Lezione del 06/10/23

Criterio del rapporto

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Se $\displaystyle\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell$: allora

$$\rho = \frac{1}{\ell}$$

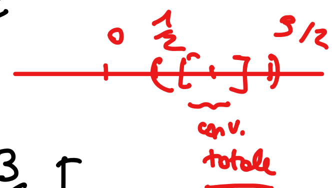
$$\lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = \rho \qquad \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

ES.

$$\sum_{n=0}^{\infty} \underbrace{2^n}_{a_n} (x-\underbrace{1}_{x_0})^n \qquad a_n = 2^n$$

$$a_{n+1} = 2^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2 \quad : \quad \rho = \frac{1}{2}$$



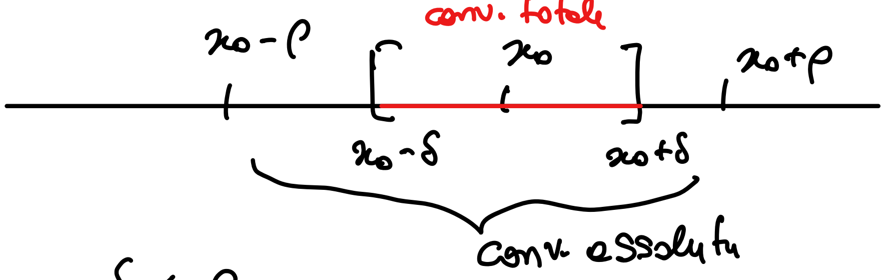
Int. conv. $\left]1-\frac{1}{2}, 1+\frac{1}{2}\right[= \left]\frac{1}{2}, \frac{3}{2}\right[$

Conv. totale? $[x_0-\delta, x_0+\delta] = [1-\delta, 1+\delta]$ conv. assoluta

$\delta < \frac{1}{2}$

NOTA

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad \rho > 0 \ , \ \left]x_0-\rho, x_0+\rho\right[$$



$[x_0-\delta, x_0+\delta]$, $\quad \delta < \rho$, $\delta \neq \rho$

conv. totale

<u>Agli estremi</u> ? $\quad \sum_{n=0}^{\infty} 2^n (x-1)^n \qquad]\frac{1}{2}, \frac{3}{2}[$

$$x=\frac{1}{2} \qquad \sum_{n=0}^{\infty} 2^n \cdot \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \cancel{2^n} \cdot \frac{(-1)^n}{\cancel{2^n}} \quad \underline{\underline{\text{oscillante}}}$$

$$x=\frac{3}{2} \qquad \sum_{n=0}^{\infty} 2^n \cdot \left(\frac{3}{2}-1\right)^n = \sum_{n=0}^{\infty} 1 = +\infty$$

ES. $\quad \sum_{n=0}^{\infty} (-1)^n\, n!\, x^n \quad$ converge solo per $x = 0 = x_0$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{(n+1)\cancel{!}}{\cancel{n!}} = +\infty = \ell$$

$$\rho = \frac{1}{\ell} = \frac{1}{\infty} = 0$$

$$\left|\frac{a_{n+1}}{a_n}\right| \qquad |(-1)^n| = 1$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)!}{(-1)^n\, n!}\right| = \frac{(n+1)!}{\cancel{n!}} = (n+1) \to \infty$$

$$\sum_{n=1}^{\infty} \frac{(1-x^2)^n}{n} \qquad \sum a_n (x-x_0)^n$$

$$y = 1-x^2$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n} \qquad a_n = 1/n \quad : \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \xrightarrow{n\to\infty} 1$$

$\rho = 1$ $\qquad (-1, 1)$

quindi la serie converge assolutamente per $y \in (-1,1)$

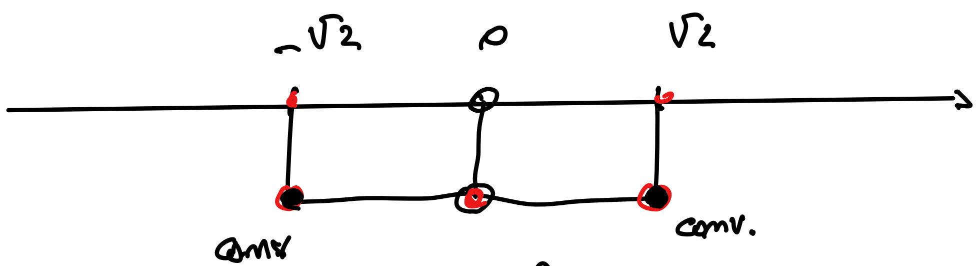
$\Rightarrow$ la serie iniziale conv. assolutamente per x tale che

$$-1 < 1-x^2 < 1$$

$$\Leftrightarrow \quad -1 < x^2-1 < 1$$

$$\Leftrightarrow \quad 0 < \underbrace{x^2}_{} < 2 \quad \Leftrightarrow \quad x \neq 0$$

$$-\sqrt{2} < x < \sqrt{2}$$



$$\sum_{n=1}^{\infty} \frac{y^n}{n} \qquad y = -1 \quad : \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ \text{converge} \ :$$

ma allora la serie in x converge quando $1-x^2 = -1$

$\Leftrightarrow x = \pm\sqrt{2}$

$y=1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$: la serie in x diverge quando

$$1-x^2 = 1 \iff x=0$$

La serie in y converge totalmente $[-\delta, \delta]$, $\delta < 1$

$(-1,1)$ $\quad y_0 = 0$

$\Longleftarrow$ $\quad -\delta \leq y \leq \delta \iff$ la serie in x converge

totalmente per x tale che

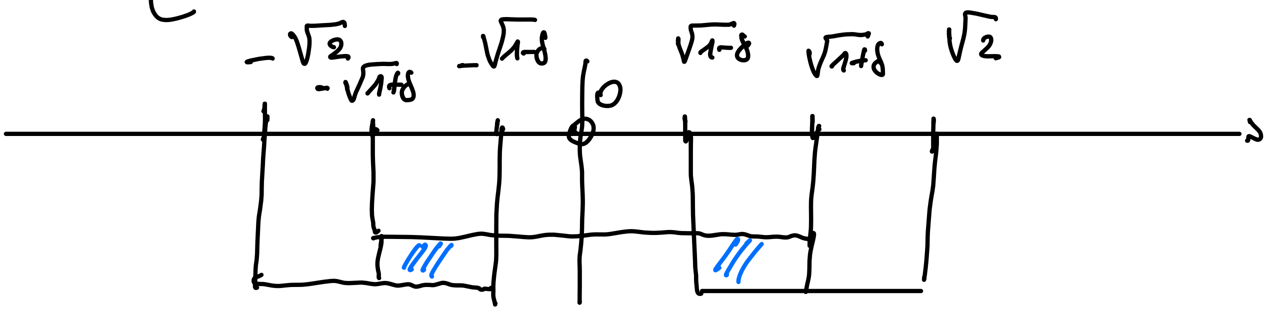
$$-\delta \leq 1-x^2 \leq \delta$$

interni:

$$\begin{cases} 1-x^2 \geq -\delta \\ 1-x^2 \leq \delta \end{cases} \iff \begin{cases} x^2-1 \leq \delta \iff x^2 \leq 1+\delta \\ x^2-1 \geq -\delta \iff x^2 \geq 1-\delta \end{cases}$$

esterni

$$\iff \begin{cases} -\sqrt{1+\delta} \leq x \leq \sqrt{1+\delta} \\ x \leq -\sqrt{1-\delta}, \quad x \geq \sqrt{1-\delta} \end{cases} \qquad 0 < \delta < 1$$



$x \in [-\sqrt{1+\delta}, -\sqrt{1-\delta}] \cup [\sqrt{1-\delta}, \sqrt{1+\delta}]$

convergenza totale

Insieme di convergenza della serie

$$\sum_{n\geq 1}^{\infty} \frac{n+1}{n+2}\,\frac{x^{3n}}{3^n} = \sum_{n=1}^{\infty} \frac{n+1}{n+2}\cdot\frac{(\overbrace{x^3}^{y})^n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n+2}\cdot\frac{y^n}{3^n} \qquad a_n = \frac{n+1}{n+2}\cdot\frac{1}{3^n}$$

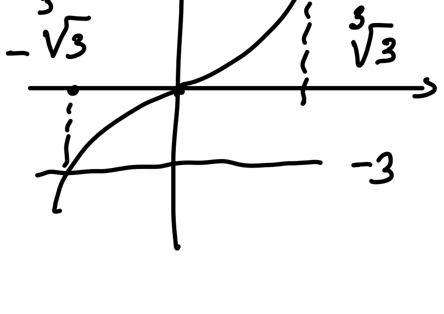
$$y_0 = 0$$

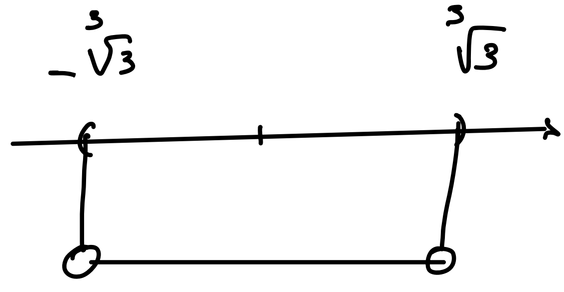
$$\left|\frac{a_{n+1}}{a_n}\right| = \overset{\nearrow 1}{\frac{n+2}{n+3}}\cdot\underbrace{\frac{1}{3^{n+1}}\cdot\cancel{3^n}}_{\frac{1}{3}}\cdot\overset{\nearrow 1}{\frac{(n+2)}{(n+1)}} \xrightarrow{n\to\infty} \frac{1}{3}$$

$\rho = \frac{1}{\frac{1}{3}} = 3$ raggio di convergenza. La serie in y converge assolutamente per $y \in (-3, 3)$

$\Leftrightarrow$ la serie in x converge assolutamente quando

$$-3 < x^3 < 3 \iff -\sqrt[3]{3} < x < \sqrt[3]{3}$$





Convergenza agli estremi? $\underbrace{y=-3}$, $y=3$

$$\sum_{n=1}^{\infty} \frac{n+1}{n+2}\cdot\frac{(-1)^n\cdot\cancel{3^n}}{\cancel{3^n}} \qquad \lim_{n\to\infty}\left|\frac{n+1}{n+2}(-1)^n\right| = 1$$

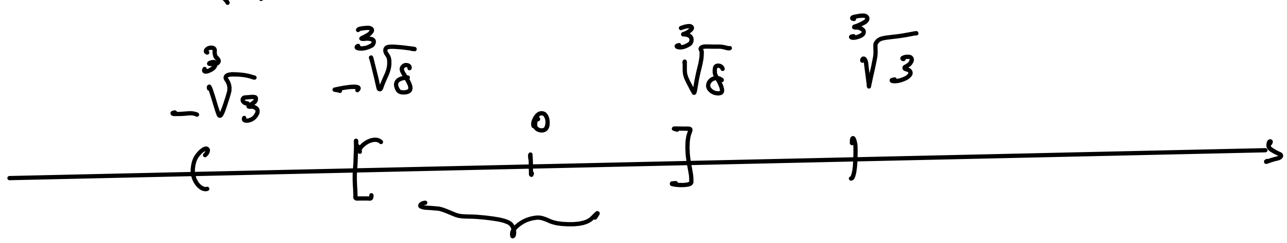
NO CONVERGENZA AGLI ESTREMI!

Convergenza totale della serie in y: $y=x^3$

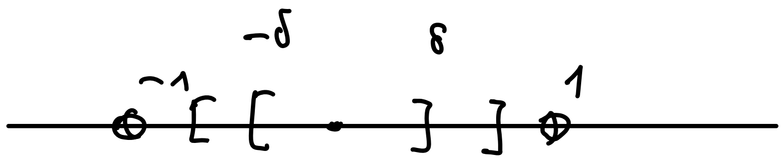
$y\in[-\delta,\delta]$, $0<\delta<3$

$$-\delta\le x^3\le\delta$$

$$\iff -\sqrt[3]{\delta}\le x\le\sqrt[3]{\delta}$$



ES. $\sum_{n=0}^{\infty} x^n$



Se vi fosse conv. totale nell'intervallo $(-1,1)$

$|x|^n\le M_n \quad \forall x\in(-1,1)$

$\sum M_n<\infty$

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{2n+1}\cdot 3^n\cdot x^{4n} \qquad y=x^4$$

$$\sum_{n=2}^{\infty} \underbrace{\frac{(-1)^n}{2n+1} \cdot 3^n}_{a_n} y^n$$

Raggio di convergenza:

$$a_{n+1} = \frac{3^{n+1}}{2n+3} \qquad |(-1)^n| = 1$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\cancel{3^n \cdot 3}\; 3^{n+1}}{2n+3} \cdot \frac{2n+1}{\cancel{3^n}} \right| \xrightarrow[n\to\infty]{} 3$$

$\rho = \frac{1}{3}$: la serie converge quando

$$-\frac{1}{3} < x^4 < \frac{1}{3} \iff x^4 < \frac{1}{3} \; ?$$

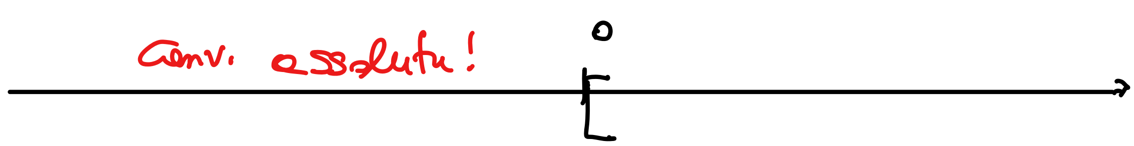
$$\iff -\sqrt[4]{\tfrac{1}{3}} < x \; \sqrt[4]{\tfrac{1}{3}}$$

$$\sum_{n=1}^{\infty} \frac{e^{nx}}{n+\log n} \qquad (y = e^x) \qquad \sum_{n=1}^{\infty} \frac{y^n}{n+\log n}$$

$$\lim_{n\to\infty} \frac{n+\log n}{(n+1)+\log(n+1)} = 1$$

La serie in y converge assolutamente quando

$$-1 < y < 1 \iff e^x < 1 \iff \boxed{x < 0}$$

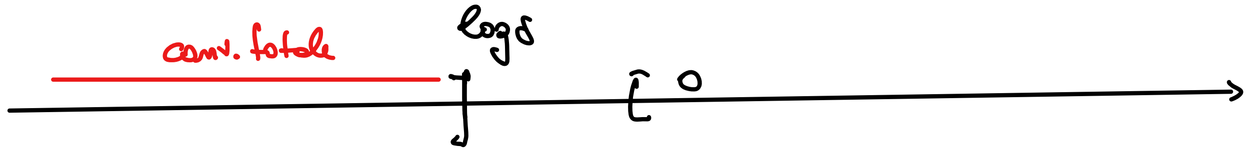


$y=1$ $\qquad \sum \frac{1}{n+\log n} \sim \sum \frac{1}{n} = +\infty$

$e^x = 1 \iff x=0$ NON CONVERGE!!

La serie in y conv. totalmente per $y \in [-\delta, \delta]$, $0<\delta<1$

$$\iff e^x \le \delta \iff x \le \log \delta$$



Oss. Se la serie $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converge in $]x_0-\rho, x_0+\rho[$, allora converge totalmente in tutti gli intervalli

$$[x_0-\delta, x_0+\delta], \quad 0<\delta<\rho.$$

Questo **non** implica che in

$]x_0-\rho, x_0+\rho[$ la convergenza

su totale

ES. La serie geometrica $\sum_{n=0}^{\infty} x^n$

converge in $]-1,1[$. Se $0<\delta<1$,

per ogni $x\in[-\delta,\delta]$ si ha

$$|x^n| \leq \underbrace{\delta^n}_{M_n}, \forall n\in\mathbb{N}, \text{ con}$$

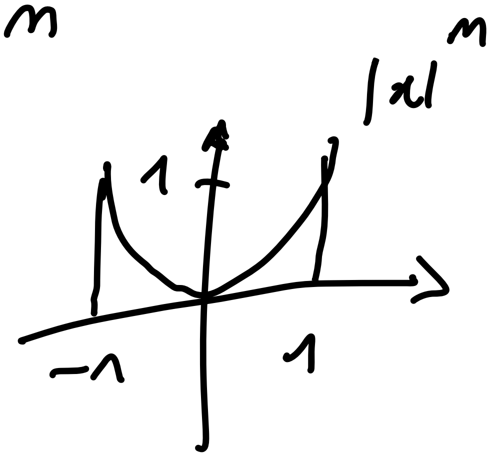
$$\sum_{n=0}^{\infty} \delta^n < \infty \quad (\text{poiché } \delta<1).$$

Se vi fosse convergenza totale in $]-1,1[$

avremmo $|x^n| \leq M_n$, $\forall x\in]-1,1[$ $\forall n\in\mathbb{N}_0$

$$\sum_{n=0}^{\infty} M_n < \infty$$

quindi $\underbrace{\sup_{x\in]-1,1[} |x|^n}_{=1} \le M_n$



$\Rightarrow M_n \geq 1$, che è assurdo.