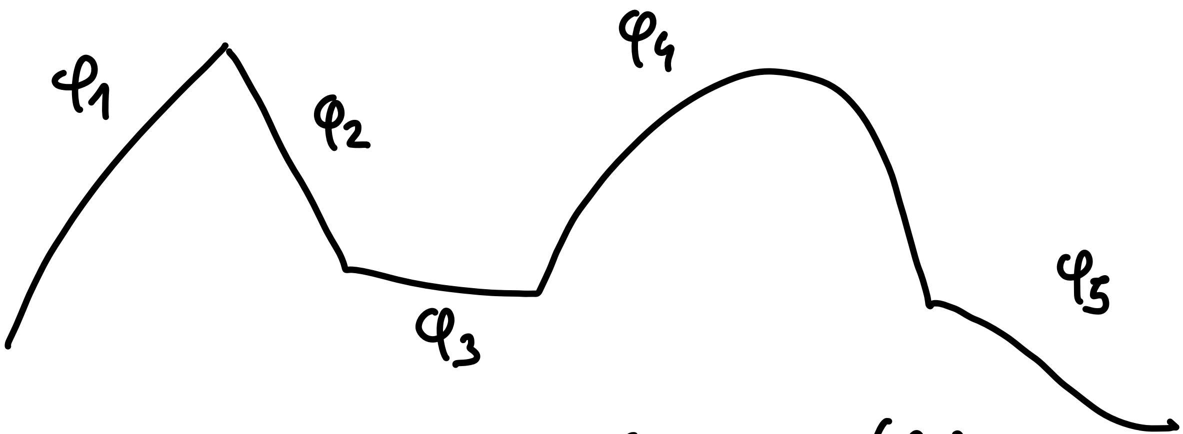
Lezione del 08/12/23

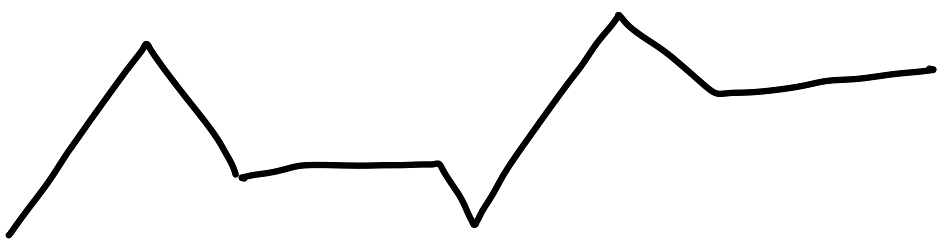
Integrale curvilineo

OSS.



$$L(\varphi) \overset{\text{def.}}{=} L(\varphi_1) + L(\varphi_2) + L(\varphi_3) + L(\varphi_4) + L(\varphi_5)$$

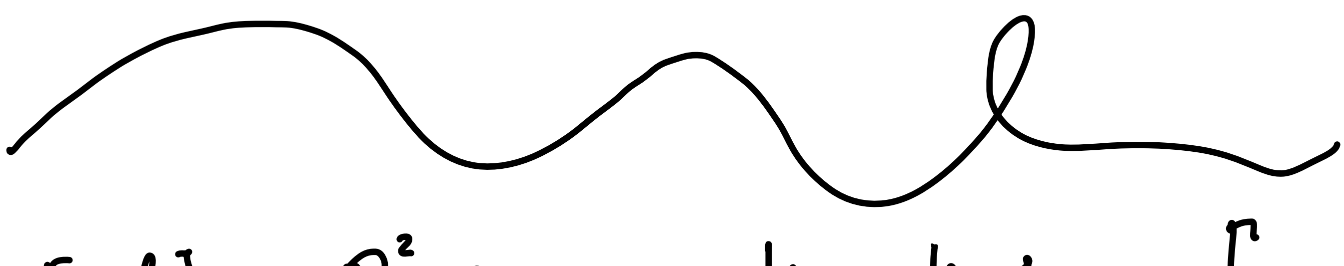
ES.



Def. γ curva regolare di sostegno $\Gamma \subseteq \mathbb{R}^2$

$f(x,y)$ definita e continua sul sostegno $\Gamma \subseteq \mathbb{R}^2$

$$f : \Gamma \to \mathbb{R}$$



$\varphi : [a,b] \to \mathbb{R}^2$ rapp. parametrica di γ

$$\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \end{cases} \qquad t \in [a,b]$$

Integrale curvilineo di $f(x,y)$ esteso a γ come ↙ (1)

$$\left[\int_\gamma f \, ds = \int_a^b f(\underbrace{\tilde{\varphi}(t)}_{\odot}) \, \underbrace{\|\tilde{\varphi}'(t)\|}_{} \, dt \right]$$

ds ↓ "ASCISSA CURVILINEA di γ"

f.ne continua

Più esplicitamente :

$$\int_\gamma f ds = \int_a^b \overset{f(x,y)}{f(\underbrace{\tilde{\varphi}_1(t)}, \tilde{\varphi}_2(t))} \cdot \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2} \, dt$$

<u>OSS.</u> γ curva in $\mathbb{R}^3$ regolare $\quad \Gamma \subseteq \mathbb{R}^3$

$f(x,y,z)$ continua su Γ

$$\int_\gamma f ds = \int_a^b f(\varphi_1(t), \varphi_2(t), \varphi_3(t)) \cdot \sqrt{(\varphi_1')^2 + (\varphi_2')^2 + (\varphi_3')^2} \, dt$$

<u>OSS.</u> (2) La (1) NON dipende dalla rapp. scelta, nel senso che se φ e ψ rappresentazioni parametriche di γ <u>equivalenti</u>, la (1) fornisce la stessa quantità; inoltre <u>NON</u> dipende dal verso indotto

della rappresentazione!

③ Se $f(x,y) = 1$, la formula diventa

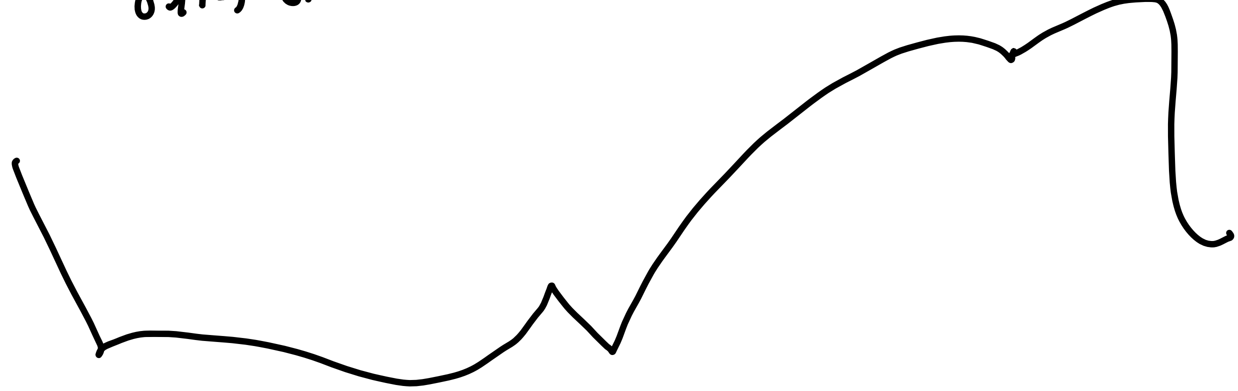
$$\int_\gamma f\,ds = \int_a^b \|\varphi'(t)\|\,dt = L(\gamma)$$

lunghezza di γ

④ a) $$\int_\gamma (\alpha f + \beta g)\,ds = \alpha \int_\gamma f\,ds + \beta \int_\gamma g\,ds$$

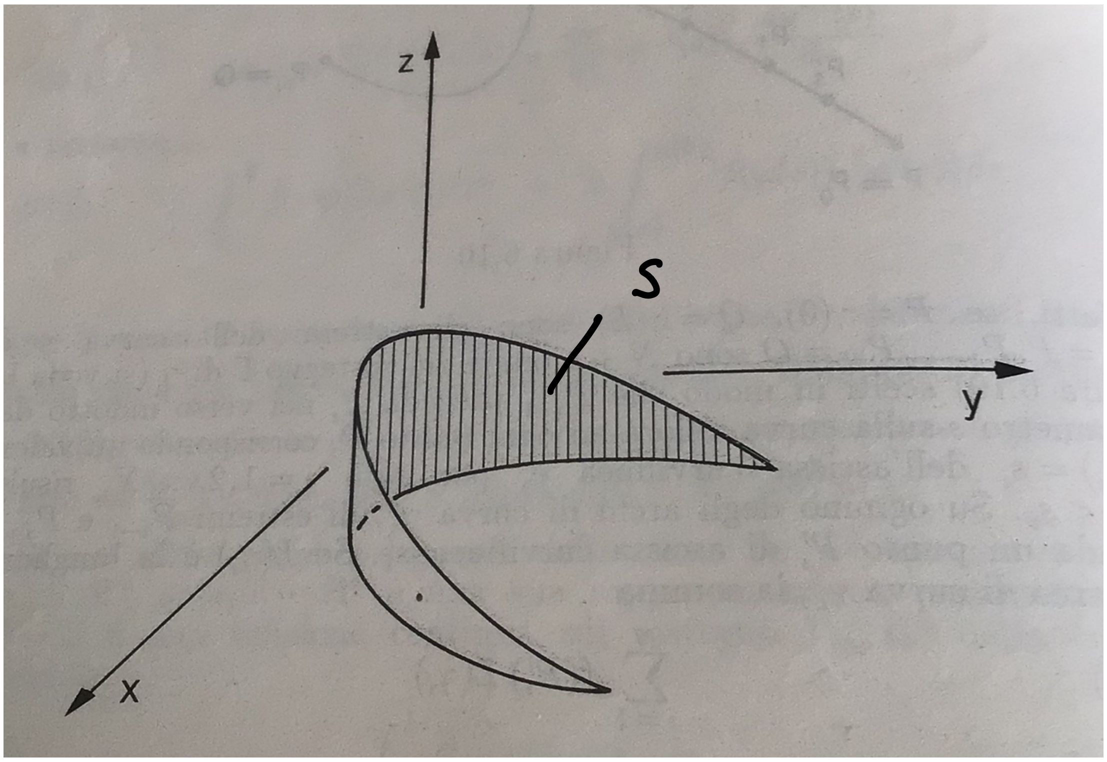
b) $$\left| \int_\gamma f\,ds \right| \leq \int_\gamma |f|\,ds$$

Def. Se γ si spezza nell'unione di curve regolari $\gamma_1, \dots, \gamma_n$

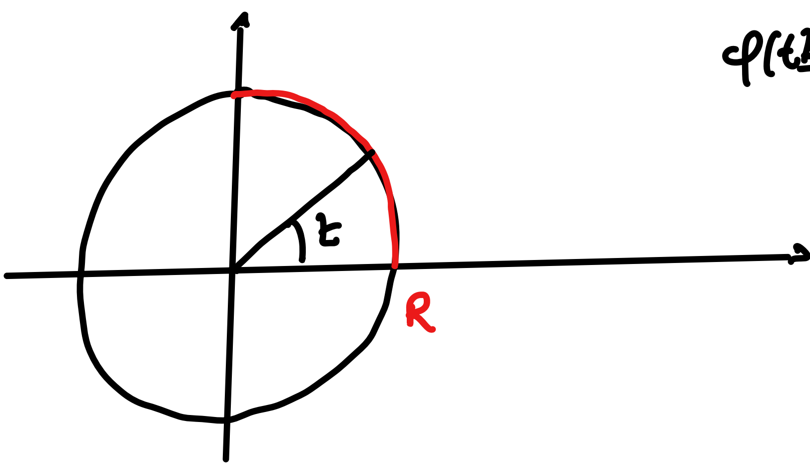


$$\int_\gamma f\,ds = \int_{\gamma_1} f\,ds + \int_{\gamma_2} f\,ds + \dots + \int_{\gamma_n} f\,ds$$

OSS. (4) Se $f \geq 0$: $\int_\gamma f\,ds$ = area(S)
$f(x,y)$



ES. (1) $\int_\gamma xy\,ds$ γ è il quarto di circonferenza $x^2+y^2=R^2$ nel quadrante positivo



$$\varphi(t)\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$$

$t \in [0, \frac{\pi}{2}]$

$$\begin{cases} x' = -R\sin t \\ y' = R\cos t \end{cases}$$

$\|\varphi'(t)\| = \sqrt{R^2} = R$

$$\int_\gamma \underset{\sim}{xy}\,ds = \int_0^{\pi/2} (R\cos t)(R\sin t)\cdot R\,dt = R^3 \int_0^{\pi/2} \cos t \sin t\,dt$$

$$= R^3 \left(\frac{\sin^2 t}{2}\right)_{t=0}^{t=\pi/2} = \frac{R^3}{2}.$$

(2) $\displaystyle\int_\gamma \sqrt{z}\,ds \qquad \gamma = \begin{cases} x = \cos t \\ y = \sin t \\ z = t^2 \end{cases} \qquad 0 \le t \le \pi$

$$\begin{cases} x' = -\sin t \\ y' = \cos t \\ z' = 2t \end{cases}$$

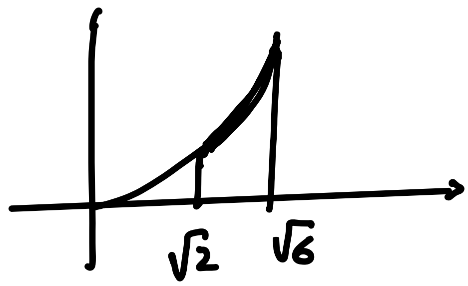
$$(x')^2 + (y')^2 + (z')^2 = 1 + 4t^2$$

$$\int_\gamma \sqrt{z}\,ds = \int_0^\pi \sqrt{t^2}\cdot\sqrt{1+4t^2}\,dt = \int_0^\pi t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{8}\int_0^\pi 8t\sqrt{1+4t^2}\,dt = \frac{1}{8}\cdot\frac{2}{3}\left[(1+4t^2)^{3/2}\right]_{t=0}^{t=\pi}$$

$$= \frac{1}{12}\left[(1+4\pi^2)^{3/2} - 1\right]$$

3) $\int_\gamma \frac{x}{y}\, ds$ $\qquad y = x^2 \qquad x \in [\sqrt{2}, \sqrt{6}]$



$$\begin{cases} x = t \\ y = t^2 \end{cases} \qquad t \in [\underbrace{\sqrt{2}}_{a}, \underbrace{\sqrt{6}}_{b}]$$

$$\begin{cases} x' = 1 . \\ y' = 2t . \end{cases}$$

$$\int_\gamma \frac{x}{y}\, ds = \int_{\sqrt{2}}^{\sqrt{6}} \frac{\cancel{t}}{t^{\cancel{2}}} \cdot \sqrt{1+4t^2} = \int_{\sqrt{2}}^{\sqrt{6}} \frac{\sqrt{1+4t^2}}{t}\, dt$$

$\underbrace{\sqrt{1+4t^2}} = u$: $\qquad 1+4t^2 = u^2$

$$4t^2 = u^2 - 1 , \qquad t^2 = \frac{u^2-1}{4}$$

$$t = \frac{\sqrt{u^2-1}}{2} .$$

$$dt = \frac{1}{2} \cdot \cancel{\frac{1}{2}} \cdot \frac{1}{\sqrt{u^2-1}} \cdot \cancel{2} u\, du$$

$$= \frac{u}{2\sqrt{u^2-1}}\, du$$

$$\int \frac{\sqrt{1+4t^2}}{t}\, dt = \int \frac{\cancel{2}u}{\sqrt{u^2-1}} \cdot \frac{u}{\cancel{2}\sqrt{u^2-1}}\, du$$

$$= \int \frac{u^2}{u^2-1}\, du$$

<u>FACILE</u> (COMPLETA

$$\int_{\gamma_2} \dots = 0 \qquad (\log 1 = 0)$$

$$y = \sqrt{1+x^2}$$

$$\gamma_3 \equiv \begin{cases} x = t \\ y = \sqrt{1+t^2} \end{cases} \qquad t \in [1, e]$$

$$\begin{cases} x' = 1 \\ y' = \dfrac{t}{\sqrt{1+t^2}} \end{cases}$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{1 + \frac{t^2}{1+t^2}} = \sqrt{\frac{1+2t^2}{1+t^2}}$$

$$\int_{\gamma_3} - = \int_1^e \frac{\cancel{\sqrt{1+t^2}} \cdot \lg(t)}{\cancel{\sqrt{2t^2+1}}} \cdot \frac{\cancel{\sqrt{1+2t^2}}}{\cancel{\sqrt{1+t^2}}}\, dt$$

$$= \int_1^e \log t \, dt = \left(t \log t\right)_1^e - \left(t\right]_1^e$$

$$= e - e + 1 = 1$$

$$\gamma_4 \equiv \begin{cases} x = e \\ y = t \end{cases} \qquad t \in [0, \sqrt{1+e^2}] \qquad \begin{cases} x' = 0 \\ y' = 1 \end{cases}$$

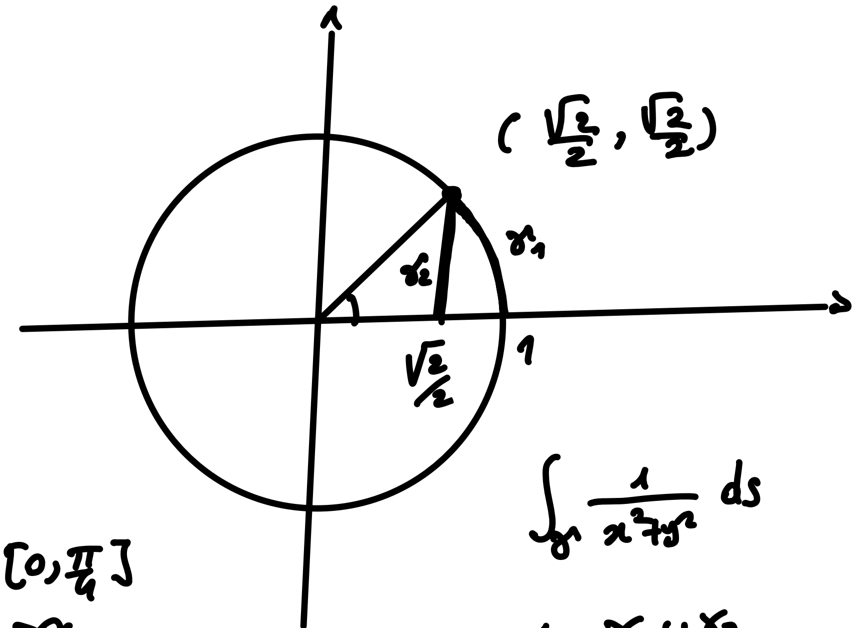
$$\int_{\gamma_4} \dots = \int_0^{\sqrt{1+e^2}} \frac{t \overbrace{\log e}^{=1}}{\sqrt{2e^2+1}}\, dt =$$

$$= \frac{1}{\sqrt{2e^2+1}} \int_0^{\sqrt{1+e^2}} t\, dt$$

$$= \frac{1}{\sqrt{2e^2+1}} \left(t^2\right)_0^{\sqrt{1+e^2}} = \frac{1+e^2}{\sqrt{1+2e^2}}$$

<u>CONCL.</u> $\displaystyle \int_{\gamma} \dots = \int_{\gamma_3} + \int_{\gamma_4} = 1 + \frac{1+e^2}{\sqrt{1+2e^2}}$

<u>ES.</u> 5



$$\gamma_1 = \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in \left[0, \frac{\pi}{4}\right]$$

$$\int_{\gamma} \frac{1}{x^2+y^2}\, ds$$

$$\gamma = \gamma_1 \cup \gamma_2$$

$$\begin{cases} x' = -\sin t \\ y' = \cos t \end{cases} \qquad (x')^2 + (y')^2 = 1$$

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

$$\int_{\gamma_1} \cdots = \int_0^{\pi/4} \frac{1}{1} \cdot 1 \, dt = \frac{\pi}{4}$$

$$\gamma_2 = \begin{cases} x = \frac{\sqrt{2}}{2} \\ y = t \end{cases} \quad t \in \left[0, \frac{\sqrt{2}}{2}\right] ; \quad \begin{cases} x' = 0 \\ y' = 1 \end{cases}$$

$$\int_{\gamma_2} \cdots = \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\frac{1}{2} + t^2} \, dt = 2 \int_0^{\frac{\sqrt{2}}{2}} \underbrace{\frac{dt}{1+2t^2}}_{\text{integrale noto}}$$

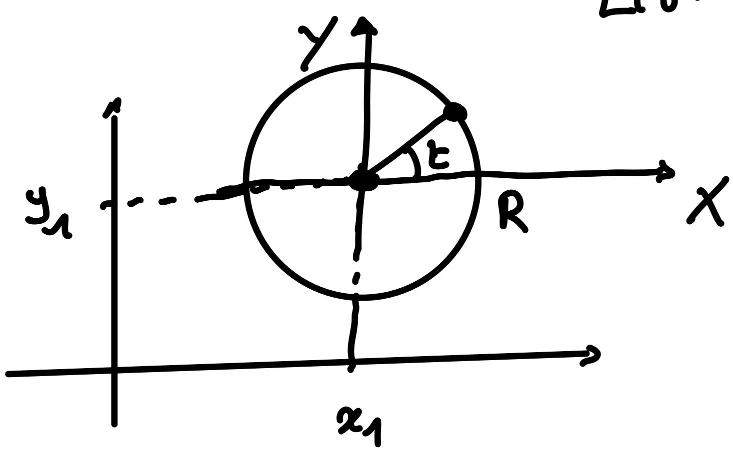
BARICENTRO DI UNA CURVA

γ regolare a tratti di $\mathbb{R}^2$: baricentro di γ

il punto (x_0, y_0) di coordinate

$$\begin{cases} x_0 = \dfrac{1}{L(\gamma)} \displaystyle\int_\gamma x \, ds \\[2ex] y_0 = \dfrac{1}{L(\gamma)} \displaystyle\int_\gamma y \, ds \end{cases}$$

ES



$$\begin{cases} x = x_1 + X \\ y = y_1 + Y \end{cases}$$

$$\begin{cases} X = R \cos t \\ Y = R \sin t \end{cases}$$

$$\gamma = \begin{cases} x = x_1 + R \cos t \\ y = y_1 + R \sin t \end{cases} \qquad t \in [0, 2\pi]$$

$$L(\gamma) = 2\pi R$$

$$\int_\gamma x \, ds$$

$$\begin{cases} x' = -R \sin t \\ y' = R \cos t \end{cases}$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{R^2} = R$$

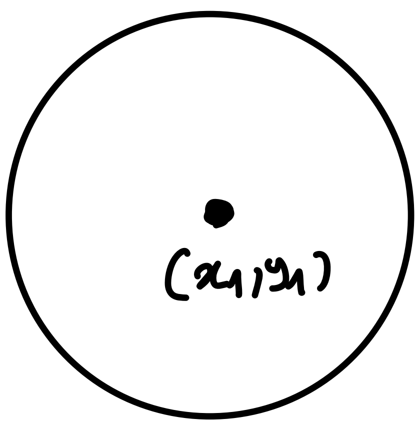
$$\int_\gamma x\,ds = \int_0^{2\pi} (x_1 + R\cos t)\cdot R\,dt$$

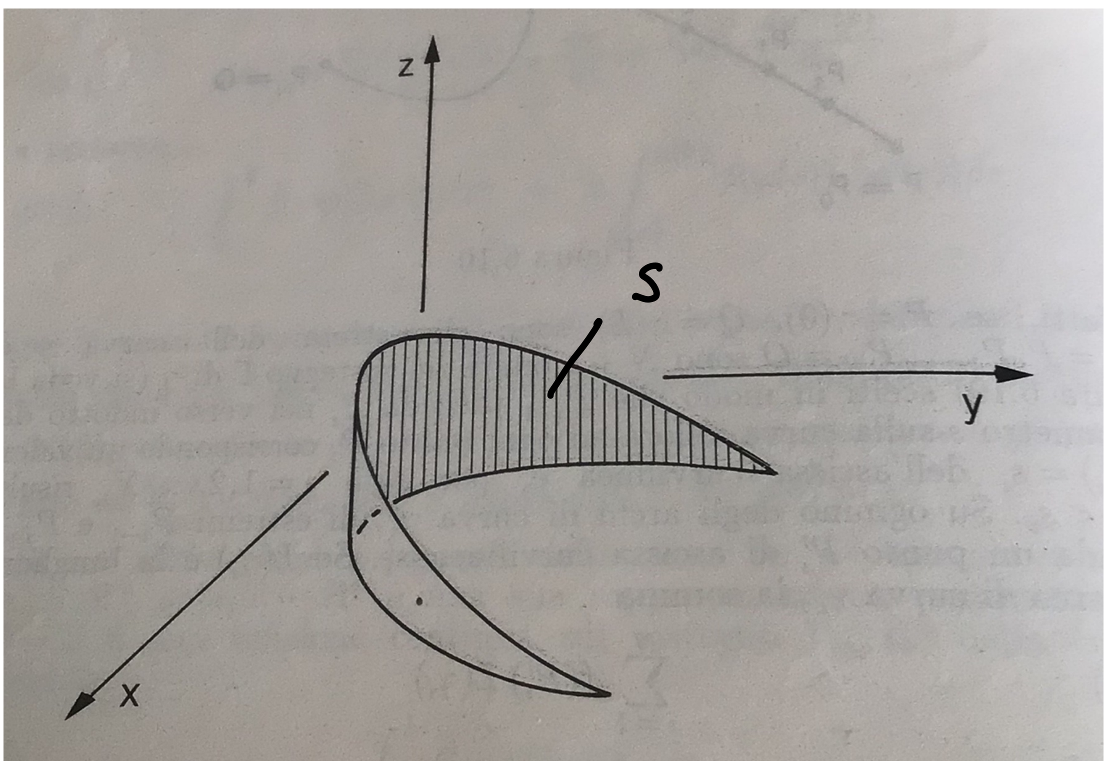
$$= x_1 \int_0^{2\pi} R\,dt + R^2 \underbrace{\int_0^{2\pi} \cos t\,dt}_{=0}$$

$$= 2\pi R x_1$$

$$x_0 = \frac{1}{L(\gamma)} \int_\gamma x\,ds = \frac{1}{\cancel{2\pi R}} \cdot \cancel{2\pi R}\, x_1 = $$

y_0 = analogamente = y_1

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