Lezioni del 10 e 12/10/2023

$$\odot \sum_{n=0}^{\infty} a_n (x-x_0)^n = f(x) \qquad \forall x \in \left]x_0-\rho, x_0+\rho\right[$$

$\{a_n\} \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$ "punto iniziale"

ρ = raggio di convergenza >0

$$\sum_{n=0}^{\infty} x^n = \underbrace{\frac{1}{1-x}}_{f(x)}, \qquad \forall x \in \left]-1, 1\right[$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

$$\frac{d}{dx}(\dots) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots + n a_n (x-x_0)^{n-1} + \dots$$

$$\left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)' = f'(x) = \sum_{n=1}^{\infty} \underbrace{n a_n (x-x_0)^{n-1}}_{\text{serie derivata}}$$

$$\int_{x_0}^{x} '' \, dt = a_0 (x-x_0) + \frac{a_1}{2}(x-x_0)^2 + \frac{a_2}{3}(x-x_0)^3 + \dots + \frac{a_n}{n+1}(x-x_0)^{n+1} + \dots$$

$$\int_{x_0}^{x} f(t)\,dt = \underbrace{\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}}_{\text{serie integrale}}$$

x_0

$$\int_{x_0}^{x_0} a_1 (t-x_0)\,dt = \frac{a_1}{2}(x-x_0)^2$$

$$\int_{x_0}^{x} a_2 (t-x_0)^2\,dt = \frac{a_2}{3}(x-x_0)^3$$

Teorema di derivazione ed integrazione termine a termine

Se $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \forall x \in \,]x_0-\rho, x_0+\rho[$, $\rho > 0$

allora le serie derivata ed integrale hanno raggio di convergenza ρ ed inoltre

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\int_{x_0}^{x} f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}$$

$$\forall x \in \,]x_0 - \rho, x_0 + \rho[$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + n x^{n-1}$$

$\forall x \in]-1,1[\qquad = \sum_{n=1}^{\infty} n x^{n-1}$

$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots + n a_n (x-x_0)^{n-1} + \dots$

$2a_2 + 3\cdot 2\, a_3(x-x_0) + \dots + n(n-1)a_n(x-x_0)^{n-2} + \dots$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

Applicando il teorema di derivazione termine a termine, si ha che $f(x)$

è derivabile infinite volte e si ha

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots\cdots(n-k+1)\cdot a_n (x-x_0)^{n-k}$$

$x = x_0$

$$f^{(k)}(x_0) = \overbrace{k(k-1)(k-2)\cdots\cdots(k-k+1)}^{k!} \cdot a_k = k!\, a_k$$

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

serie di Taylor di p.to iniziale x_0

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$$+ \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots$$

$$\nearrow \quad o\left((x-x_0)^n\right)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$f \in C^{\infty}(I)$$

$$f(x) \overset{??}{\underset{??}{=}} \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}}(x-x_0)^n$$

in un intorno di x_0

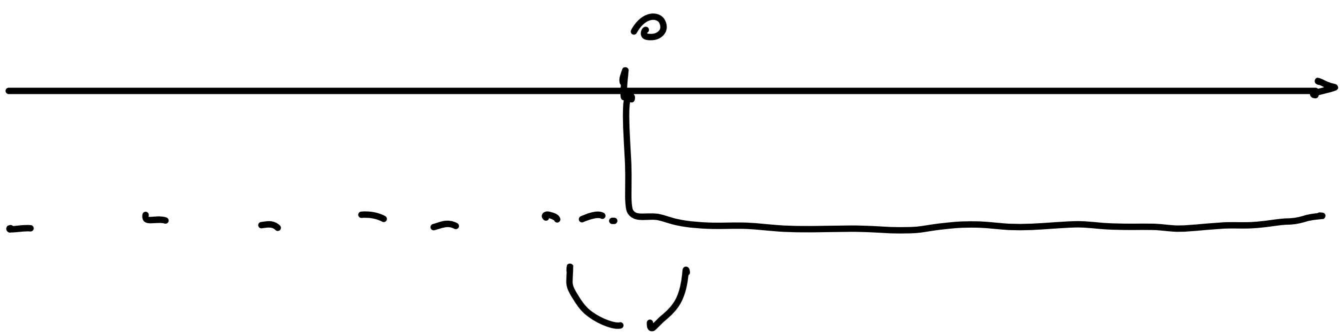
serie di Taylor di $f(x)$, di punto iniziale x_0

$$f(x) = \begin{cases} e^{-1/x^2} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-1/x^2} = 0 = f(0)$$

$$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0$$

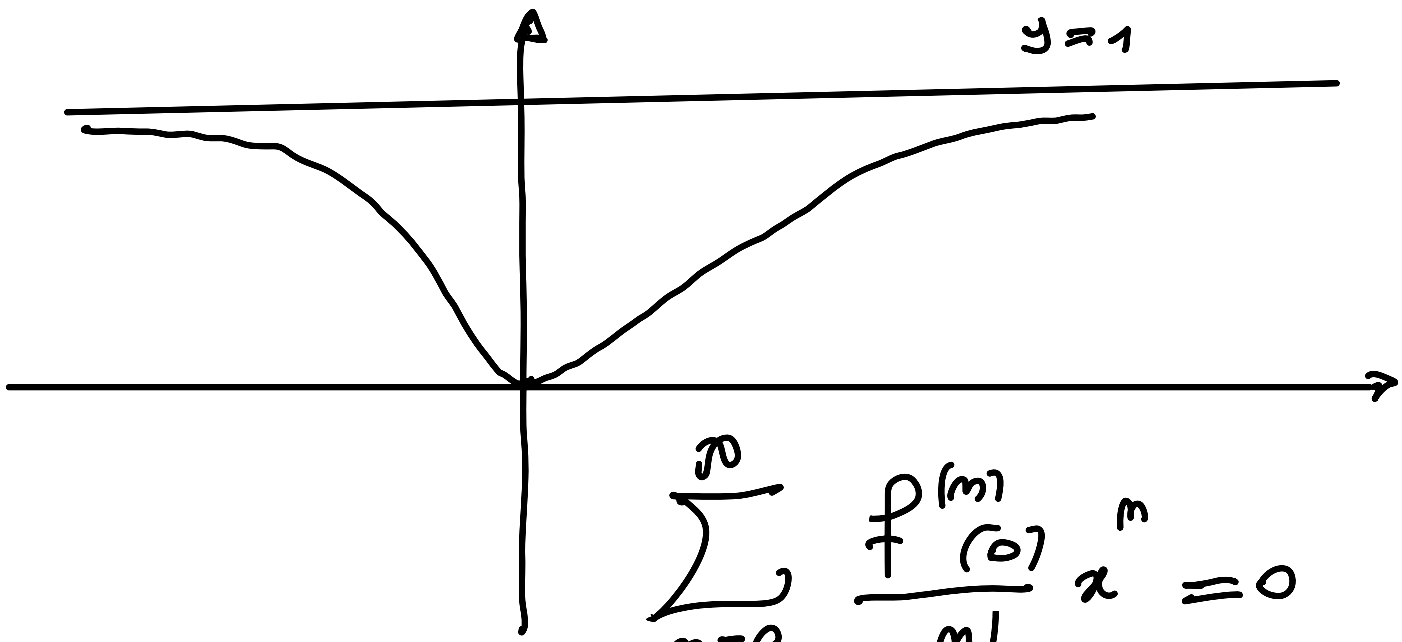
$$x \neq 0 : \quad f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3} \gtrless 0 \iff x \gtrless 0$$



$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} e^{-1/x^2} = 1$$

$y = 1$ asintoto orizzontale

$$f^{(n)}(0) = 0 \quad \forall n \geq 0$$



$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

$$\neq f(x)$$

<u>Condizione sufficiente per la sviluppabilità in serie di Taylor</u>

$f \in C^{\infty}(I)$, $x_0 \in I$ e si abbia $\forall x \in I$, $\forall n \in \mathbb{N}$

$$|f^{(n)}(x)| \leq M h^n$$

$M, h > 0$ costanti

In particolare, $h = 1$ può essere!

$$|f^{(n)}(x)| \leq M$$

Allora, $f(x)$ è sviluppabile in serie in I

e $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall x \in I$$

ES. (1) $f(x) = e^x$ $\quad x_0 = 0$

$f^{(n)}(x) = e^x \qquad \sum \frac{f^{(n)}(0)}{n!} x^n$

$f^{(n)}(0) = e^0 = 1 \quad \forall n \in \mathbb{N}_0$

$$e^x \overset{?}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad : \quad \overbrace{[-\delta, \delta]}^{I}, \delta > 0$$

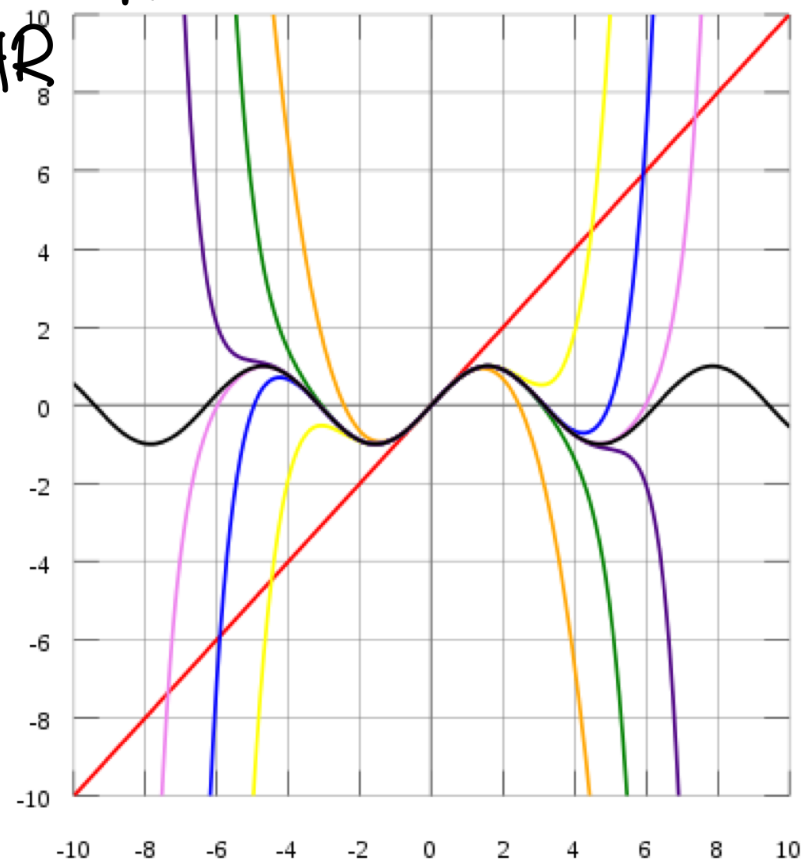
$$|f^{(n)}(x)| = e^x \leq e^\delta = M \quad \forall x \in [-\delta, \delta]$$

$\Rightarrow$ condiz. sufficiente $\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\underline{\forall x \in \mathbb{R}}$

$\forall x \in [-\delta, \delta]$

$n=0$

$x \in \mathbb{R}$



$f(x) = \sin x$

$f(x) = \sin x \qquad f'(x) = \cos x, \qquad f''(x) = -\sin x$

$f'''(x) = -\cos x$

$\left| f^{(n)}(x) \right| \leq 1 \qquad f^{(iv)}(x) = \sin x$

$\dots$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \dots + \frac{f^{(n)}(0)}{n!}, \dots$$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0, \quad f'''(0) = -1, \quad f^{(iv)}(0) = 0, \quad f^{(v)}(0) = 1$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

$$-1 < x < 1$$

$x \leftrightarrow -x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x dt - \int_0^x t\,dt + \int_0^x t^2\,dt + \dots + (-1)^n \int_0^x t^n\,dt + \dots$$

per l'integrazione termine a termine $\qquad \forall x \in \,]-1,1[$

$$\Big(\log|1+t|\Big)_{t=0}^{t=x} = \log(1+x)$$

$$-1 < x < 1$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots$$

$$-1 < x < 1 \qquad (\rho = 1)$$

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \underline{-1<x<1}$$

SVILUPPO IN SERIE DI MACLAURIN DI $\log(1+x)$

$\underline{\underline{x=1}}$
$$\log 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$-1 < x < 1$$

$$x \leftrightarrow -x^2$$

$$\forall x \in]-1,1[$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \dots + (-1)^n x^{2n} + \dots$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x dt - \int_0^x t^2\,dt + \int_0^x t^4\,dt + \dots + (-1)^n \int_0^x t^{2n}\,dt + \dots$$

$$\int \frac{1}{1+t^2}\,dt = \operatorname{arctg} t + C$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \forall x \in]-1,1[$$

sviluppo in serie di Maclaurin di arctg x

$$\frac{\pi}{4} = \operatorname{arctg} 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{n\, 5^n}{2^n + 3^n} (\overbrace{4 - |x+2|}^{y})^n \qquad y = 4 - |x+2|$$

$$\sum_{n=0}^{\infty} \underbrace{\frac{n\, 5^n}{2^n + 3^n}} y^n \qquad y_0 = 0$$

$$a_n = \frac{n\, 5^n}{2^n + 3^n} \qquad a_{n+1} = \frac{(n+1)\, 5^{n+1}}{2^{n+1} + 3^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\, 5^{n+1}}{2^{n+1} + 3^{n+1}} \cdot \frac{2^n + 3^n}{n\, 5^n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 5 \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{\downarrow\ 1} \cdot \frac{2^n + 3^n}{2^{n+1} + 3^{n+1}}$$

$$= 5 \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{5}{3}$$

$\rho = 3/5$: la serie in y converge assolutamente

quando $$-\frac{3}{5} < y < \frac{3}{5}$$

$$\Leftrightarrow \quad -\frac{3}{5} < 4 - |x+2| < \frac{3}{5}$$

$$\Leftrightarrow \quad -\frac{3}{5} < |x+2| - 4 < \frac{3}{5}$$

$$\Leftrightarrow \quad 4 - \frac{3}{5} < |x+2| < 4 + \frac{3}{5}$$

$$\Leftrightarrow \quad \frac{17}{5} < |x+2| < \frac{23}{5}$$

$$\Leftrightarrow \begin{cases} |x+2| < \frac{23}{5} \\ |x+2| > \frac{17}{5} \end{cases} \Leftrightarrow \begin{cases} -\frac{23}{5} < x+2 < \frac{23}{5} \\ x+2 < -\frac{17}{5} \ \cup \\ \cup \ x+2 > \frac{17}{5} \end{cases}$$

Agli estremi: $y = -\frac{3}{5}$: $\displaystyle\sum_{n=0}^{\infty} \frac{n\, 5^n}{2^n + 3^n} \left(-\frac{3}{5}\right)^n$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n\, 3^n}{2^n+3^n}$$

$$\lim_{n\to\infty} n \left(\frac{3^n}{2^n+3^n}\right) = +\infty$$

($n \to +\infty$, $\frac{3^n}{2^n+3^n} \to 1$)

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n \log \frac{n+1}{n}}\right)^n}_{a_n} x^n \qquad \left(\frac{1}{n \log(n+1)}\right)^n = \left(\frac{1}{\log(1+1/n)^n}\right)^n$$

(Studio agli estremi: GUARDA SOTTO)

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{n \log \frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{n \log(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{1}{\log(1+1/n)^n} = 1 \Rightarrow \rho = 1$$

($(1+1/n)^n \to e$)

La serie converge assolutamente per $-1 < x < 1$

" converge totalmente per $-\delta \leq x \leq \delta$, $\delta \in (0,1)$

Studio agli estremi ??

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(\log n)} (2x+1)^n =$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{2^n \left(x+\frac{1}{2}\right)^n}{\log(\log n)} \qquad x_0 = -\frac{1}{2}$$

CRIT. RAPPORTO: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{\log(\log(n+1))} \cdot \frac{\log(\log n)}{2^n}$ $\searrow$ 2

$$\lim_{x\to+\infty} \frac{\log(\log x)}{\log(\log(x+1))} \overset{H}{=} \lim_{x\to+\infty} \frac{\frac{1}{\log x}\cdot\frac{1}{x}}{\frac{1}{\log(x+1)}\cdot\frac{1}{x+1}} = 1$$

$\rho = \frac{1}{2}$: la serie iniziale converge assolutamente

$$]x_0-\rho, x_0+\rho[=]-\tfrac{1}{2}-\tfrac{1}{2}, -\tfrac{1}{2}+\tfrac{1}{2}[=]-1, 0[$$

Conv. totale in $[x_0-\delta, x_0+\delta] = [-\frac{1}{2}-\delta, -\frac{1}{2}+\delta]$

$$0 < \delta < \tfrac{1}{2}$$

Agli estremi? $x = -1$

$$\sum_{n=2}^{\infty} (-1)^n \frac{2^n \left(-\frac{1}{2}\right)^n}{\log(\log n)} = \sum_{n=2}^{\infty} (-1)^{2n} \frac{1}{\log(\log n)}$$

($2^n\left(-\frac{1}{2}\right)^n = (-1)^n \frac{1}{2^n}\cdot 2^n$, $\;(-1)^{2n} = 1$)

$$= \sum_{n=2}^{\infty} \frac{1}{\log(\log n)}$$

$x=0$

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(\log n)} \qquad \frac{1}{\log(\log x)}$$

$\underbrace{\log(\log n)}$ decrescente ??

Studio agli estremi di $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n \log \frac{n+1}{n}} \right)^n x^n$

Per $x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left[n \log\left(\frac{n+1}{n}\right) \right]^n}$

$$a_n = \frac{1}{\left[\log(1+1/n)^n \right]^n}$$

$\downarrow$ e

$$\lim_{n \to +\infty} \left[\log\left(1+\frac{1}{n}\right)^n \right]^n \qquad \text{F.I. } 1^{\infty}$$

$$= \lim_{n \to +\infty} e^{n \log\left(\log\left(1+\frac{1}{n}\right)^n\right)}$$

$$\lim_{x \to +\infty} \frac{\log\left(\log\left(1+\frac{1}{x}\right)^x\right)}{1/x} =$$

$= \text{RICORDIAMO CHE} \quad \log(1+t) \sim t$

$$= \lim_{x \to +\infty} \frac{\log\left(1 + \left(\overbrace{\log\left(1+\frac{1}{x}\right)^x - 1}^{t}\right)\right)}{1/x}$$

$$= \lim_{t \to 0} \frac{\log\left(1 + \left(\log(1+t)^{1/t} - 1\right)\right)}{t}$$

$$= \lim_{t \to 0^+} \frac{\log(1+t)^{1/t} - 1}{t} = \lim_{t \to 0^+} \frac{\frac{1}{t}\log(1+t) - 1}{t}$$

Formula di Taylor: $\quad f(t) = \log(1+t) = f'(0)t + f''(0)\frac{t^2}{2} + o(t^2)$

$$= t - \frac{t^2}{2} + o(t^2)$$

Quindi $\displaystyle \lim_{x \to +\infty} \frac{\log\left(\log\left(1+\frac{1}{x}\right)^x\right)}{1/x} =$

$$\Rightarrow \quad = \lim_{t\to 0^+} \frac{\cancel{1} - \frac{t}{2} + o(t^2)/t \cancel{-1}}{t} = -\frac{1}{2}$$

Allora $\lim_{n\to +\infty} \left[\log\left(1+\frac{1}{n}\right)^n\right]^n = \lim_{n\to+\infty} e^{n \log\left(\log\left(1+\frac{1}{n}\right)^n\right)}$

$$= e^{-\frac{1}{2}}$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\left[\log\left(1+\frac{1}{n}\right)^n\right]^n} = \sqrt{e} \Rightarrow$ la serie **NON CONVERGE** (DIVERGE A $+\infty$)

Per $x=-1$ abbiamo la serie

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n \log\left(\frac{n+1}{n}\right)}\right)^n$$

e non converge nemmeno questa.

Esercizi sulle serie numeriche

Studiare il carattere delle seguenti serie, di cui riportiamo per semplicità solo il termine generale

1) $\dfrac{n^{n+1}}{(n-1)!}$; 2) $\dfrac{n^{n/2}}{n!}$; 3) $\dfrac{5^{\sqrt{n}}}{4^n}$

4) $2^n\left(\dfrac{n-1}{n}\right)^{n^2}$; 5) $4^n\left(\dfrac{n-1}{n}\right)^{\frac{n^3+3}{n+1}}$

6) $\dfrac{2^{\sqrt{n}}\, n^n}{n!}$ 7) $\dfrac{3^n}{(1+\lg n)\, n!}$

8) $\dfrac{1+\sin n}{n^n}$; 9) $\dfrac{(-1)^n}{2\sqrt{n}-1}$;

10) $(-1)^n\left(\pi - 2\,\mathrm{arctg}\, n\right)$

Esercizi sulle serie di funzioni

Determinare l'insieme di convergenza delle seguenti serie di funzioni

1) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{tg}\, x)^n$; 2) $\displaystyle\sum_{n=0}^{\infty} \frac{3n}{2^n}(\log x)^n$

3) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{\pi}\,\mathrm{arctg}\, x\right)^n$; 4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9^n}\left(\frac{6 \arccos x}{\pi}\right)^{2n}$

5) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{e^{-2nx^2}}{(2n)!}$; 6) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n(n+1)}$.

RISPOSTE

SERIE NUMERICHE

1) DIVERGE ; 2) CONVERGE ; 3) CONVERGE

4) CONVERGE ; 5) DIVERGE ; 6) DIVERGE

7) CONVERGE : BASTA OSSERVARE CHE,

PER $n \geq 2$, $1+\log n \geq 1+\log 2$, quindi

la serie è maggiorata da quella di

$3^n/n!$ che CONVERGE

8) CONVERGE : $1+\sin n \leq 2$ e la

serie $\sum 1/n^n$ converge

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^{n+1}} \cdot n^n = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1} \longrightarrow 0 < 1$$

$\left(1+\frac{1}{n}\right)^n \to e$, $\frac{1}{n+1} \to 0$

9) CONVERGE PER LEIBNIZ

10) CONVERGE

SERIE DI FUNZIONI

1) CONVERGE $\forall x \in \mathbb{R}$;

2) CONVERGE PUNTUALMENTE IN $[-1, -\frac{1}{3})$

3) " " " " $]-1, 1[$

4) " " " $]0, 1]$

5) CONVERGE $\forall x \in \mathbb{R}$

" "

6)