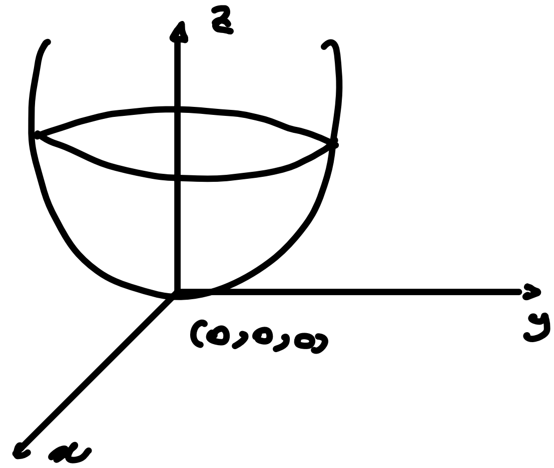
Lezioni del 02/11/2023 e
del 03/11/2023

Es. $f(x,y) = x^2+y^2$

$f_x = 2x, \quad f_y = 2y$

$\nabla f(0,0) = (0,0)$



$\nabla f(x_0,y_0) = \underline{0}$ si dice che (x_0,y_0) è un punto critico o stazionario

$(x_0,y_0) \in \mathring{X}$ estremo relativo $\Longrightarrow$ (x_0,y_0) stazionario

$\not\Longleftarrow$

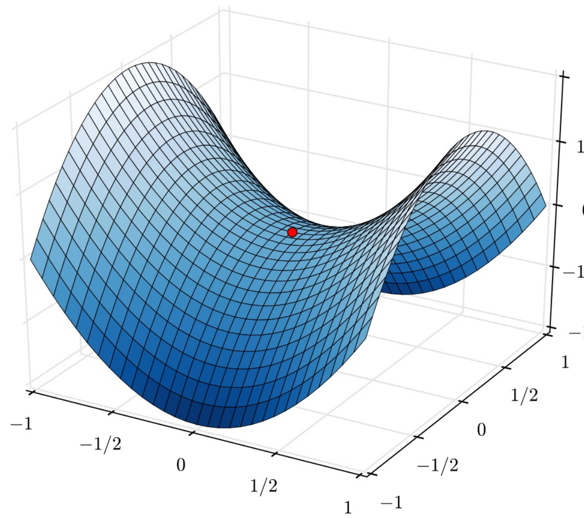
Es.

$f(x,y) = x^2 - y^2 \qquad X = \mathbb{R}^2$

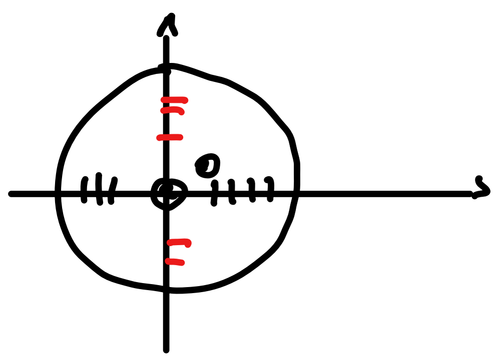
$z = x^2 - y^2$

$\nabla f = (2x, -2y)$

$\nabla f(0,0) = (0,0)$



$(0,0)$ punto critico



$x \neq 0$

$y = 0$

$f(x,0) = x^2 > 0 = f(0,0)$

$f(0,y) = -y^2 < 0 = f(0,0)$

$\Longrightarrow$ $(0,0)$ né di minimo né di massimo

$y \neq 0$

$(0,0)$ punto di sella

Def. $(x_0,y_0) \in \mathring{X}$ è di sella se

$$Df(x_0,y_0) = \underline{0}$$

e (x_0,y_0) né di minimo né di massimo relativo.

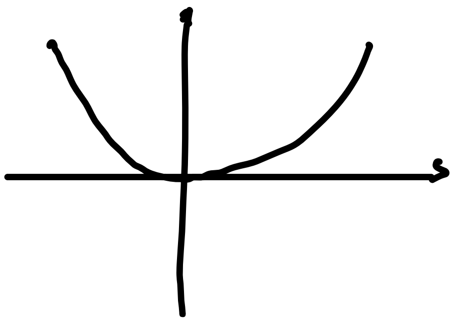
$f = f(x)$: x_0 minimo relativo $\Longrightarrow$ $f'(x_0)=0$

$f''(x_0) \geq 0$

$f(x) = x^4$

$f' = 4x^3$

$f'' = 12x^2$



x_0 massimo relativo $\Longrightarrow$ $f'(x_0)=0$

$f''(x_0) \leq 0$

CONDIZIONE NECESSARIA AL SECONDO ORDINE

$f \in C^2$

$$D^2 f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$Hf(x,y) = \det D^2 f(x,y) = f_{xx} f_{yy} - \left(f_{xy}\right)^2$$

determinante Hessiano

Teorema $f \in C^2$ in un intorno di (x_0, y_0)

Se (x_0,y_0) di minimo relativo allora $\Rightarrow$ $\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \geq 0 \\ f_{xx}(x_0,y_0) \geq 0 \end{cases}$

(x_0,y_0) di massimo relativo $\Rightarrow$ $\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \geq 0 \\ f_{xx}(x_0,y_0) \leq 0 \end{cases}$

(x_0,y_0) di sella $\Rightarrow$ $\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \leq 0 \end{cases}$

ES. $f(x,y) = x^2+y^2$ $(0,0)$ di minimo

$f_x = 2x, \quad f_y = 2y$

$f_x(0,0) = f_y(0,0) = 0$

$f_{xx} = 2, \quad f_{xy} = 0 = f_{yx}, \quad f_{yy} = 2$

$Hf = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \qquad f_{xx}(0,0) > 0$

$$f = x^2 - y^2 \qquad f_x = 2x, \quad f_y = -2y$$

$(0,0)$ **sella**

$$f_{xx} = 2, \quad f_{xy} = 0 = f_{yx}$$

$$f_{yy} = -2$$

$$Hf = -4 < 0$$

CONDIZIONI SUFFICIENTI AL 2° ORDINE

$$\begin{cases} f'(x_0) = 0 \\ f''(x_0) \gneq 0 \end{cases} \Rightarrow x_0 \text{ di minimo relativo}$$

$f \in C^2$ in un intorno di (x_0, y_0). Allora:

$$\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \gneq 0 \\ f_{xx}(x_0,y_0) \gneq 0 \end{cases} \Rightarrow (x_0,y_0) \text{ di minimo relativo}$$

$$\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \gneq 0 \\ f_{xx}(x_0,y_0) \lneq 0 \end{cases} \Rightarrow (x_0,y_0) \text{ di massimo relativo}$$

$$\begin{cases} f_x(x_0,y_0) = f_y(x_0,y_0) = 0 \\ Hf(x_0,y_0) \underset{\neq}{<} 0 \end{cases} \Rightarrow (x_0,y_0) \text{ di sella.}$$

OSS. Se $Hf(x_0,y_0) = 0$? NULLA SI PUÒ DIRE!!

(x_0,y_0) minimo, massimo, sella.

1° PASSO Punti critici: $\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$

FORNISCE i punti critici (x_0,y_0).

2° PASSO Calcolare $Hf(x_0,y_0)$ e controllarne il segno $\left(\begin{matrix} >0 \\ <0 \end{matrix} \right)$

ES. CLASSIFICARE I PUNTI CRITICI DI

$$f(x,y) = x(x-1)^2 - y^2$$

$$f_x = (x-1)^2 + 2x(x-1) =$$

$$= x^2 - 2x + 1 + 2x^2 - 2x = 3x^2 - 4x + 1$$

$f_y = -2y$

$$\begin{cases} f_x = 0 \iff 3x^2 - 4x + 1 = 0 \\ f_y = 0 \iff -2y = 0 \iff y = 0 \end{cases}$$

$x = \frac{1}{3}$

$x = 1$

$$\iff \begin{cases} x = \frac{1}{3} \\ y = 0 \end{cases} \cup \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$A = \left(\frac{1}{3}, 0\right)$, $B = (1, 0)$ punti critici.

$f_{xx} = 6x - 4$; $f_{xy} = 0 = f_{yx}$

$f_{yy} = -2$

$f_{xx}\left(\frac{1}{3}, 0\right) = 6 \cdot \frac{1}{3} - 4 = 2 - 4 = -2$

$f_{xy}\left(\frac{1}{3}, 0\right) = 0 = f_{yx}\left(\frac{1}{3}, 0\right)$

$f_{yy} = -2$

$$Hf\left(\tfrac{1}{3},0\right) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$$

$$f_{xx}\left(\tfrac{1}{3},0\right) = -2 < 0 \Rightarrow$$

$\Rightarrow A = \left(\tfrac{1}{3},0\right)$ di massimo relativo

$$f_{xx}(1,0) = 6-4 = 2 \qquad f_{xy} = f_{yx} = 0$$

$$f_{yy} = -2$$

$$Hf(1,0) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$$

$\Rightarrow B = (1,0)$ di $\underline{\underline{\text{SELLA}}}$.

$$f(x,y) = 2x^3 + y^3 - 3x^2 - 3y$$

$$f_x = 6x^2 - 6x \quad ; \quad f_y = 3y^2 - 3$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \iff \begin{cases} x^2 - x = 0 \\ y^2 - 1 = 0 \end{cases} \quad x^2-x=0 \Rightarrow x=0,\ x=1$$

$$\iff \begin{cases} x = 0 \\ y^2 - 1 = 0 \end{cases} \cup \begin{cases} x = 1 \\ y^2 - 1 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = -1 \end{cases} \cup \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = -1 \end{cases} \cup \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$A = (0,-1)$, $B = (0,1)$, $C = (1,-1)$, $D = (1,1)$

punti critici

$$f_{xx} = 12x - 6 \qquad f_{xy} = 0 = f_{yx}$$

$$f_{yy} = 6y$$

COMPLETARE !!

ES.

$$f(x,y) = x^3 + y^3 - (1+x+y)^3$$

$$f_x = 3x^2 - 3(1+x+y)^2$$

$$f_y = 3y^2 - 3(1+x+y)^2$$

$$f_{xx} = 6x - 6(1+x+y)$$

$$f_{xy} = -6(1+x+y) = f_{yx}$$

$$f_{yy} = 6y - 6(1+x+y)$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \iff \begin{cases} x^2 - (1+x+y)^2 = 0 \\ y^2 - (1+x+y)^2 = 0 \end{cases} \iff$$

SOTTRAENDO MEMBRO A MEMBRO

$$\iff \begin{cases} x^2 - y^2 = 0 \\ x^2 - (1+x+y)^2 = 0 \end{cases} \iff \cancel{x = \pm\sqrt{y}}$$

$$\iff \begin{cases} (x-y)(x+y) = 0 \\ x^2 - (1+x+y)^2 = 0 \end{cases}$$

$$\iff \begin{cases} x - y = 0 \\ x^2 - (1+x+y)^2 = 0 \end{cases} \cup \begin{cases} x + y = 0 \\ x^2 - (1+x+y)^2 = 0 \end{cases}$$

$$\begin{cases} y = x \\ x^2 - (1+2x)^2 = 0 \end{cases} \cup \begin{cases} y = -x \\ x^2 - 1 = 0 \end{cases}$$

$$\iff \begin{cases} y = x \\ x^2 - 1 - 4x^2 - 4x = 0 \end{cases} \cup \begin{cases} y = -x \\ x^2 - 1 = 0 \end{cases}$$

$$\begin{cases} y = x \\ -3x^2 - 4x - 1 = 0 \end{cases} \cup \begin{cases} y = -x \\ x^2 - 1 = 0 \end{cases}$$

$$3x^2 + 4x + 1 = 0$$

$$\begin{cases} y = x \\ x_{1,2} = \dfrac{-2 \pm 1}{3} \begin{array}{l} \nearrow -1 \\ \searrow -\frac{1}{3} \end{array} \end{cases} \cup \begin{cases} y = -x \\ x = \pm 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} y = -1 \\ x = -1 \end{cases} \cup \begin{cases} y = -\frac{1}{3} \\ x = -\frac{1}{3} \end{cases}$$

$$\cup \begin{cases} y = 1 \\ x = -1 \end{cases} \cup \begin{cases} y = -1 \\ x = 1 \end{cases}$$

$A = (-1, -1)$, $B = \left(-\frac{1}{3}, -\frac{1}{3}\right)$

$C = (-1, 1)$, $D = (1, -1)$

p.ti critici

$$f(x,y) = 2\log(2+x^2+y^2) - xy$$

$$\begin{cases} f_x = \dfrac{2}{2+x^2+y^2}\cdot 2x - y = \dfrac{4x}{2+x^2+y^2} - y = 0 \\ f_y = \qquad\qquad = \dfrac{4y}{2+x^2+y^2} - x = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \cup \begin{cases} x,y\neq 0 \\ \dfrac{4x}{2+x^2+y^2} = y \\ \dfrac{4y}{2+x^2+y^2} = x \end{cases}$$

$\underbrace{\qquad}_{O}$

$O = (0,0)$

$$\Leftrightarrow \begin{cases} \dfrac{2+x^2+y^2}{4x} = \dfrac{1}{y} \\ \dfrac{4y}{2+x^2+y^2} = x \end{cases}$$

$$\Leftrightarrow \begin{cases} 2+x^2+y^2 = 4x/y \\ \cancel{4}y \cdot \dfrac{y}{\cancel{4}x} = x \Leftrightarrow y^2 - x^2 = 0 \end{cases}$$

$$\begin{cases} 2+x^2+y^2 = 4\frac{x}{y} \\ (y-x)(y+x)=0 \end{cases} \iff \begin{cases} 2+x^2+y^2 = 4\frac{x}{y} \\ y=x \end{cases}$$

$$\cup \begin{cases} 2+x^2+y^2 = 4\frac{x}{y} \\ y=-x \end{cases}$$

$$\iff \begin{cases} 2+2x^2 = 4 \iff x^2=1 \\ y=x \end{cases}$$

$$\cup \begin{cases} 2+2x^2 = -4 \quad \underline{\text{IMPOSSIBILE}} \\ y=-x \end{cases}$$

$$\iff \begin{cases} x=-1 \\ y=x=-1 \end{cases} \cup \begin{cases} x=1 \\ y=1 \end{cases}$$

$O=(0,0)$, $A=(-1,-1)$, $B=(1,1)$ punti critici.

R. $(0,0)$ minimo relativo

$(1,1)$ sella

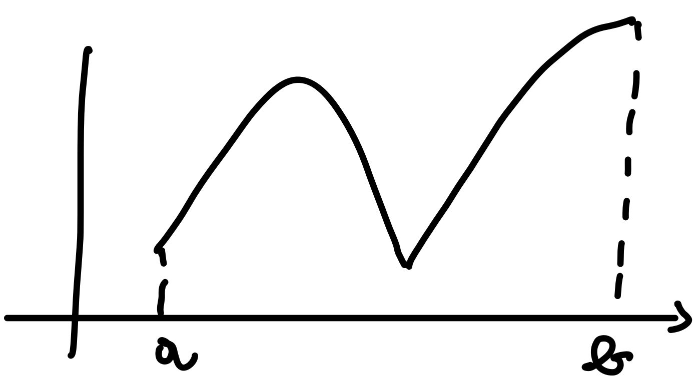
$(-1,-1)$ "

ESTREMI ASSOLUTI ?

$f=f(x)$ $f:[a,b]\to\mathbb{R}$ continua :

$\exists\, \bar{x}, \bar{\bar{x}}$: $f(\bar{x}) = \min_{[a,b]} f(x)$ WEIERSTRASS

$$f(\bar{\bar{x}}) = \max_{[a,b]} f(x)$$



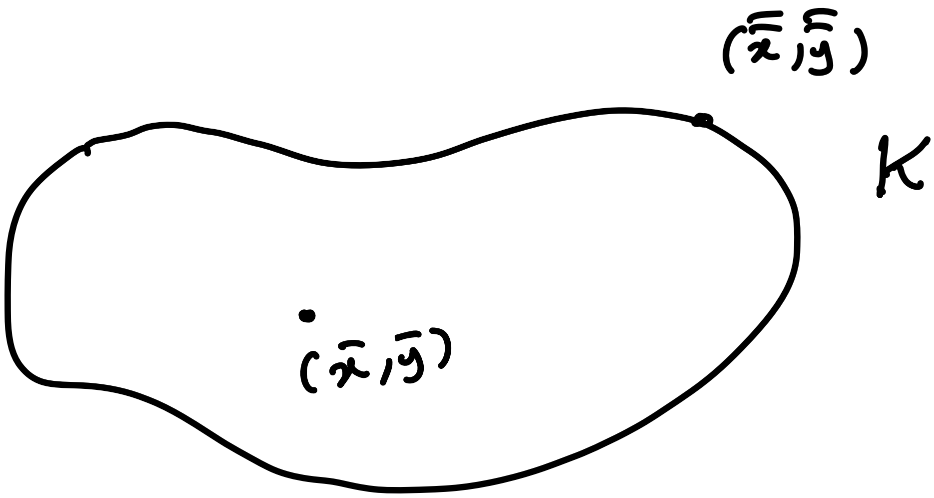
$f = f(x,y)$, $f : K \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ continua:

K compatto

$\exists\ (\bar{x},\bar{y}),\ (\bar{\bar{x}},\bar{\bar{y}}) \in K$: $f(\bar{x},\bar{y}) = \min\limits_{K} f$

$f(\bar{\bar{x}},\bar{\bar{y}}) = \max\limits_{K} f$

WEIERSTRASS

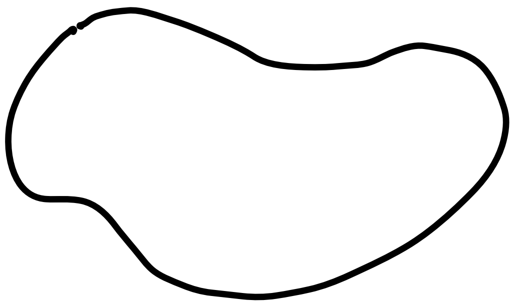


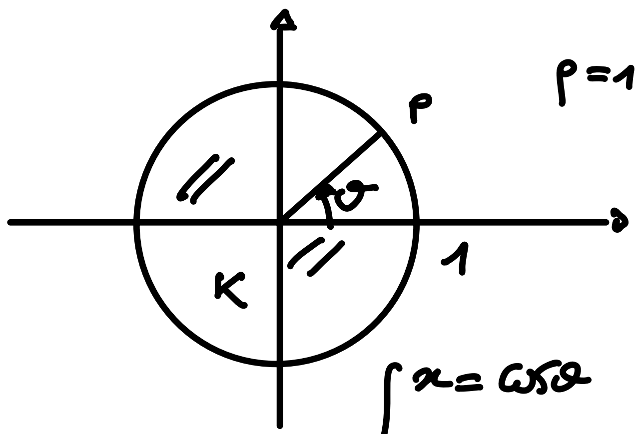
1) CALCOLO DEI PUNTI CRITICI IN K

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

NON È NECESSARIO CLASSIFICARE I PUNTI CRITICI : NON CALCOLARE $Hf(x,y)$!!

2) Studio di $f(x,y)$ su ∂K



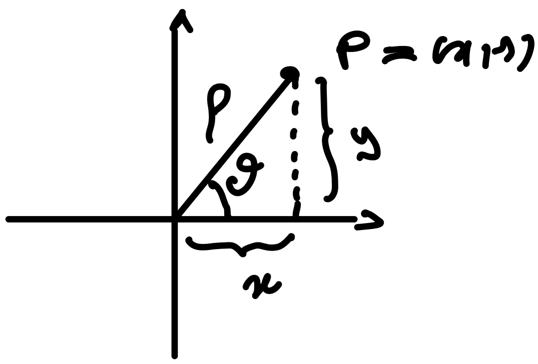


$\rho = 1$

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$

$\theta \in [0, 2\pi]$

$$x^2 + y^2 = 1$$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

TRASFORMAZIONE IN COORDINATE POLARI

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

EQUAZIONI PARAMETRICHE DELLA CIRCONFERENZA

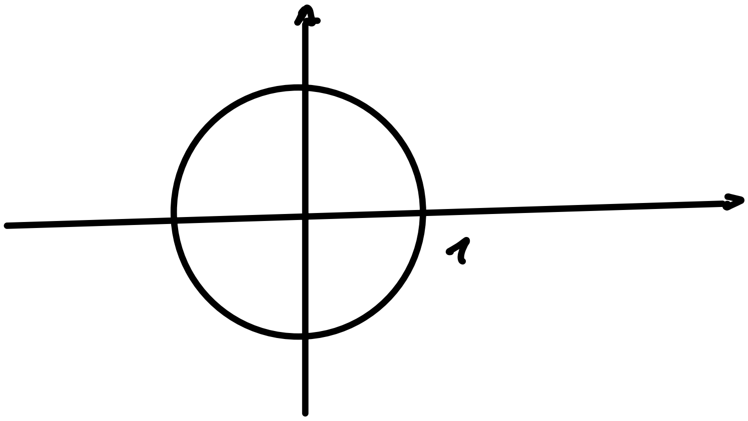
Studiare f su ∂K significa determinare minimo e massimo assoluto di f su ∂K:

$$g(\theta) = f(\cos\theta, \sin\theta) \quad , \theta \in [0, 2\pi]$$

IN GENERALE

$$\partial K \equiv \begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad t \in [a, b]$$

ES. $f(x,y) = xy$ ESTREMI ASSOLUTI IN $C(0,1)$



$$\begin{cases} f_x = y = 0 \\ f_y = x = 0 \end{cases} \iff (x,y) = (0,0) \quad : \quad f(0,0) = 0$$

Studio di f su $\partial C \equiv \begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $\quad t \in [0, 2\pi]$

$$g(t) = f(\cos t, \sin t) = \cos t \sin t \quad , \quad t \in [0, 2\pi]$$

$$= \frac{1}{2} \sin(2t)$$

$g(t)$ massima: quando $\sin(2t) = 1 \iff 2t = \frac{\pi}{2} + 2k\pi \overset{(*)}{\iff}$

per cui il massimo di f su $\partial C = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\overset{(*)}{\iff} \quad t = \frac{\pi}{4} + k\pi \quad : k = 0 \longrightarrow t = \frac{\pi}{4}$

$k \in \mathbb{Z} \quad : k = 1 \longrightarrow t = \frac{\pi}{4} + \pi = \frac{5}{4}\pi$

$\sin 2t = 2 \sin t \cos t \qquad t \in \underbrace{[0, 2\pi]}$

$t = \frac{\pi}{4} \longrightarrow \begin{cases} x = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ y = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{cases}$ $\qquad A = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$t = \frac{5}{4}\pi \quad \begin{cases} x = -\frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \end{cases}$ $\qquad B = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$g(t)$ è minima quando $\sin(2t) = -1 \iff \boxed{2t = \frac{3}{2}\pi + 2k\pi}$

il minimo di f su $\partial C = \frac{1}{2}\cdot(-1) = -\frac{1}{2}$

$\iff t = \frac{3}{4}\pi + k\pi \ , k \in \mathbb{Z}$

$k = 0 \longrightarrow t = \frac{3}{4}\pi$

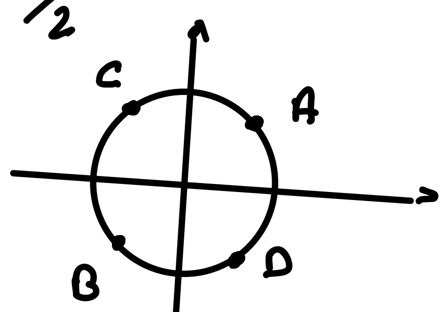
$k = 1 \longrightarrow t = \frac{3}{4}\pi + \pi = \frac{7}{4}\pi$

$\begin{cases} x = \cos\frac{3}{4}\pi = -\frac{\sqrt{2}}{2} \\ y = \sin\frac{3}{4}\pi = \frac{\sqrt{2}}{2} \end{cases}$ $\qquad \begin{cases} x = \cos\frac{7}{4}\pi = \frac{\sqrt{2}}{2} \\ y = \sin\frac{7}{4}\pi = -\frac{\sqrt{2}}{2} \end{cases}$

$C = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad D = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ p.ti di minimo assoluto

$0 \ , \underbrace{-\frac{1}{2}, \ \frac{1}{2}}_{\partial C}$, A, B p.ti di massimo assoluto

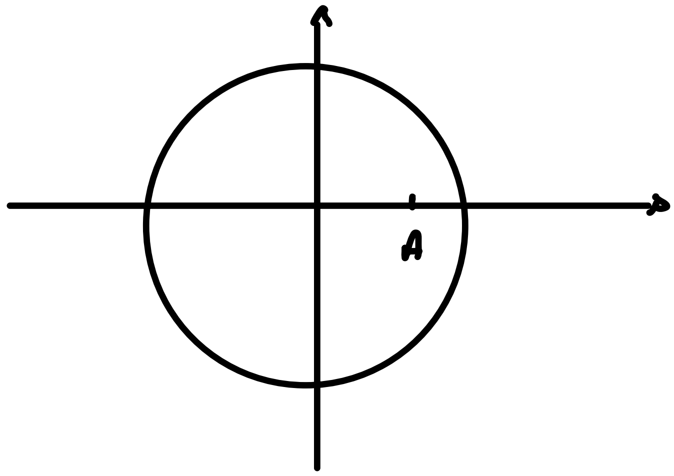
$\min_C f = -\frac{1}{2}, \quad \max_C f = \frac{1}{2}$



DETERMINARE GLI ESTREMI ASSOLUTI DI

$$f(x,y) = y^2 - x^2(x-1) \quad \text{in} \quad C(0,1)$$

$x^2+y^2=1$



$y^2 = 1-x^2$

$$\begin{cases} f_x = -2x(x-1) - x^2 = -3x^2+2x = 0 \\ f_y = 2y = 0 \end{cases}$$

$$\begin{cases} 3x^2-2x=0 \\ y=0 \end{cases} \iff \begin{cases} x=0 \\ y=0 \end{cases} \cup \begin{cases} x=\frac{2}{3} \\ y=0 \end{cases}$$

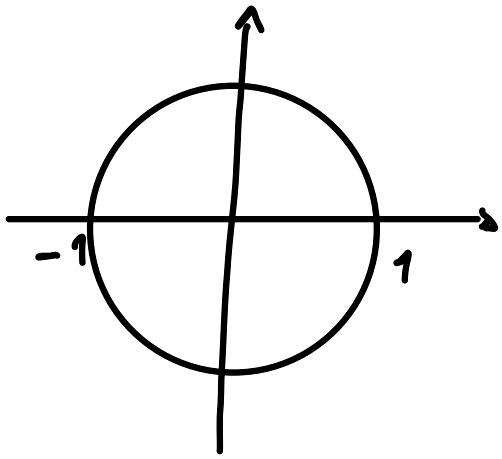
$O=(0,0)$, $A=(\frac{2}{3},0) \in C$

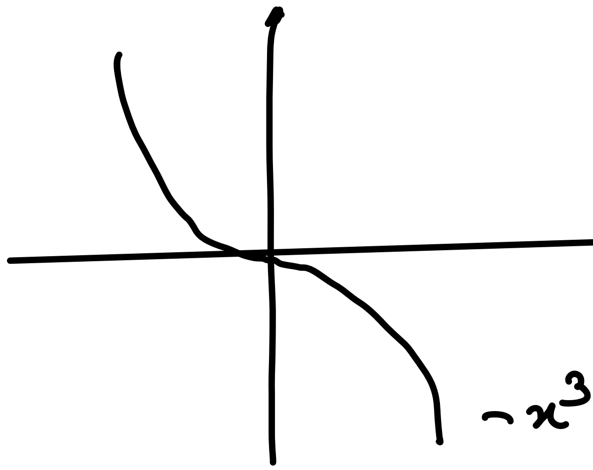
punti critici

$f(0,0) = 0$; $f(\underset{x}{\frac{2}{3}}, \underset{y}{0}) = \frac{4}{27}$ DA PARTE

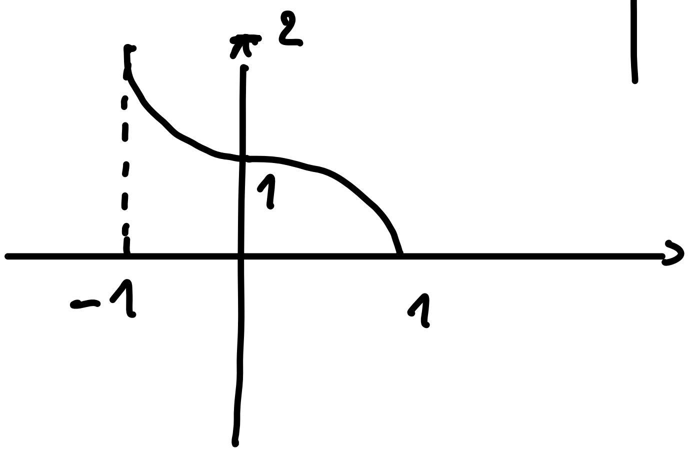
$x^2+y^2=1 \iff y^2=1-x^2$: su ∂C

$$g(x) = 1-x^2-x^2(x-1) = 1-\cancel{x^2}-x^3+\cancel{x^2}$$
$$= 1-x^3, \quad x\in[-1,1]$$





$1-x^3$



$\min\limits_{[-1,1]} g = 0$.

$\max\limits_{[-1,1]} g = 2$.

$x=1$ minimo assoluto di $g(x)$

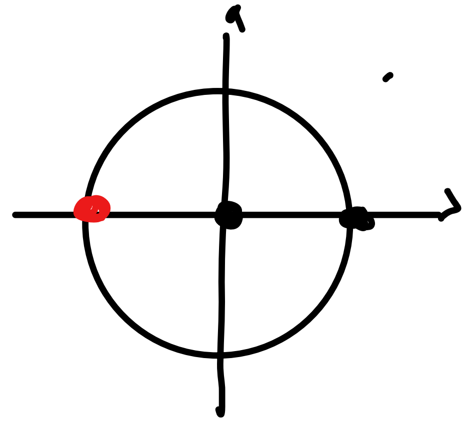
$x=-1$ massimo assoluto " $g(x)$

$$x^2+y^2=1 \iff y=0$$

$$x=\pm 1$$

$(1,0)$ p.to di minimo di f su ∂C

$(-1,0)$ " " massimo di f su ∂C



$\underset{\sim}{0}, \quad \frac{4}{27}, \quad 2$

$$\min_C f = 0, \qquad \max_C f = 2$$

$\downarrow$

in $(0,0)$ e in $(1,0)$

p.ti di minimo assoluto

$\searrow$

$(-1,0)$

p.to di massimo assoluto

ALTERNATIVAMENTE

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t\in[0,2\pi]$$

$t \in [0, 2\pi]$

$$g(t) = f(\cos t, \sin t) =$$

$$= \sin^2 t - \cos^2 t\,(\cos t - 1) =$$

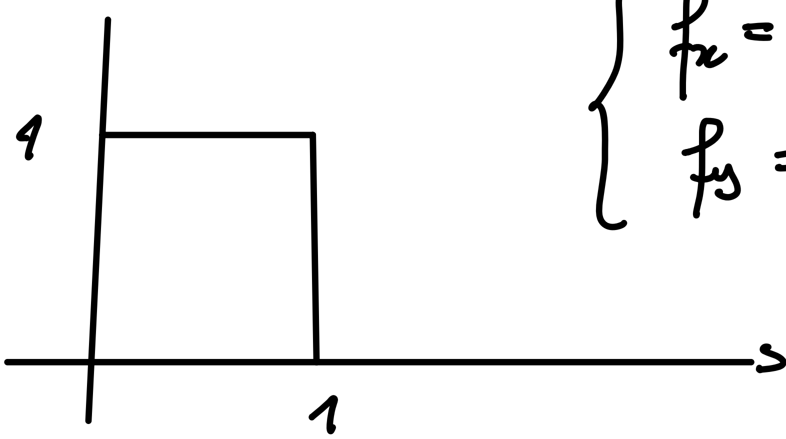
$$= \sin^2 t - \cos^3 t + \cos^2 t$$

$$= 1 - \cos^3 t \qquad t \in [0, 2\pi]$$

ES.

$$f = 2xy - y^2 - x^4$$

$$Q = [0,1] \times [0,1]$$

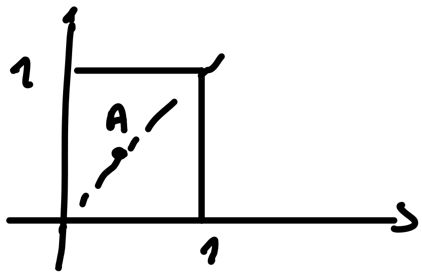


$$\begin{cases} f_x = 2y - 4x^3 = 0 \\ f_y = 2x - 2y = 0 \end{cases}$$

$$\begin{cases} y-2x^3=0 \\ y=x \end{cases} \iff \begin{cases} x-2x^3=0 \\ y=x \end{cases}$$

$$\iff \begin{cases} x(2x^2-1)=0 \\ y=x \end{cases} \iff \begin{cases} x=0 \\ y=0 \end{cases} \cup \cancel{\begin{cases} x=-\frac{\sqrt{2}}{2} \\ y=-\frac{\sqrt{2}}{2} \end{cases}}$$

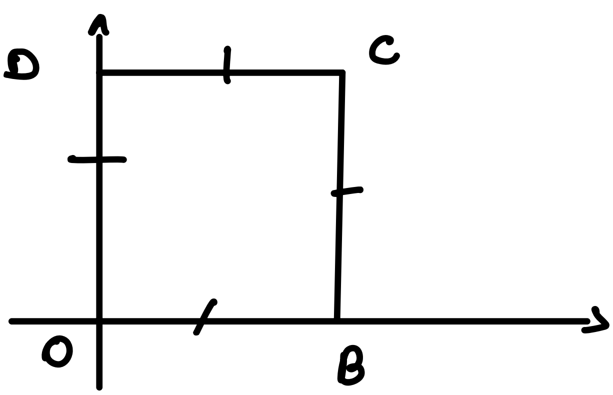
NO!



$$A = \begin{cases} x=\frac{\sqrt{2}}{2} \\ y=\frac{\sqrt{2}}{2} \end{cases}$$

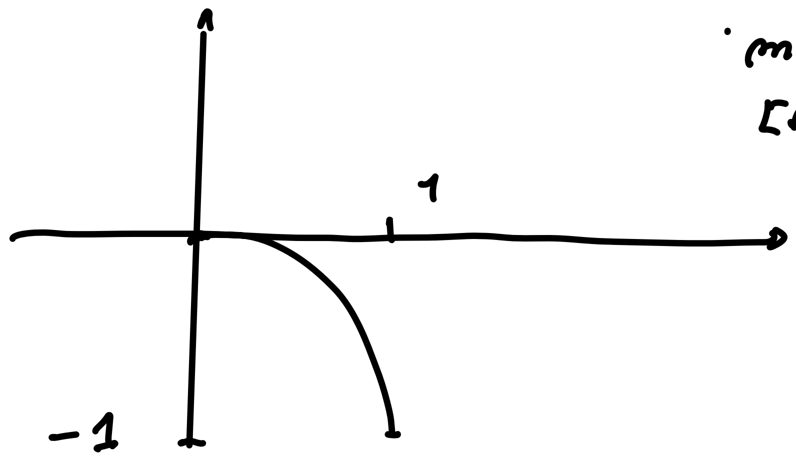
$$f(0,0)=0, \qquad f\Big(\underset{x}{\frac{\sqrt{2}}{2}}, \underset{y}{\frac{\sqrt{2}}{2}}\Big)=\frac{1}{4}$$

Su ∂Q ?



Su $\overline{OB}$: $\begin{cases} x\in[0,1] \\ y=0 \end{cases}$

$g_1(x) = f(x,0) = -x^4 \quad , \quad x \in [0,1]$



$\min\limits_{[0,1]} g_1 = -1$

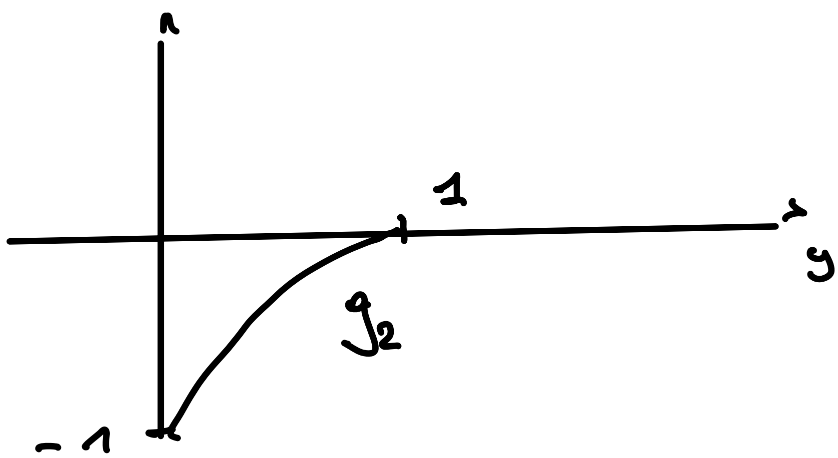
$\max\limits_{[0,1]} g_1 = 0$

Su $\overline{BC}$: $\begin{cases} x = 1 \\ y \in [0,1] \end{cases}$

$g_2(y) = f(1,y) = 2y - y^2 - 1 \quad , \quad y \in \underbrace{[0,1]}$

$g_2(0) = -1 \, , \quad g_2(1) = 2 - 1 - 1 = 0$

$g_2'(y) = 2 - 2y \geq 0 \iff [\, y \leq 1$ SEMPRE VERO!



$\begin{cases} \min\limits_{[0,1]} g_2 = -1 \\ \max\limits_{[0,1]} g_2 = 0 \end{cases}$

Su $\overline{CD}$: $\begin{cases} x \in [0,1] \\ y = 1 \end{cases}$ "

" $\overline{OD}$ = $\begin{cases} x = 0 \\ y \in [0,1] \end{cases}$ "