lambda y \qquad\qquad y' = \underbrace{x^2}_{a(x)} y + \overbrace{\log x}^{b(x)}$$

Se $b(x) = 0$, l'equazione è OMOGENEA.

$$y' = a(x) y \quad ; \quad \frac{dy}{dx} = a(x) y$$

$y \neq 0$ $\qquad \displaystyle\int \frac{dy}{y} = \int a(x)\,dx$

$$\log|y| = G(x) + c, \quad c \in \mathbb{R}$$

$y' = x^2 y \ ; \ y \neq 0 \quad \displaystyle\int \frac{dy}{y} = \int x^2\,dx$ $\qquad |a| = b \iff a = \pm b$

$$\log|y| = \frac{x^3}{3} + c$$

$$|y| = e^{x^3/3 + c} = e^c e^{x^3/3}$$

$$y = \underbrace{(\pm e^c)}_{k} e^{x^3/3}$$

$$\boxed{y = k e^{x^3/3}} \ , \ k \in \mathbb{R}$$

$y = 0$ è soluzione: $\quad y' = 0 \ ; \quad x^2 y = 0$

$y' = -e^x y \ ; \quad \displaystyle\int \frac{dy}{y} = -\int e^x\,dx$

$$\log|y| = -e^x + c$$

$$|y| = e^{-e^x + c} = e^c \cdot e^{-e^x}$$

$$y = k\, e^{-e^x} \quad , k \in \mathbb{R}$$

$$\begin{cases} y' = -e^x y \\ y(0) = 2 \end{cases}$$

$\downarrow$

$$2 = y(0) = k\, e^{-1}$$

$$\Leftrightarrow \quad k = 2e$$

$$y = 2e\, e^{-e^x} =$$
$$= 2\, e^{1 - e^x}$$

$$y' = -(\sin 2x)\, y \quad : \quad \int \frac{dy}{y} = -\int \sin(2x)\, dx$$

$$\log|y| = -\frac{1}{2}\int 2 \sin(2x)\, dx = \frac{1}{2}\cos(2x) + c$$

$$y = k\, e^{\frac{1}{2}\cos(2x)} \quad , k \in \mathbb{R}$$

$$y' = \lambda y \qquad y = c\, e^{\lambda x} \qquad \frac{dy}{y} = \lambda\, dx$$

$$\log|y| = \lambda x + c$$

$$y = k e^{\lambda x} \quad ||$$

$$k$$

$$y' = a(x)\, y + b(x) \qquad b(x) \neq 0$$

$$y' = \underbrace{x^2}_{a(x)}\, y + \overbrace{\log x}^{b(x)} \qquad ?$$

$a(x)$ primitiva: $A'(x) = a(x)$

$$y = y(x)$$

$$y' = a(x)\, y + b(x) \qquad e^{-A(x)}$$

$$y' e^{-A(x)} = \widetilde{a(x)}\, e^{-A(x)}\, y + e^{-A(x)}\, b(x)$$

$$y' e^{-A(x)} - A'(x)\, e^{-A(x)}\, y = e^{-A(x)}\, b(x)$$

$$\frac{d}{dx}\left(e^{-A(x)}\, y\right) = e^{-A(x)}\, b(x) \qquad \int$$

$$e^{-A(x)}\, y = \int e^{-A(x)}\, b(x)\, dx$$

$$\boxed{y(x) = e^{A(x)} \int e^{-A(x)} b(x)\,dx}$$

$$\begin{cases} y' = \overset{a(x)}{\widetilde{2x}}\,y + e^{x^2}\underbrace{\cos x}_{b(x)} \\ y(0) = 1 \end{cases}$$

$a(x) = 2x$

$b(x) = e^{x^2}\cos x$

$A(x) = x^2$

$$y(x) = e^{x^2} \int \overbrace{e^{-x^2} e^{x^2}}^{1} \cos x\,dx$$

$$= e^{x^2} \int \cos x\,dx = e^{x^2}(\sin x + c)$$

$c \in \mathbb{R}$

$$1 = y(0) = c \iff \boxed{c = 1}$$

$$y = e^{x^2}(\sin x + 1)$$

$$y' = \overset{a(x)}{\widetilde{(\cos x)}}\,y + e^{\sin x}\overbrace{\log x}^{b(x)}$$

$A(x) = \sin x$

$$y = e^{\sin x} \int \underbrace{e^{-\sin x} \cdot e^{\sin x}}_{=1} \log x\,dx$$

$$= e^{\sin x} \int 1 \cdot \log x \, dx = \quad \text{(P. PARTI)}$$

$$= e^{\sin x} \left[x \log x - x + c \right]$$

$\boxed{y(1)=2}$ $$2 = y(1) = e^{\sin 1} \left[-1 + c \right]$$

$$c - 1 = 2e^{-\sin 1} \iff c = 1 + 2e^{-\sin 1}$$

$$y = e^{\sin x} \left[x \log x - x + 1 + 2e^{-\sin 1} \right]$$

$$\begin{cases} y' = (x+1)\dfrac{y}{x} + x(1-x) = \overbrace{\dfrac{x+1}{x}}^{a(x)} y + \overbrace{x(1-x)}^{b(x)} \\[2ex] y(1) = e \qquad\qquad x > 0 \end{cases}$$

$$A(x) = \int \frac{x+1}{x} \, dx = \int \left(1 + \frac{1}{x}\right) dx = x + \underbrace{\log x}\, + c$$

$$y = x e^{x} \int \cancel{x}(1-x) e^{-x} \cdot \underbrace{e^{-\log x}}_{\frac{1}{\cancel{x}}} \, dx$$

$$= x e^{x} \int (1-x) e^{-x} \, dx = x e^{x} \left[x e^{-x} + C \right]$$

$$e = y(1) = e \left[e^{-1} + C \right] = 1 + Ce \iff \boxed{C = \frac{e-1}{e}}$$

$$\int (1-x)e^{-x}\,dx = -e^{-x} - \int x e^{-x}\,dx$$

$$-\int x e^{-x}\,dx = \int x\,(-e^{-x})\,dx = x e^{-x} - \int e^{-x}\,dx \quad (*)$$

$$f(x) = x \qquad g'(x) = -e^{-x}$$

$$f'(x) = 1 \qquad g(x) = e^{-x}$$

$$\overset{(*)}{=} \; x e^{-x} + e^{-x} + c$$

$$\int (1-x)\,e^{-x}\,dx = \cancel{-e^{-x}} + x e^{-x} \cancel{+e^{-x}} + c$$

SOLUZIONE: $y = x e^{x}\left[x e^{-x} + \frac{e-1}{e}\right]$

$$y(1) = e\left[e^{-1} + \frac{e-1}{e}\right] =$$

$$= \cancel{1} + e - \cancel{1} = e$$

EQUAZIONI NONLINEARI $\qquad y' = \underbrace{x^3}_{p(x)}\,\underbrace{y^4}_{q(y)}$

A VARIABILI SEPARABILI

$$y' = f(x)\, g(y) \qquad (h(x,y))$$

$$y' = y^2 = \underbrace{1}_{f(x)} \cdot \underbrace{y^2}_{g(y)} \qquad y' = \log x \cdot \frac{1}{\sin y}$$

$$y' = a(x)\, y \qquad \int \frac{dy}{y} = \int a(x)\, dx$$

$$y' = f(x)\, g(y)$$

$$\frac{dy}{dx} = f(x)\, g(y) \quad : \quad \boxed{g(y) \neq 0}$$

$$\underbrace{\int \frac{dy}{g(y)}}_{G(y)} = \int f(x)\, dx$$

$$G(y) = F(x) + c$$

$1/g(y)$ $\quad f(x)$

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

$y \neq 0$

$$\frac{dy}{dx} = y^2 \quad ; \int \frac{dy}{y^2} = \int dx$$

$y = 0$ É SOLUZIONE!! ⟶ INTEGRALE SINGOLARE

$$-\frac{1}{y} = x + c \quad , \quad c \in \mathbb{R}$$

$$\frac{1}{y} = c - x \iff y = \frac{1}{c-x}$$

$c \in \mathbb{R}$

$$1 = y(0) = \frac{1}{c} \iff \boxed{c = 1}$$

$$y = \frac{1}{1-x}$$

$\underline{\underline{y(0)=0}}$: l'unica è l'integrale singolare

$\underline{y = 0}$

$y' = 1 \cdot (1 + y^2)$ $\qquad f(x) = 1, \quad g(y) = 1 + y^2$

$\dfrac{dy}{dx} = 1 + y^2$ $\quad ; \quad 1 + y^2 \neq 0$ SEMPRE!

$\displaystyle\int \frac{dy}{1+y^2} = \int dx \quad : \quad \operatorname{arctg} y = \overbrace{x + c}^{\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

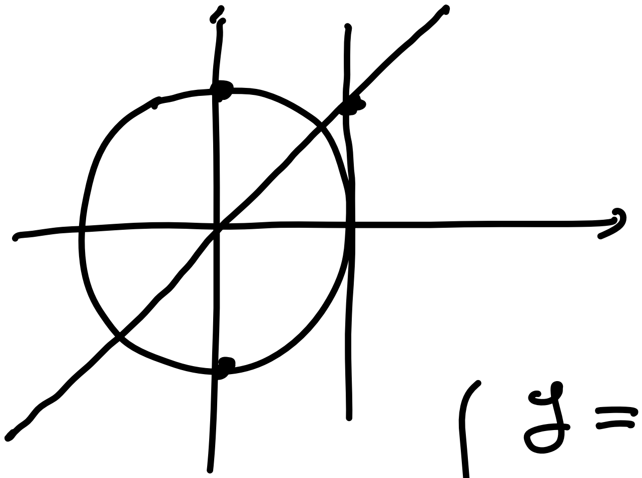
$y = \operatorname{tg}(x + c), \; c \in \mathbb{R}$

$y' = \cos^2 y \quad ; \quad \dfrac{dy}{dx} = \cos^2 y$

$\underline{\cos y \neq 0}$

$\displaystyle\int \frac{dy}{\cos^2 y} = \int dx$

$\left[\operatorname{tg} y = x + c, \quad c \in \mathbb{R}\right]$



$y =$

$c \in \mathbb{R}$

$$\cos y = 0 \iff \begin{cases} y = \operatorname{arctg}(x+c) + k\pi \quad k \in \mathbb{Z} \\ y = \underbrace{\frac{\pi}{2} + k\pi}, \quad k \in \mathbb{Z} \end{cases}$$

SOLUZIONI!

$y' = 0 \quad ; \quad \cos^2 y = 0$

$y' = f(x, y)$

$(\operatorname{tg} x)\, y' = y \qquad x \in \left(0, \frac{\pi}{2}\right)$

LINEARE

$y' = \frac{y}{\operatorname{tg} x} \;:\; \underbrace{y \neq 0}$

$y = 0$

$\in$ SOLUZIONE

$$\int \frac{dy}{y} = \int \frac{1}{\operatorname{tg} x}\, dx$$

$$\log|y| = \int \frac{\cos x}{\sin x}\,dx = \log|\sin x| + c$$

$$e^{\cdot} \quad \cdot\ e^{c}$$

$$|y| = e^{c}\,|\sin x|, \qquad c \in \mathbb{R}$$

$$e^{c}\cdot|\sin x|$$

$$y = \underbrace{(\pm e^{c})}_{k}\,\sin x$$

$$kc$$

$$y = k\sin x$$

$$\underline{\underline{k=0}}\ !!$$

$$xy' = \operatorname{tg} y \quad : \quad y' = \frac{1}{x}\operatorname{tg} y$$

$$\operatorname{tg} y \neq 0$$

$$\frac{dy}{\operatorname{tg} y} = \frac{1}{x}\,dx$$

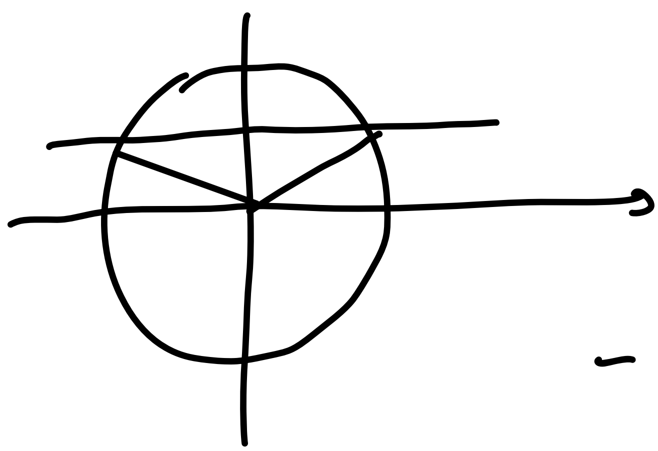
$$\int \frac{\cos y}{\sin y} = \int \frac{dx}{x} = \log|x| + c$$

$$\log|\sin y| = \log|x| + c$$

$$\boxed{\sin y = kx, \quad k \in \mathbb{R}}$$

$k = 0 \longrightarrow \sin y = 0 \iff y = k\pi$

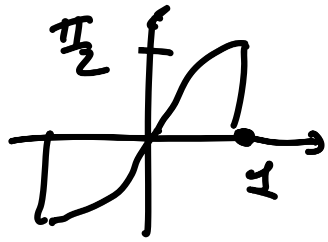
$\operatorname{tg} y = 0 \iff y = 0 + k\pi = k\pi, \ k \in \mathbb{Z}$



$-1 \leq kx \leq 1$

$$\begin{cases} y = \arcsin(\widetilde{kx}) + 2k\pi \\ y = \pi - \arcsin(kx) + 2k\pi \end{cases}$$

$$\begin{cases} y' = \dfrac{\operatorname{tg} y}{x} \\ y(1) = \dfrac{\pi}{2} \end{cases}$$



$$\sin y = kx \qquad , k \in \mathbb{R}$$

$$1 = \sin \underbrace{y(1)}_{y(1)=\frac{\pi}{2}} = k \iff \boxed{k = 1}$$

$$\sin y = 1 \cdot x = x$$

$$\iff \underbrace{y = \arcsin x}$$

$$\begin{cases} xy' = 1 + y^2 \\ y(1) = 1 \end{cases}$$