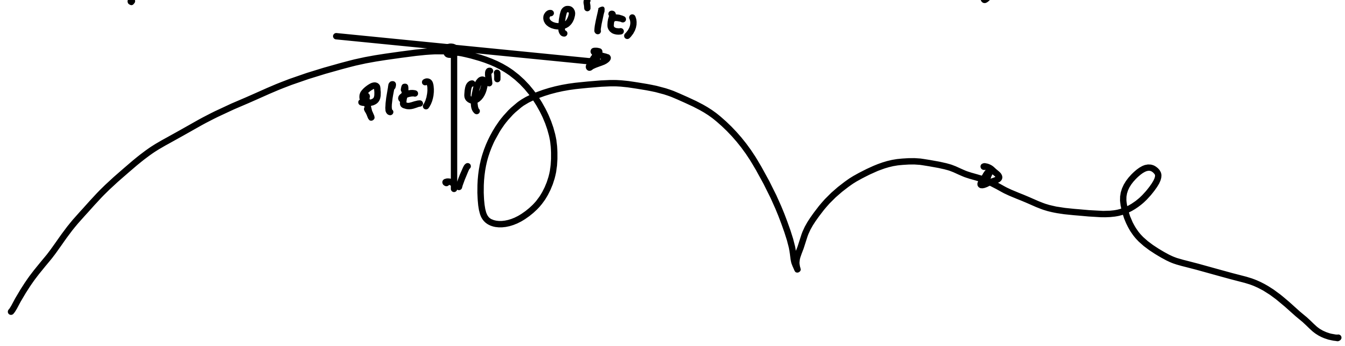
Lezioni del 30/11/23 - 01/12/23

Curve in $\mathbb{R}^n$

$$P(t) = (x(t), y(t), z(t)) \quad , t \in I$$

$$\varphi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3 \qquad = \varphi(t)$$



$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad t \in I \qquad \text{eq. parametriche delle traiettorie}$$

$$\varphi'(t) = \text{velocità} = (x'(t), y'(t), z'(t))$$
$$= \underline{v}(t)$$

<u>Uniforme</u> : $\qquad \|\varphi'(t)\|^2 = \text{costante}$

$$[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2 = \text{costante}$$

$$\frac{d}{dt}\left(\quad \text{---} \quad \right) = 0$$

$$x'(t)\,x''(t) + y'(t)\,y''(t) + z'(t)\,z''(t) = 0$$

$$\varphi''(t) = \underline{a}(t) = \text{accelerazione} = (x''(t), y''(t), z''(t))$$

$$\varphi'(t) \cdot \varphi''(t) = 0$$

$$\Rightarrow \varphi''(t) \perp \varphi'(t)$$

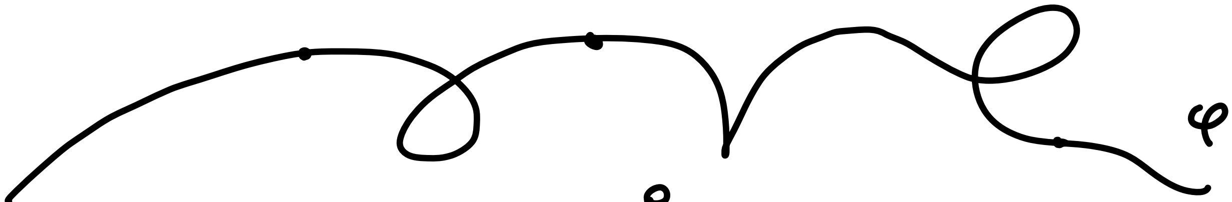
__Def.__ Una curva in $\mathbb{R}^n$ è un'applicazione continua

$$\varphi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$t \in I \longrightarrow \varphi(t) = (\varphi_1(t), \dots, \varphi_n(t)) \in \mathbb{R}^n$$

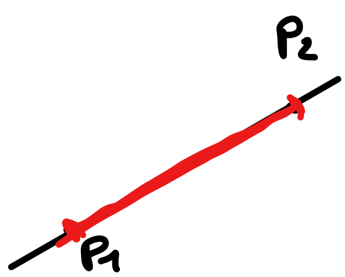
$$\begin{cases} x_1 = \varphi_1(t) \\ x_2 = \varphi_2(t) \\ \dots \\ x_n = \varphi_n(t) \end{cases}$$ equazioni parametriche di φ

$$\varphi(I) = \text{SOSTEGNO DI } \varphi \subseteq \mathbb{R}^n$$
$$= \{\varphi(t) : t \in I\}$$



$\varphi(I)$

__ESEMPI__



$P_1 = (x_1, y_1, z_1)$

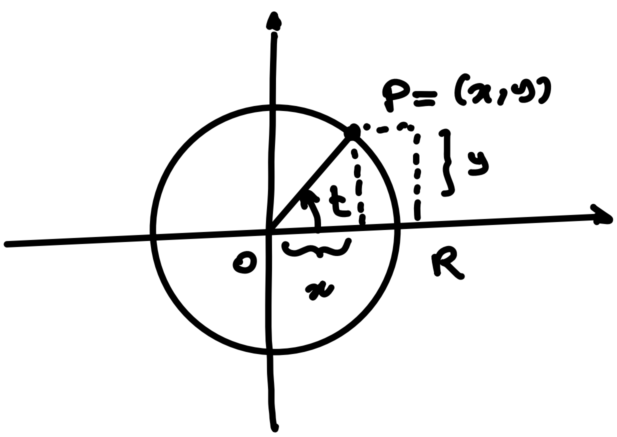
$P_2 = (x_2, y_2, z_2)$

$$\begin{cases} x = t x_2 + (1-t) x_1 \\ y = t y_2 + (1-t) y_1 \\ z = t z_2 + (1-t) z_1 \end{cases} \qquad t \in \mathbb{R}$$

$(t \in [0,1])$

$$\varphi(t) = \left(t x_2 + (1-t) x_1 ,\ t y_2 + (1-t) y_1 ,\ t z_2 + (1-t) z_1 \right)$$

CIRCONFERENZA



$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [0, 2\pi]$$

$$\varphi(t) = (R\cos t, R\sin t) \qquad t \in [0, 2\pi]$$

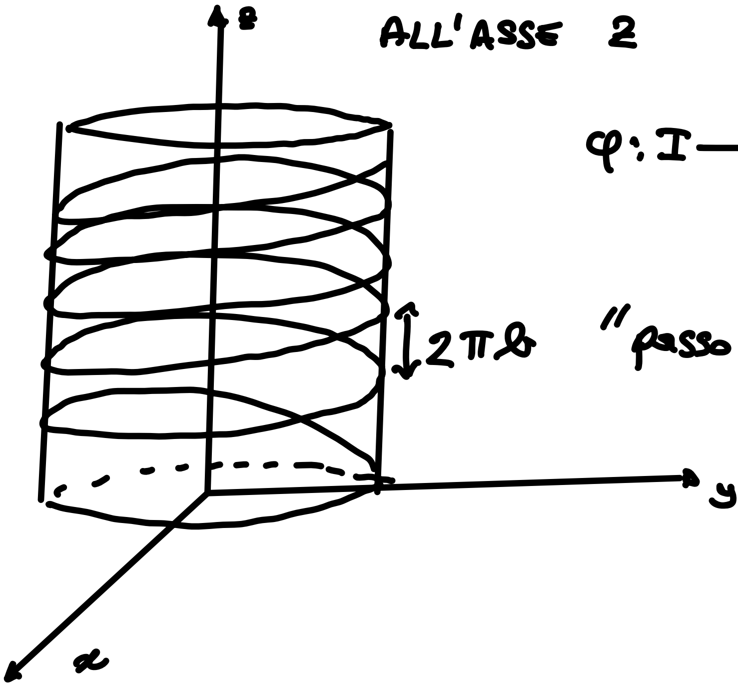
$$\psi(t) = (R\cos t, R\sin t) \qquad t \in [0, 4\pi]$$

due curve DIVERSE MA CON LO STESSO SOSTEGNO!

3) $\varphi(t)$ $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$ ELICA CILINDRICA $\quad a > 0 \quad b \neq 0 \quad t \in \mathbb{R}$

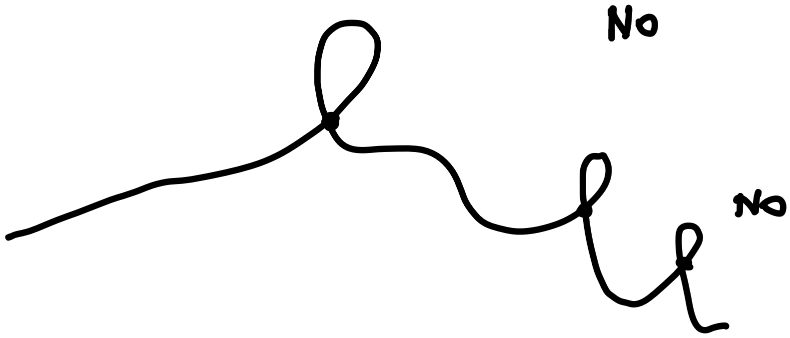
$$x^2 + y^2 = a^2$$ CILINDRO DI CURVA DIRETTRICE LA CIRCONF. E GENERATRICI PARALLELE

ALL'ASSE z



$\varphi : I \to \mathbb{R}^3$

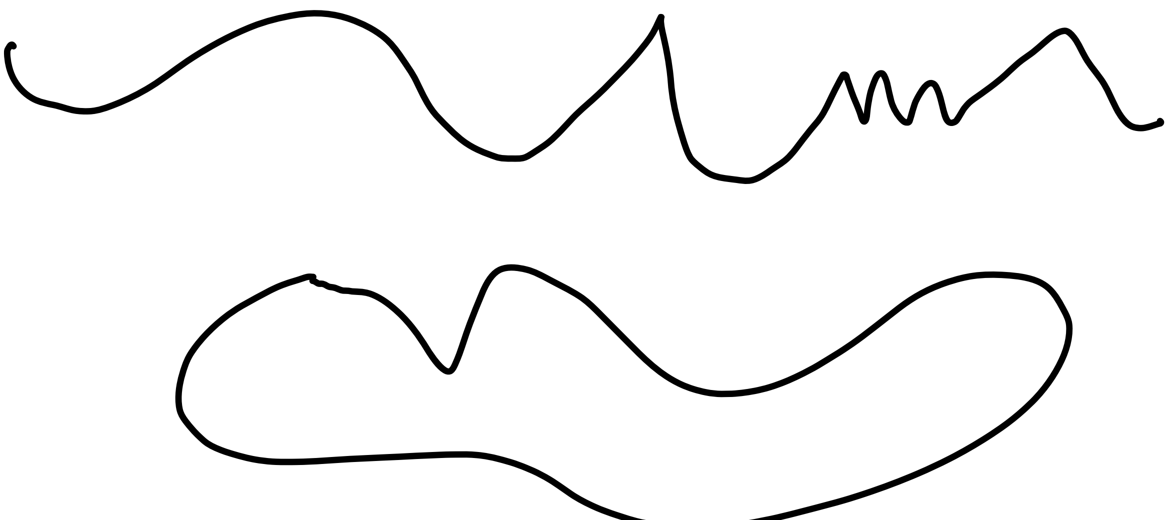
<u>Curva semplice</u>



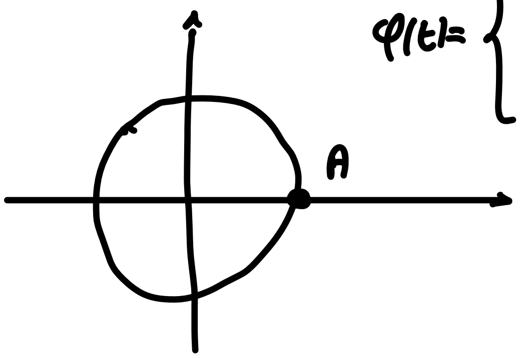
$\varphi : I \to \mathbb{R}^n$ semplice se

$\forall t_1, t_2 \in I$, $t_1 \neq t_2$ e "uno dei due interno ad I"

si ha $\quad \varphi(t_1) \neq \varphi(t_2)$.



Es. (RETTA)



$$\varphi(t) = \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [0, 2\pi]$$

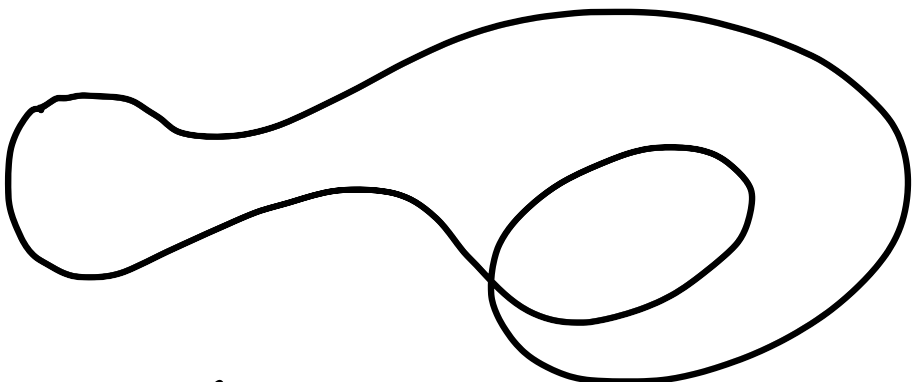
SI

$$\varphi(t) = \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [0, 4\pi]$$

NO

Def. (Curva chiusa) $\quad \varphi : [a, b] \longrightarrow \mathbb{R}^n$

chiusa se $\underline{\varphi(a) = \varphi(b)}$



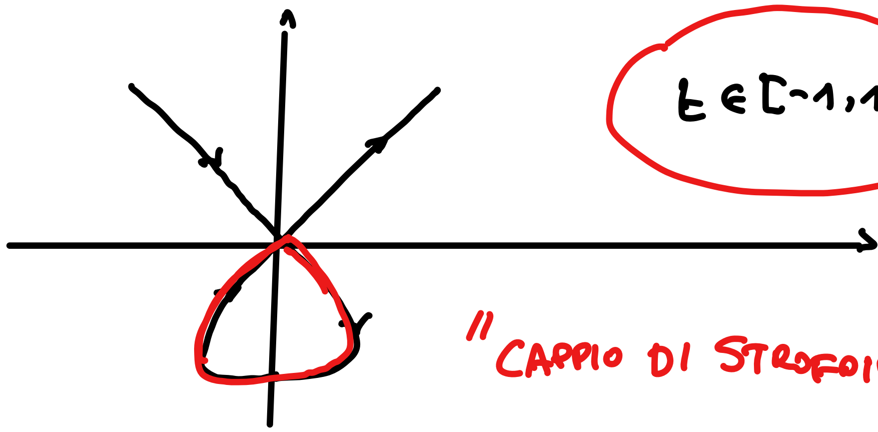
Es $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in \underline{[0, 2\pi]}$

$$\varphi(0) = (R, 0) = \varphi(2\pi)$$

"STROFOIDE"

Es. $\quad \varphi(t) = \begin{cases} x = t^3 - t \\ y = t^2 - 1 \end{cases} \qquad t \in \mathbb{R}$

$t = -1 : \quad (0, 0) \; ; \quad t = 1 : (0, 0)$



$t \in [-1,1]$

"CAPPIO DI STROFOIDE"

Curve regolari

$$\varphi : [a,b] \longrightarrow \mathbb{R}^2 \quad \text{"regolare"}$$

se

1. $\varphi \in C^1([a,b])$

$$\varphi(t) = (\underset{\substack{\Uparrow \\ C^1}}{\varphi_1(t)}, \underset{\substack{\Uparrow \\ C^1}}{\varphi_2(t)})$$

2. $\varphi'(t) \neq 0 \quad \forall t \in]a,b[$

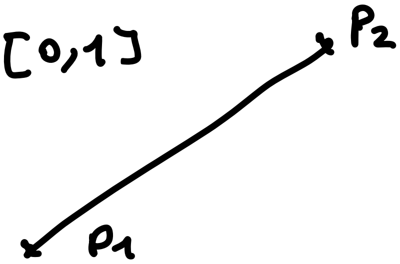
$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t))$$

$$\iff \|\varphi'(t)\| \underset{\neq}{>} 0$$

$$[\varphi_1'(t)]^2 + [\varphi_2'(t)]^2 \underset{\neq}{>} 0$$

ES.

$$\begin{cases} x = \overbrace{t x_2 + (1-t) x_1}^{\varphi_1} \\ y = \underbrace{t y_2 + (1-t) y_1}_{\varphi_2} \end{cases} \qquad t \in [0,1]$$



$\varphi_1'(t) = x_2 - x_1$; $\varphi_2' = y_2 - y_1$

$\varphi'(t) \neq 0$

CIRCONFERENZA

$$\varphi(t) = \begin{cases} x = \overbrace{R\cos t}^{\varphi_1} \\ y = \underbrace{R\sin t}_{\varphi_2} \end{cases} \qquad t \in [0, 2\pi]$$

$$\varphi_1' = -R\sin t, \quad \varphi_2' = R\cos t$$

$$[\varphi_1']^2 + [\varphi_2']^2 = R^2 > 0$$

ELICA

$$\begin{cases} x = a\cos t & \varphi_1 \\ y = a\sin t & \varphi_2 \\ z = bt & \varphi_3 \end{cases} \qquad t \in [0, 2\pi]$$

$$\varphi_3' = b \neq 0$$

<u>OSS.</u> La definizione di curva regolare si estende a

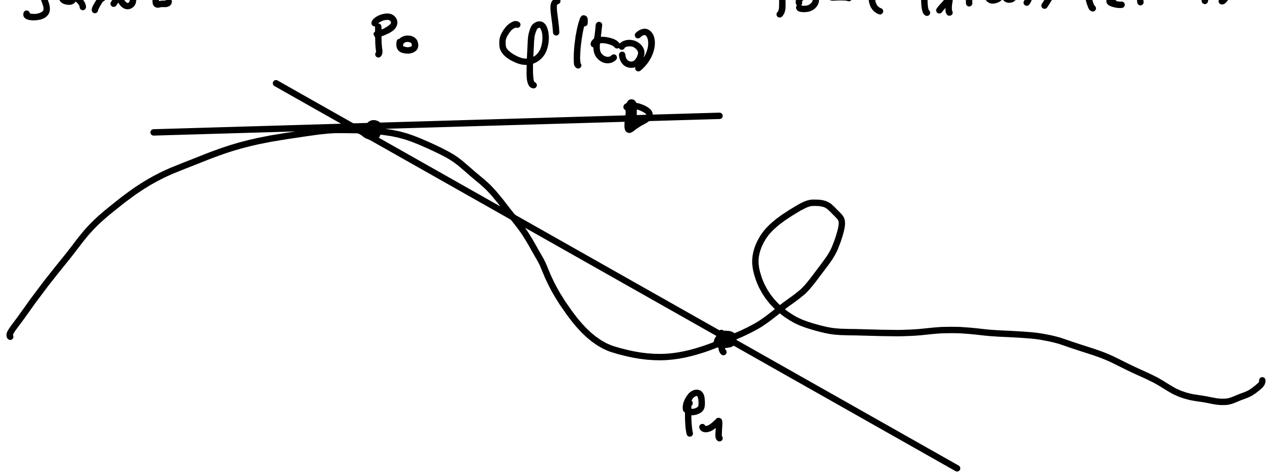
$$\mathbb{R}^3 \quad (\mathbb{R}^n)$$

$$\left([\varphi_1'(t)]^2 + [\varphi_2'(t)]^2 + [\varphi_3'(t)]^2 > 0 \right)$$

$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t), \varphi_3'(t)), \quad \forall t \in (a, b)$$

$\varphi : [a,b] \longrightarrow \mathbb{R}^2$ regolare ; $P_0 = \varphi(t_0)$

$t_0 \in \,]a,b[$ $\quad P_0 = (\varphi_1(t_0), \varphi_2(t_0))$



$t_1 \to t_0$??

$P_1 = \varphi(t_1)$
$= (\varphi_1(t_1), \varphi_2(t_1))$

$$\overline{P_0P_1} \quad \begin{cases} x = \varphi_1(t_0) + \dfrac{\varphi_1(t_1) - \varphi_1(t_0)}{t_1 - t_0}(t - t_0) \\[2ex] y = \varphi_2(t_0) + \dfrac{\varphi_2(t_1) - \varphi_2(t_0)}{t_1 - t_0}(t - t_0) \end{cases}$$

$$t_1 \to t_0 : \quad \begin{cases} x = \varphi_1(t_0) + \varphi_1'(t_0)(t - t_0) \\ y = \varphi_2(t_0) + \varphi_2'(t_0)(t - t_0) \end{cases}$$

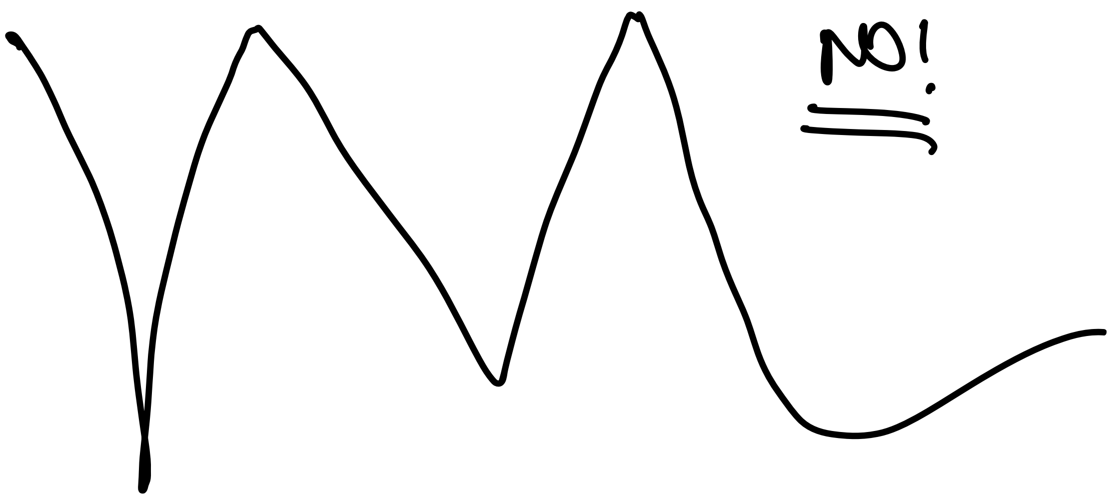
[EQUAZIONI PARAMETRICHE DELLA RETTA TANGENTE A φ in P_0]

$$\varphi'(t_0) = (\varphi_1'(t_0), \varphi_2'(t_0))$$

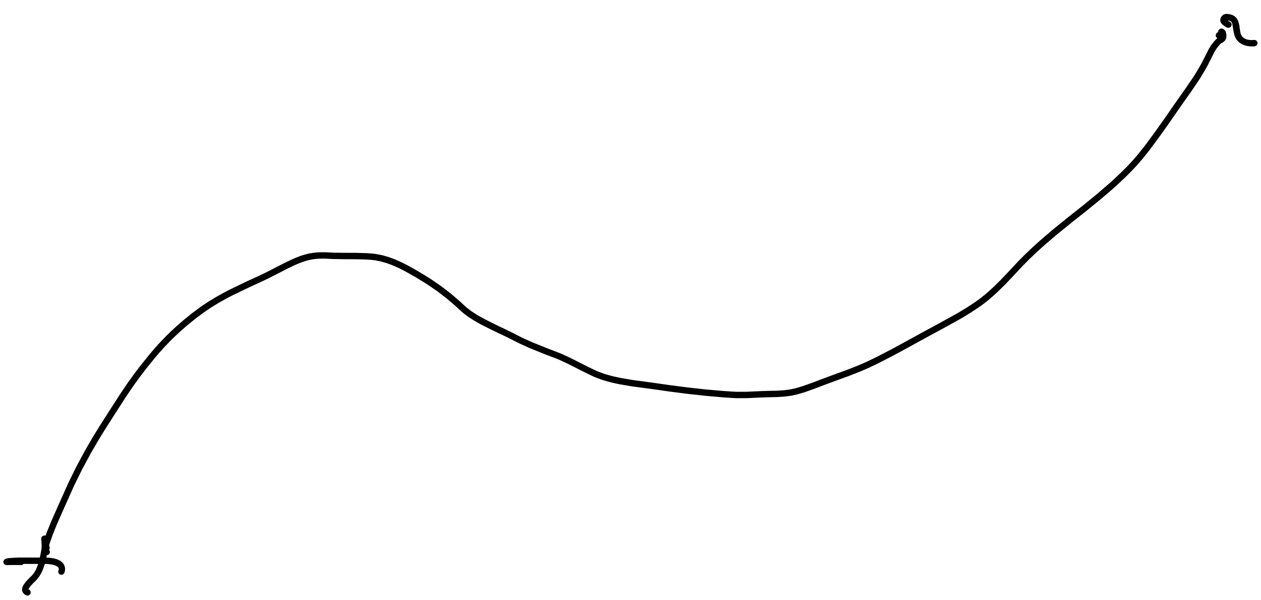
VETTORE TANGENTE A φ in P_0

$$T(t_0) = \frac{\varphi'(t_0)}{\|\varphi'(t_0)\|}$$

versore tangente



NO!

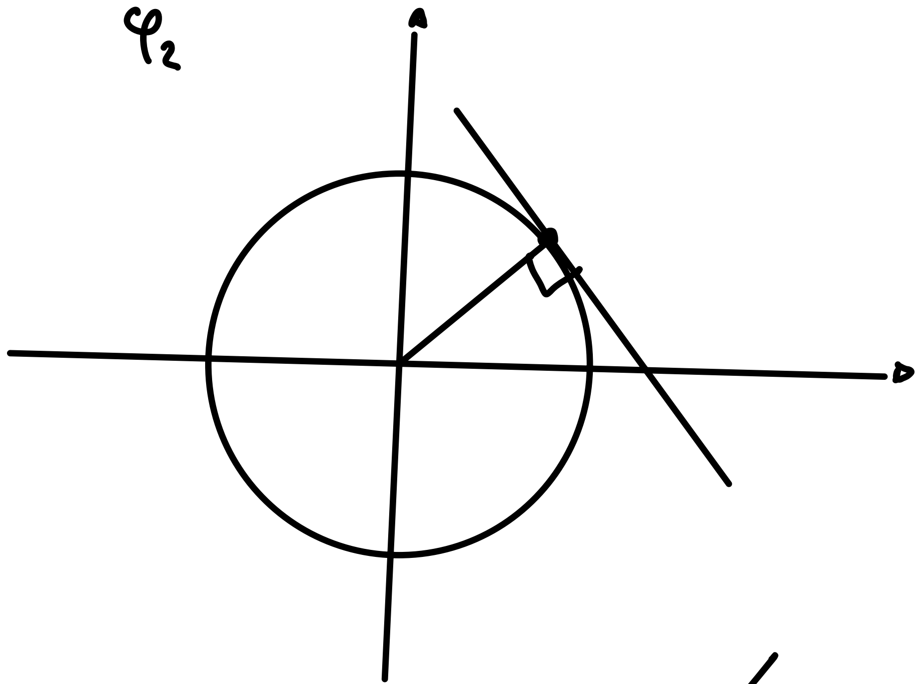


ES

$$\begin{cases} x = \underbrace{\cos t}_{\varphi_1} \\ y = \underbrace{\sin t}_{\varphi_2} \end{cases} \qquad t \in [0, 2\pi]$$

$$\underbrace{t_0 = \frac{\pi}{4}}$$

$$\varphi_0 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



$$\varphi_1' = -\sin t$$

$$\varphi_2' = \cos t$$

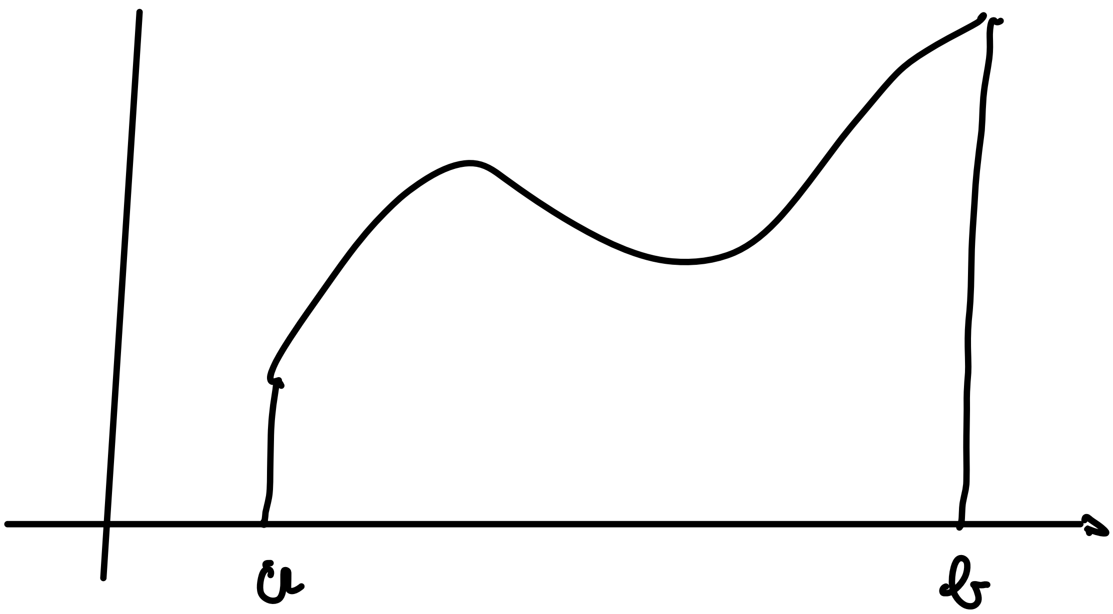
$$\varphi_1'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\varphi_2'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$+ \begin{cases} x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(t - \overset{t_0}{\overset{\parallel}{\frac{\pi}{4}}}\right) \\ y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(t - \frac{\pi}{4}\right) \end{cases}$$

$x + y = \sqrt{2}$

ES_ $f = f(x)$ $f \in C^1([a,b])$



$y = f(x)$ "EQ. CARTESIANA GRAFICO DI $f(x)$"

$$\varphi(t) = \begin{cases} x = t \\ y = \underbrace{f(t)}_{\varphi_2(t)} \end{cases} \qquad t \in [a,b]$$

$$\varphi_1'(t) = 1 \neq 0$$

φ curva che ha per SOSTEGNO IL GRAFICO DI $f(x)$

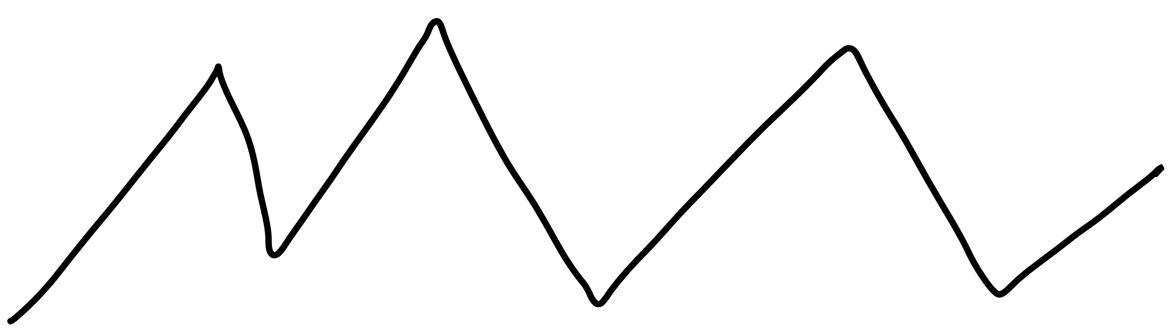
$x_0 \in [a,b]$: $y = f(x_0) + f'(x_0)(x - x_0)$

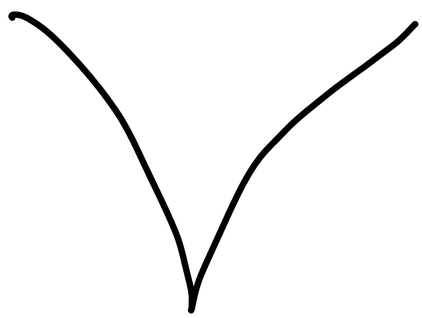
$t_0 = x_0$

$$:\begin{cases} x = t_0 + (t - t_0) = t \\ y = \varphi_2(t_0) + \varphi_2'(t_0)(t - t_0) \\ \quad = f(t_0) + f'(t_0)(t - t_0) \end{cases}$$

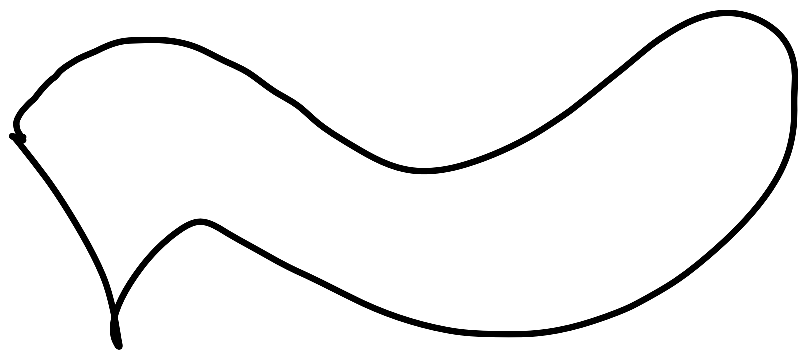
$$y = f(x_0) + f'(x_0)(x - x_0)$$

Def.





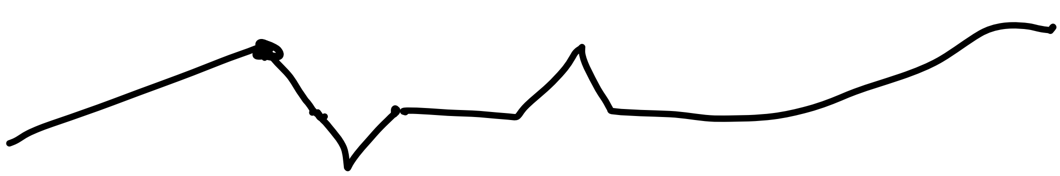
"GENERALMENTE REGOLARI"



OSS. Se $\varphi : [a,b] \longrightarrow \mathbb{R}^3$ regolare : $\varphi'(t) \neq 0 \quad \forall t \in]a,b[$.

EQ. RETTA TANGENTE

$$\begin{cases} x = \varphi_1(t_0) + \varphi_1'(t_0)(t-t_0) \\ y = \varphi_2(t_0) + \varphi_2'(t_0)(t-t_0) \\ z = \varphi_3(t_0) + \varphi_3'(t_0)(t-t_0) \end{cases}$$



Curve piane

$$f(x,y) = 0$$

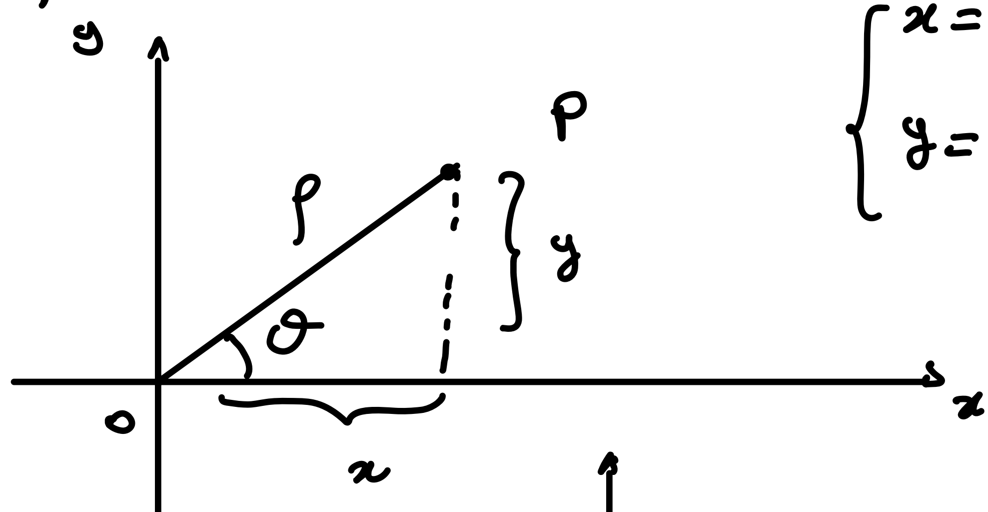
(ES. $\underbrace{x^2+y^2}_{f(x,y)} - 1 = 0$ ✓)

$$\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \end{cases} \quad t \in [a,b]$$

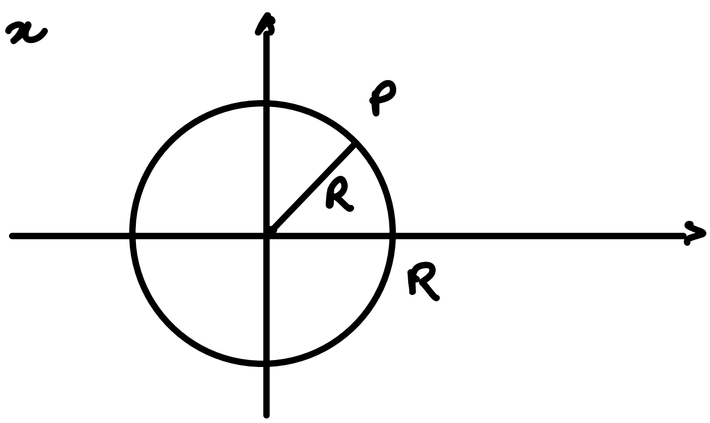
RAPP. PARAMETRICA

(ES. $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $t \in [0, 2\pi]$)

Rappresentazione polare di una curva piana



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

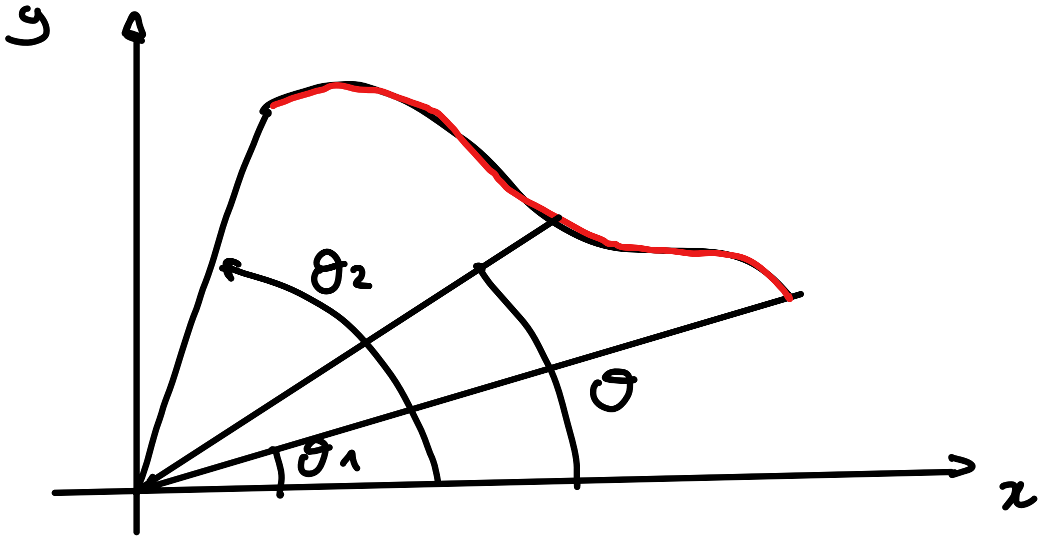


$\rho = R$ equazione polare della circonferenza

$$x^2 + y^2 = R^2$$

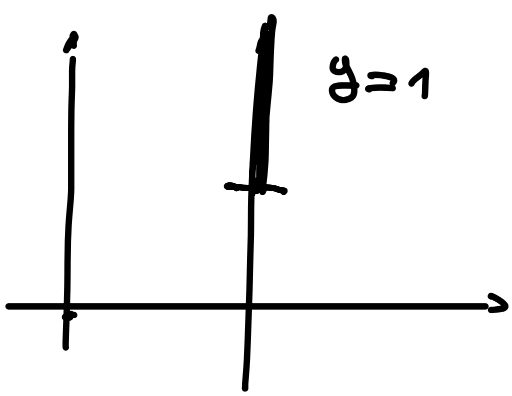
$$\rho^2 \cos^2\vartheta + \rho^2 \sin^2\vartheta = R^2$$

$$\rho^2 = R^2 \iff \boxed{\rho = R}$$



$\longrightarrow \quad \rho = \rho(\vartheta) \quad , \quad \vartheta \in [\vartheta_1, \vartheta_2]$

EQUAZIONE POLARE DI UNA CURVA PIANA



$$\rho \sin\vartheta = 1$$

$$\rho = \frac{1}{\sin\vartheta}$$

$$\rho = \rho(\vartheta)$$

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \end{cases}$$

$$\begin{cases} x = \rho(\vartheta)\cos\vartheta \\ y = \rho(\vartheta)\sin\vartheta \end{cases} \qquad \vartheta \in [\vartheta_1, \vartheta_2]$$

RAPP. PARAMETRICA

$$\rho \in C^1([\vartheta_1, \vartheta_2])$$

$$\begin{cases} x' = \rho'\cos\vartheta - \rho\sin\vartheta \\ y' = \rho'\sin\vartheta + \rho\cos\vartheta \end{cases}$$

$$\begin{cases} (x')^2 = (\rho')^2\cos^2\vartheta + \rho^2\sin^2\vartheta - 2\rho\rho'\sin\vartheta\cos\vartheta \\ (y')^2 = (\rho')^2\sin^2\vartheta + \rho^2\cos^2\vartheta + 2\rho\rho'\sin\vartheta\cos\vartheta \end{cases}$$

$$[x'(\vartheta)]^2 + [y'(\vartheta)]^2 = \rho^2(\vartheta) + [\rho'(\vartheta)]^2$$

Curva è regolare $\Longleftrightarrow$ $[\rho^2(\vartheta)] + [\rho'(\vartheta)]^2 > 0$

$\forall \vartheta \in]\vartheta_1, \vartheta_2[$

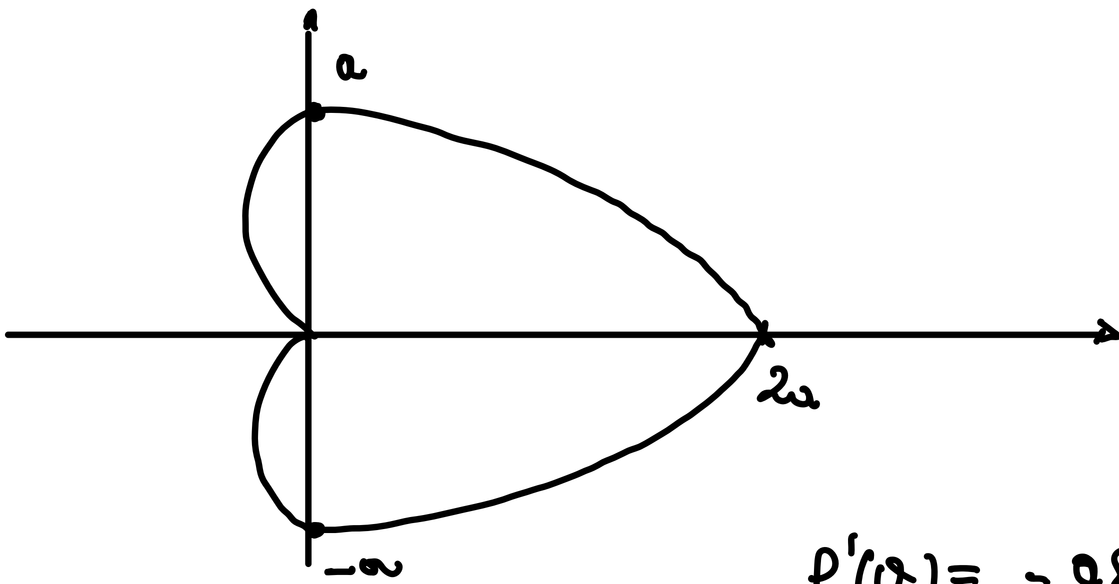
<u>ES.</u> $\rho = a(1+\cos\vartheta)$

$\vartheta = \frac{\pi}{2}$

$\vartheta \in [0, 2\pi]$

$\vartheta = \pi$

<u>CARDIOIDE</u>



$\rho'(\vartheta) = -a\sin\vartheta$

$\rho^2 = a^2(1+\cos^2\vartheta + 2\cos\vartheta)$

$\rho^2 + (\rho')^2 = a^2[1+\cos^2\vartheta + 2\cos\vartheta + \sin^2\vartheta]$

$= 2a^2(1+\cos\vartheta)$

$= 0$ quando $\underline{\vartheta = \pi}$

NO REGOLARE!

$\rho = a(1+\cos\theta) \qquad \theta \in [-\pi, \pi]$

REGOLARE !

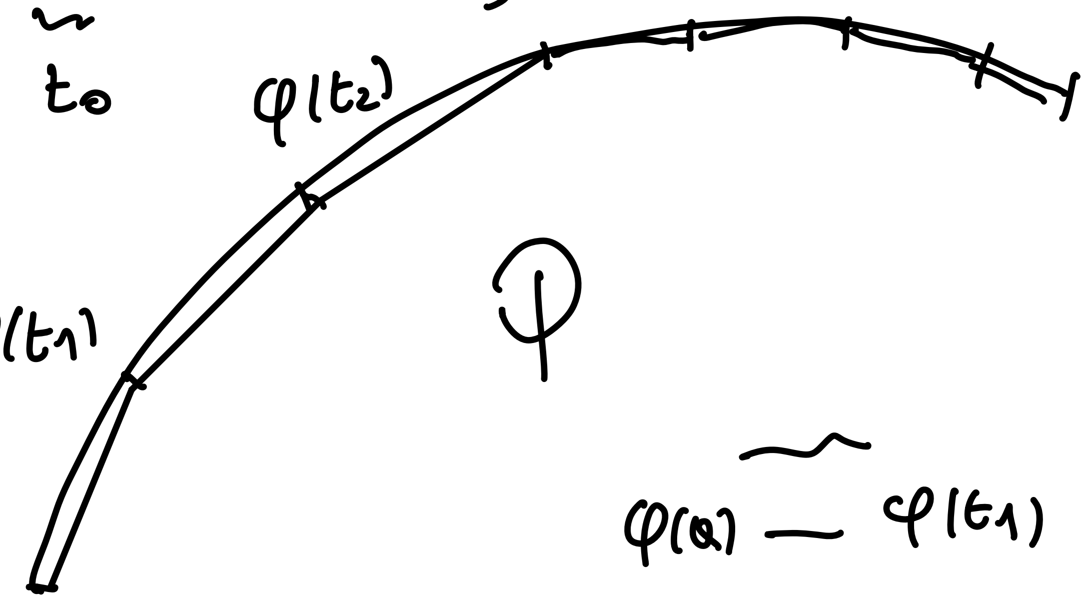
Lunghezza di un arco di curva

$$\varphi : [a,b] \longrightarrow \mathbb{R}^2 \quad (\mathbb{R}^3, \mathbb{R}^n)$$

$t_1 \quad t_2 \quad b = t_n$

$a \sim t_0 \quad t_3 \quad \dots$

$\varphi(t_{n-1})$



$\varphi(t_2)$

$\varphi(b) \sim \varphi(t_n)$

$\varphi(t_1)$

$\varphi(a)$

$\widetilde{\varphi(a) - \varphi(t_1)}$

$\varphi(t_1) - \varphi(t_2)$

$\dots$

$\varphi(t_{n-1}) - \varphi(t_n) = \varphi(b)$

$$\ell(P) = \|\varphi(t_1) - \varphi(t_0)\| + \|\varphi(t_2) - \varphi(t_1)\| + \dots + \|\varphi(t_N) - \varphi(t_{N-1})\|$$

$$= \sum_{i=1}^{N} \|\varphi(t_i) - \varphi(t_{i-1})\|$$

"Poligonale"

$$\boxed{L(\varphi) = \sup \left\{ \ell(P) : P \text{ inscritta a } \varphi \right\}}$$

Lunghezza di un arco di curva

$$L(\varphi) \in [0, +\infty]$$

$L(\varphi) < +\infty$: φ è rettificabile.

Teorema (RETTIFICABILITÀ DELLE CURVE DI CLASSE C^1)

$\varphi : [a,b] \longrightarrow \mathbb{R}^2$ $(\mathbb{R}^3)$ di classe C^1.

Allora φ è rettificabile e

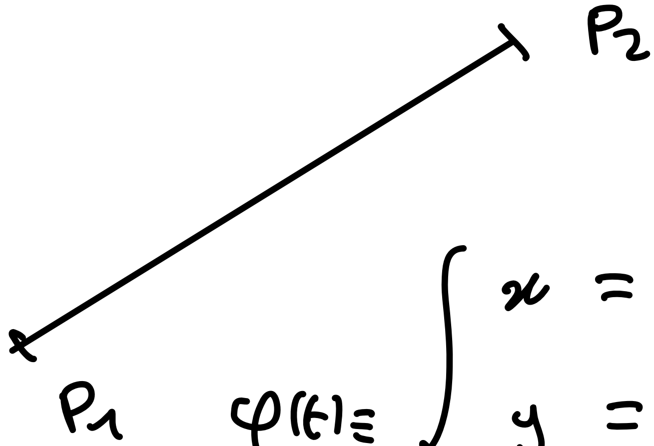
$$L(\varphi) = \int_a^b \|\varphi'(t)\| \, dt .$$

Se $\varphi(t) = (\varphi_1(t), \varphi_2(t))$

$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t))$$

$$L(\varphi) = \int_a^b \sqrt{[\varphi_1'(t)]^2 + [\varphi_2'(t)]^2} \, dt$$

ES. Segmento di estremi $P_1 = (x_1, y_1, z_1)$
$P_2 = (x_2, y_2, z_2)$



$$\varphi(t) \equiv \begin{cases} x = \overbrace{t x_2 + (1-t)x_1}^{\varphi_1} \\ y = t y_2 + (1-t) y_1 \;\}\, \varphi_2 \\ z = t z_2 + \underbrace{(1-t) z_1}_{\varphi_3} \end{cases}$$

$$t \in [\underset{a}{\underset{\shortparallel}{0}}, \underset{b}{\underset{\shortparallel}{1}}]$$

$$\varphi_1'(t) = x_2 - x_1$$

$$\varphi_2'(t) = y_2 - y_1$$

$$\varphi_3'(t) = z_2 - z_1$$

$$L(\varphi) = \int_0^1 \sqrt{\underbrace{(x_2-x_1)^2 + (y_2-y_1)^2}_{d(P_1,P_2)} + (z_2-z_1)^2}\, dt$$

$$= d(P_1, P_2).$$

ES.

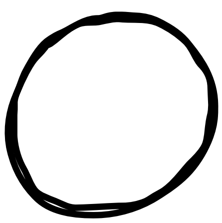
$$\varphi(t) = \begin{cases} x = \overbrace{R\cos t}^{\varphi_1(t)} \\ y = \underbrace{R\sin t}_{\varphi_2(t)} \end{cases} \qquad t \in [\underset{a}{0}, \overset{b}{2\pi}]$$

$$\varphi_1'(t) = -R\sin t, \quad \varphi_2'(t) = R\cos t$$

$$\sqrt{(\varphi_1')^2 + (\varphi_2')^2} = \sqrt{R^2} = R$$

$$L(\varphi) = \int_0^{2\pi} R\,dt = 2\pi R$$

Se $t \in [0, 4\pi]$: $L(\varphi) = 4\pi R$



OSS. Se φ è semplice, $L(\varphi) =$ lunghezza geometrica

ES

$$\varphi(t) = (\widetilde{\cos t}, \sin t, t), \quad t \in [0, 2\pi]$$

$$\varphi'(t) = (-\sin t, \cos t, 1)$$

$$\|\varphi'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

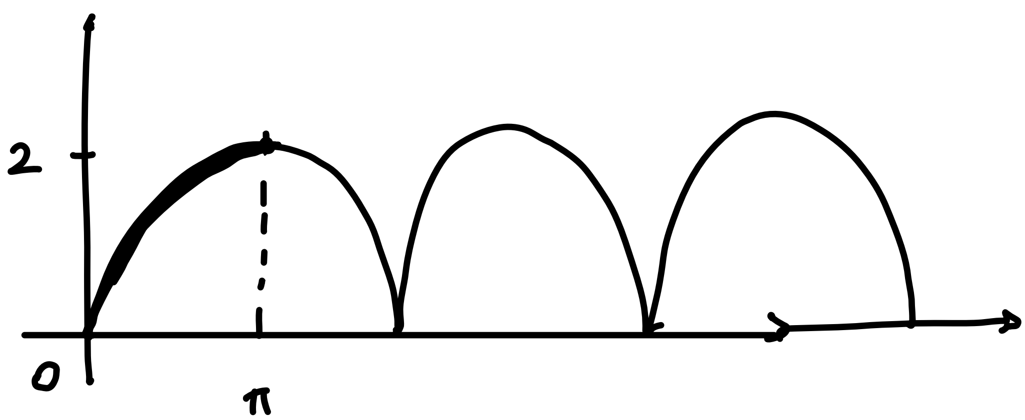
$$L(\varphi) = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

ES.

$$\varphi(t) = (t - \sin t, 1 - \cos t)$$

CICLOIDE

$$t \in [0, \pi]$$



$$\varphi'(t) = (1 - \cos t, \sin t)$$

$$\sqrt{(1-\cos t)^2+\sin^2 t} = \sqrt{2-2\cos t}$$

$a=0,\ b=\pi$ $$= \sqrt{2}\,\sqrt{1-\cos t}$$

$$L(\varphi) = \int_0^{\pi} \sqrt{2}\,\sqrt{1-\cos t}\,dt$$

$$\sin^2 \tfrac{t}{2} = \frac{1-\cos t}{2} \quad : \quad \sin \tfrac{t}{2} = \sqrt{\frac{1-\cos t}{2}}$$

$$\cos^2 \tfrac{t}{2} = \frac{1+\cos t}{2}$$

$$\downarrow \qquad \downarrow$$

$$\sqrt{2}\,\sin \tfrac{t}{2} = \sqrt{1-\cos t}$$

$$L(\varphi) = \int_0^{\pi} \sqrt{2}\cdot\sqrt{2}\,\sin \tfrac{t}{2}\,dt$$

$$= 2\int_0^{\pi} \sin \tfrac{t}{2}\,dt = 4\int_0^{\pi} \left(\sin \tfrac{t}{2}\right)\cdot \tfrac{1}{2}\,dt$$

$$= -a \left[\cos \frac{t}{2} \right]_0^{\pi} = a.$$

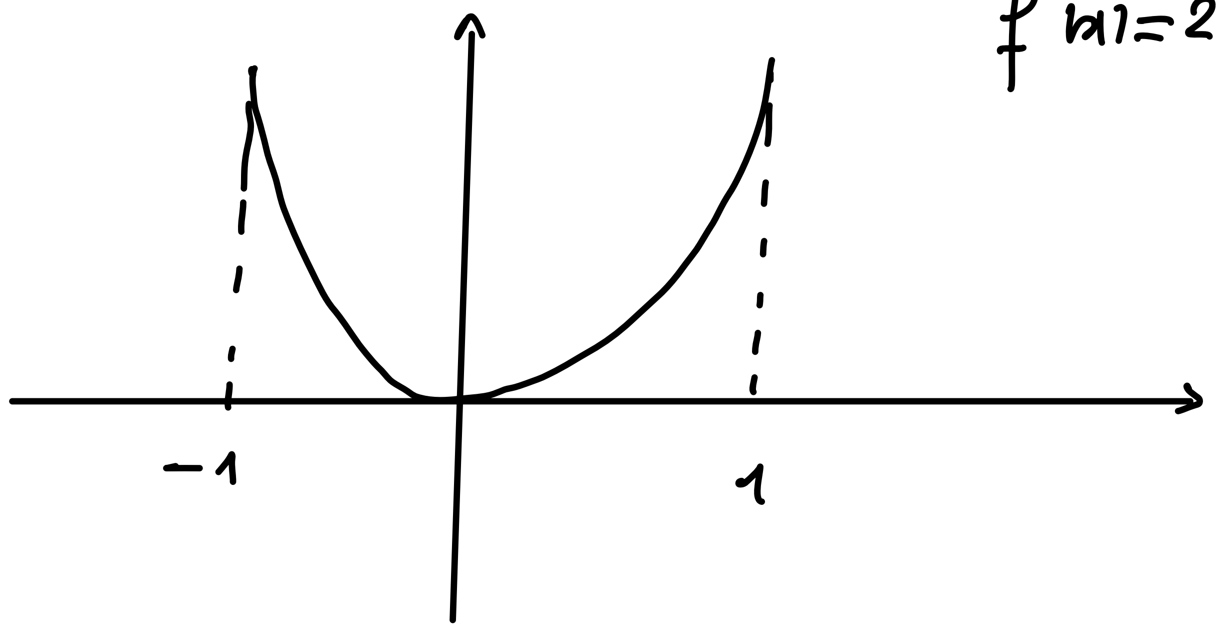
$y = f(x) \qquad f \in C^1([a,b])$

$$\begin{cases} x = t \\ y = f(t) \end{cases} \quad \underline{\underline{t \in [a,b]}} \quad \begin{cases} x' = 1 \\ y' = f'(t) \end{cases}$$

$$L(\varphi) = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

ES $\quad f(x)=x^2 \quad , \quad x\in[\overset{a}{-1},\overset{b}{1}]$

$f'(x)=2x$



$$L(\varphi)=\int_{-1}^{1}\sqrt{1+4x^2}\,dx$$

$t=2x$

$dt=2dx$

$dx=\frac{1}{2}dt$

$$=\int_{-1}^{1}\sqrt{1+(\underbrace{2x}_{t})^2}\,dx$$

$$=\frac{1}{2}\int_{-2}^{2}\sqrt{1+t^2}\,dt$$

$$\int\sqrt{1+t^2}\,dt=\int 1\cdot\sqrt{1+t^2}\,dt \qquad = (*)$$

$$f(t) = \sqrt{1+t^2} \qquad g'(t) = 1$$

$$f'(t) = \frac{t}{\sqrt{1+t^2}} \qquad g(t) = t$$

$$\overset{(*)}{=} t\sqrt{1+t^2} - \int \frac{(1+t^2)-1}{\sqrt{1+t^2}}\, dt$$

$$= t\sqrt{1+t^2} - \int \frac{1+t^2}{\sqrt{1+t^2}}\, dt + \int \frac{dt}{\sqrt{1+t^2}}$$

$$= t\sqrt{1+t^2} - \int \underbrace{\sqrt{1+t^2}}\, dt + \int \frac{dt}{\sqrt{1+t^2}}$$

?

$$\int \sqrt{1+t^2}\, dt = \frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$\rho = \rho(\theta) \quad , \quad \theta \in [\theta_1, \theta_2]$

$$L(\varphi) \overset{?}{=} \int_{\theta_1}^{\theta_2} \sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}\, d\theta$$

$$[x'(\theta)]^2 + [y'(\theta)]^2 = [\rho^2(\theta)] + [\rho'(\theta)]^2$$

ES. Calcolare la lunghezza della cardioide: $\rho = a(1+\cos\theta)$, $\theta \in [-\pi, \pi]$.