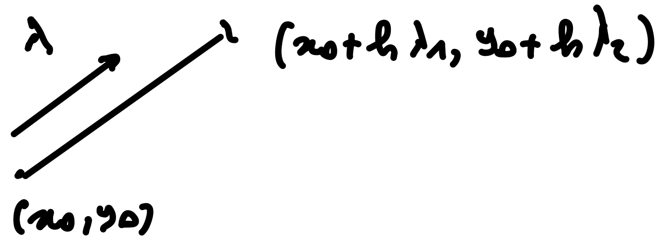
Lezione del 31/10/2023

DERIVATA DIREZIONALE

$f(x,y)$, $\quad f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x_0, y_0) \in A$

$\underline{\lambda} = (\lambda_1, \lambda_2)$ direzione : $\|\underline{\lambda}\| = 1$

$$\frac{\partial f}{\partial \underline{\lambda}}(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0 + h\lambda_1, y_0 + h\lambda_2) - f(x_0,y_0)}{h}$$



$\underline{\lambda} = \underline{e}_1 = (1,0) \qquad \frac{\partial f}{\partial \underline{\lambda}}(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)$

$\underline{\lambda} = \underline{e}_2 = (0,1) \qquad \frac{\partial f}{\partial \underline{\lambda}}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0)$

Teorema (FORMULA DEL GRADIENTE)

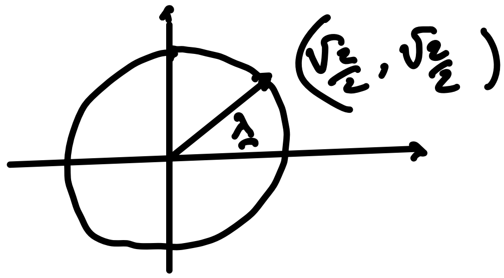
$f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ differenziabile. Allora, per ogni $(x_0,y_0) \in A$, per ogni direzione $\underline{\lambda} = (\lambda_1, \lambda_2)$, esiste

$$\frac{\partial f}{\partial \underline{\lambda}}(x_0,y_0) \quad e \quad \frac{\partial f}{\partial \underline{\lambda}}(x_0,y_0) = \nabla f(x_0,y_0)\cdot \underline{\lambda}$$

$$= f_x(x_0,y_0)\,\lambda_1 + f_y(x_0,y_0)\,\lambda_2$$

ES. $f(x,y) = x^2 - 3x + 4xy + 5$

$(x_0,y_0) = (1,0) \qquad \underline{\lambda} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



$f_x = 2x - 3 + 4y \qquad f_y = 4x$

$f_x(1,0) = 2 - 3 = -1 \qquad f_y(1,0) = 4$

$$\frac{\partial f}{\partial \underline{\lambda}} = \nabla f(x_0,y_0)\cdot\underline{\lambda} = -1\cdot\frac{\sqrt{2}}{2} + \cancel{4}^{\,2}\cdot\frac{\sqrt{2}}{\cancel{2}} = 2\sqrt{2} - \frac{\sqrt{2}}{2}.$$

OSS.

$$\frac{\partial f}{\partial \lambda}(x_0,y_0) = \nabla f\cdot\underline{\lambda}$$

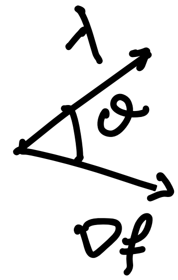
$$\left|\frac{\partial f}{\partial \lambda}\right| = |\nabla f\cdot\underline{\lambda}| \le \|\nabla f\|\cdot\underbrace{\|\underline{\lambda}\|}_{=1} = \|\nabla f\|$$

$|a| \le b \iff -b \le a \le b$

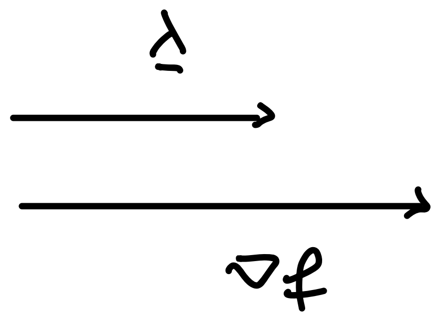
$$-\|\nabla f\| \le \frac{\partial f}{\partial \lambda} \le \|\nabla f\|$$

$\frac{\partial f}{\partial \lambda}$ è MASSIMA QUANDO

$$\|\nabla \cancel{f}\| \cos\theta = \frac{\partial f}{\partial \lambda} = \|\nabla \cancel{f}\|$$

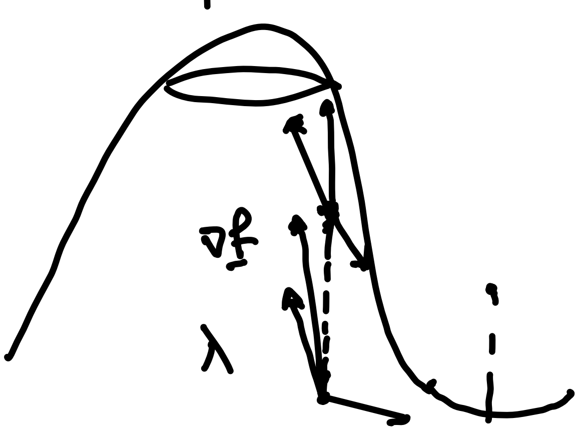


$$\cos\theta = 1 \iff \theta = 0$$

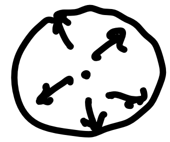


$$\underline{\lambda} = \frac{\nabla f}{\|\nabla f\|}$$: $\frac{\partial f}{\partial \lambda}$ massima nella direzione e nel verso di ∇f.

$\frac{\partial f}{\partial \lambda}$ minima quando $\underline{\lambda} = -\frac{\nabla f}{\|\nabla f\|}$



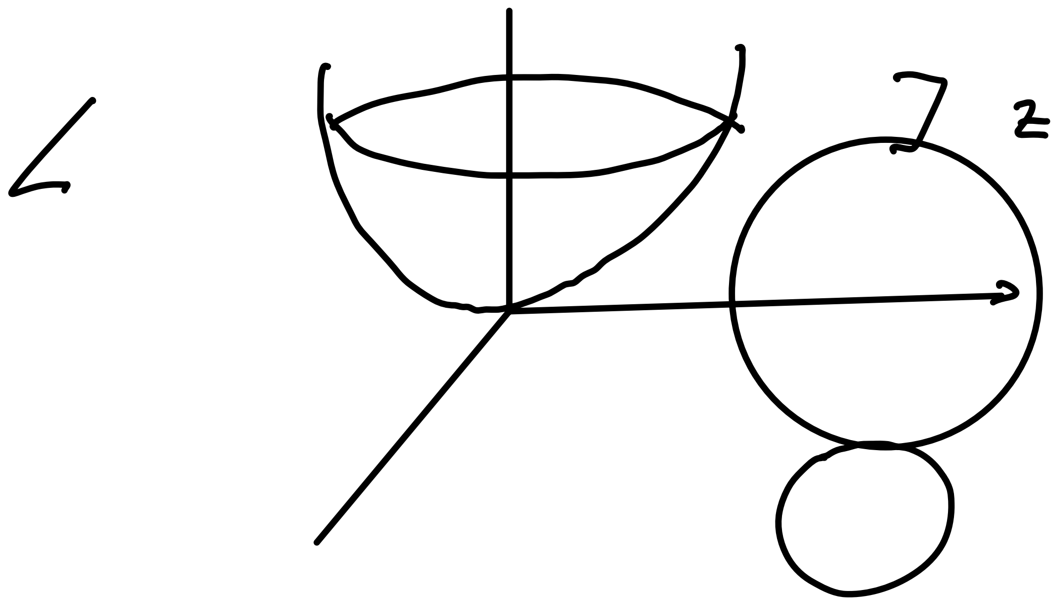
(x_0, y_0)

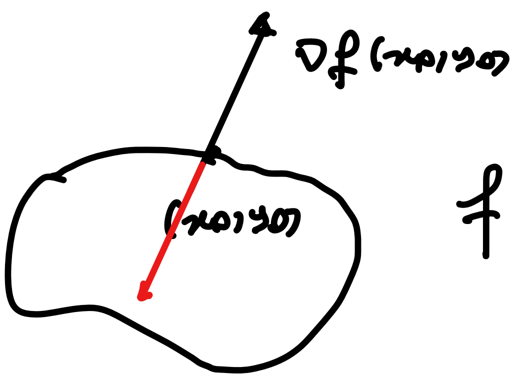


MINIMO

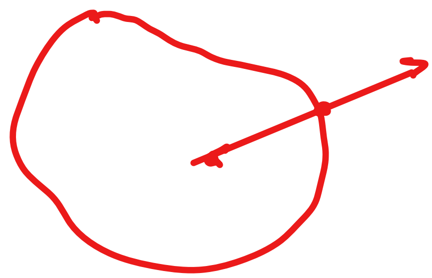
2^a CONSEGUENZA

(CARATTERIZZAZIONE DI ORTOGONALITÀ DEL GRADIENTE ALLE CURVE DI LIVELLO)





$f(x,y) = f(x_0, y_0)$



ES. $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y) =$

$= 2x\,\underline{e}_1 + 2y\,\underline{e}_2$

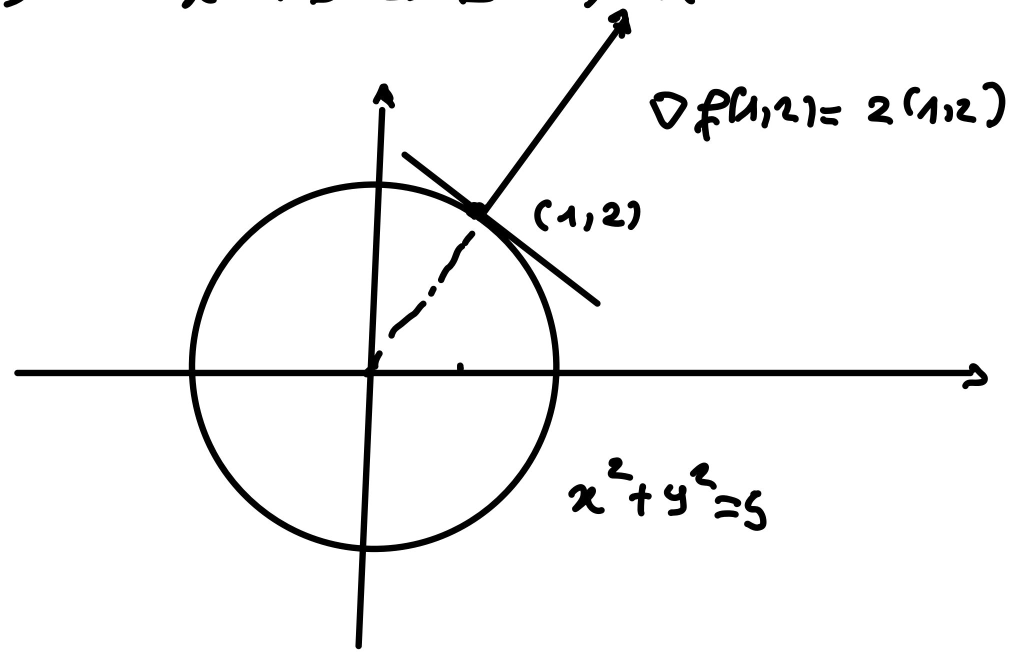
$(x_0, y_0) = (1, 2)$

$\nabla f(1,2) = (2, 4) = 2(1,2)$

$f(x_0,y_0) = f(\underset{x}{1},\underset{y}{2}) = 1 + 4 = 5$

Curva di LIVELLO per $(1,2)$: $\quad f(x,y) = f(x_0,y_0)$

$\iff \quad x^2 + y^2 = 5 \quad , \quad R = \sqrt{5}$



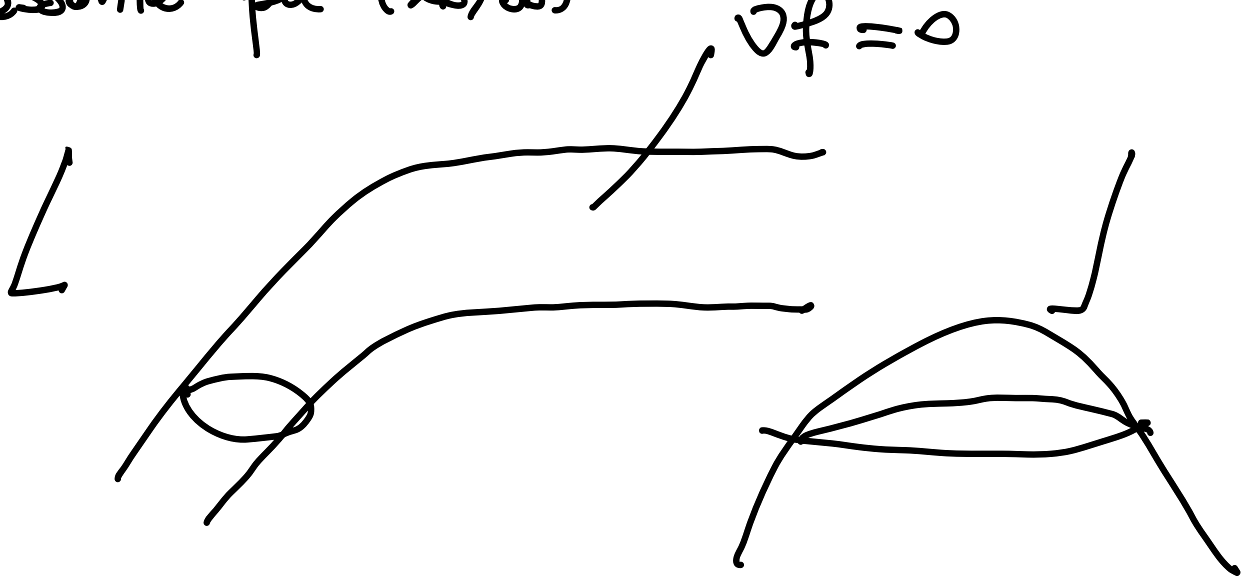
<u>**Teorema**</u> $\quad f : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ è

differenziabile in (x_0,y_0) e $\nabla f(x_0,y_0) \neq \underline{0}$

allora $\nabla f(x_0,y_0)$ è ORTOGONALE ALLA

CURVA DI LIVELLO $\quad f(x,y) = f(x_0,y_0)$

passante per (x_0, y_0)



MASSIMI E MINIMI RELATIVI

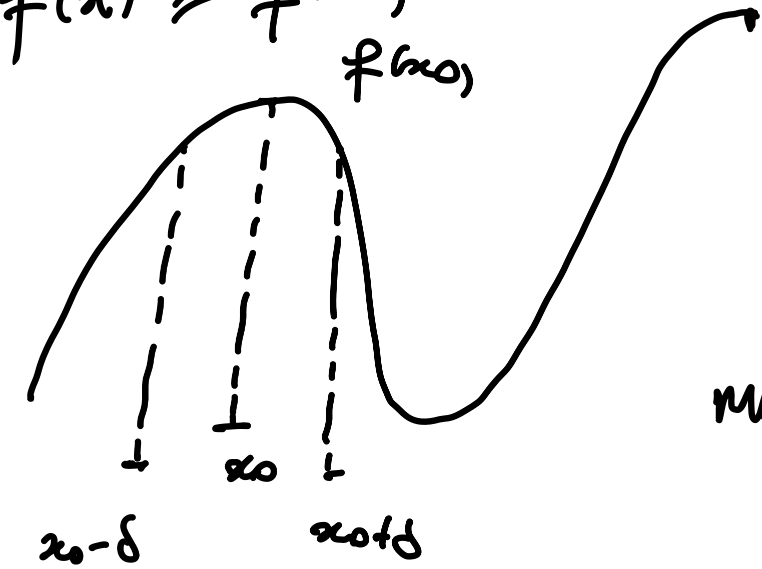
$$f = f(x) \quad , \quad f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

$x_0 \in I$: x_0 è di MAX RELATIVO (MIN. REL.)

se $\exists \delta > 0$: $f(x) \leq f(x_0)$ $\quad \forall x \in I \cap]x_0 - \delta, x_0 + \delta[$

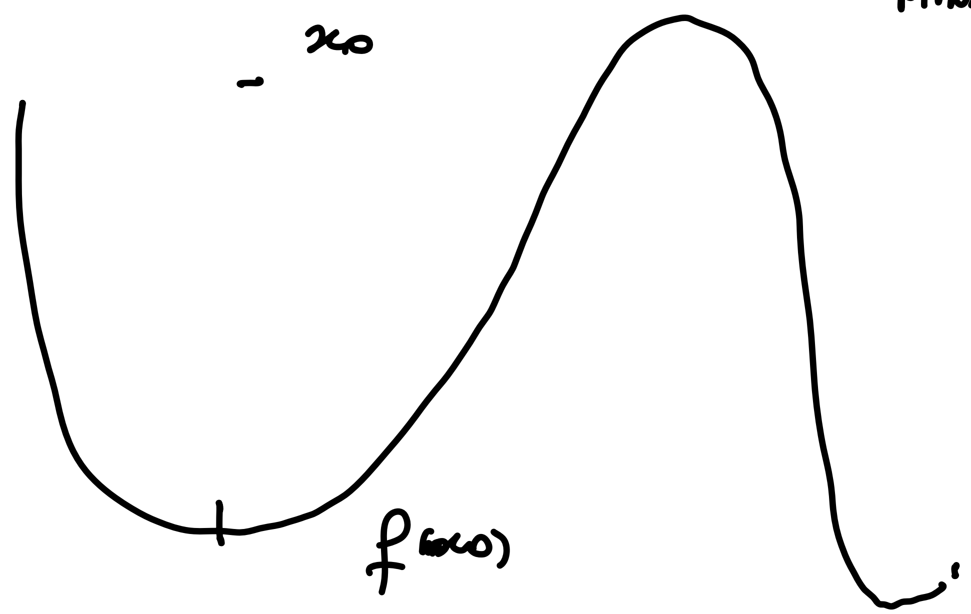
$(f(x) \geq f(x_0)$ $)$

$(f(x) \leq f(x_0))$



MAX ASS.

MIN AS.



Def.

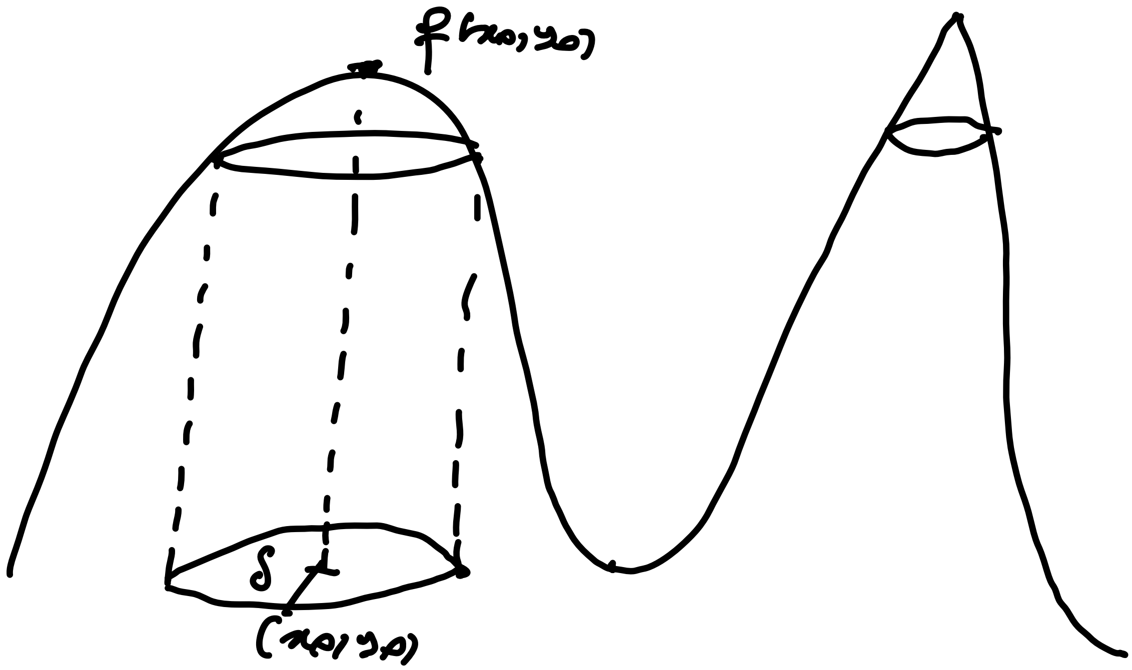
$$f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$(x_0, y_0) \in X$: si dice che (x_0, y_0) è

di MAX RELATIVO PER $f(x,y)$ se

$\exists \delta > 0 : f(x,y) \leq f(x_0,y_0)$

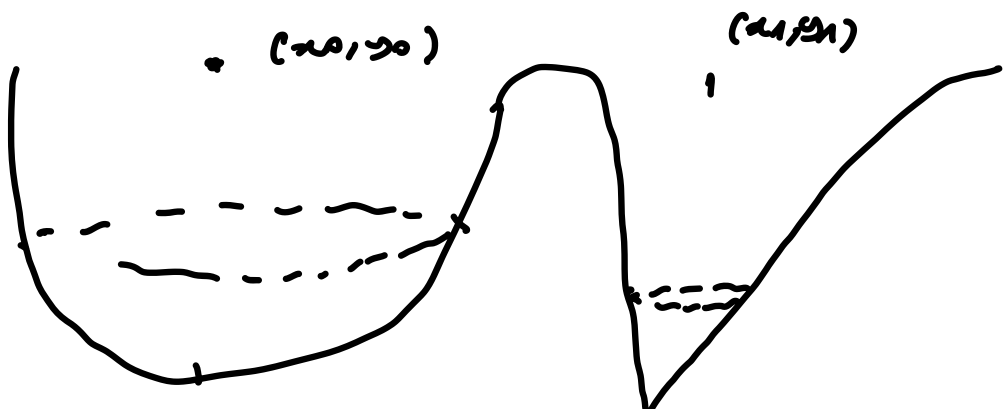
$\forall (x,y) \in X \cap I_\delta(x_0,y_0)$

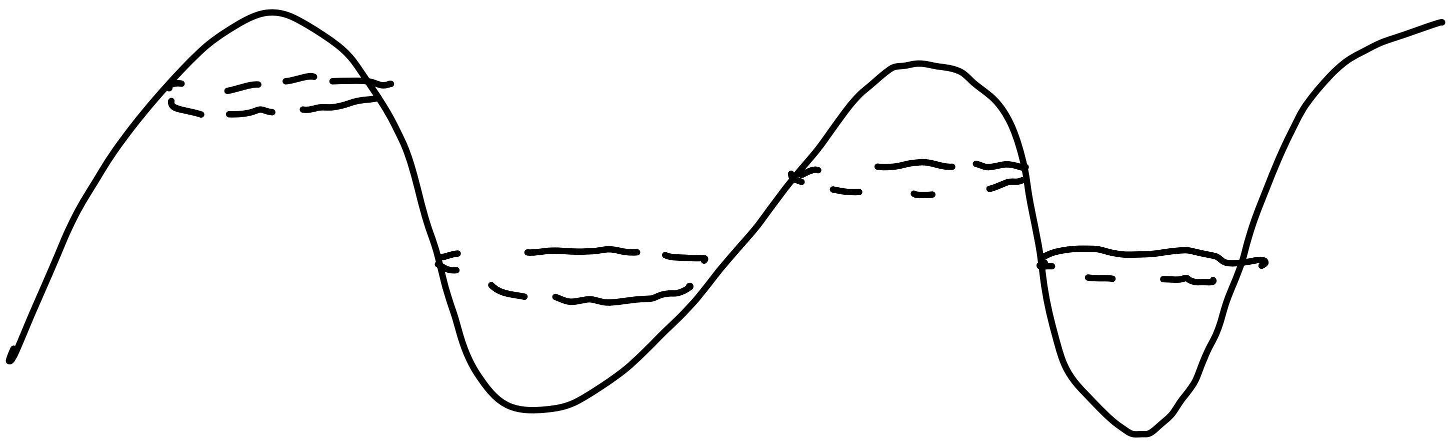


$(x_0,y_0) \in X$ di minimo relativo

$\exists \delta > 0 : f(x,y) \geq f(x_0,y_0)$

$\forall (x,y) \in X \cap I_\delta(x_0,y_0)$



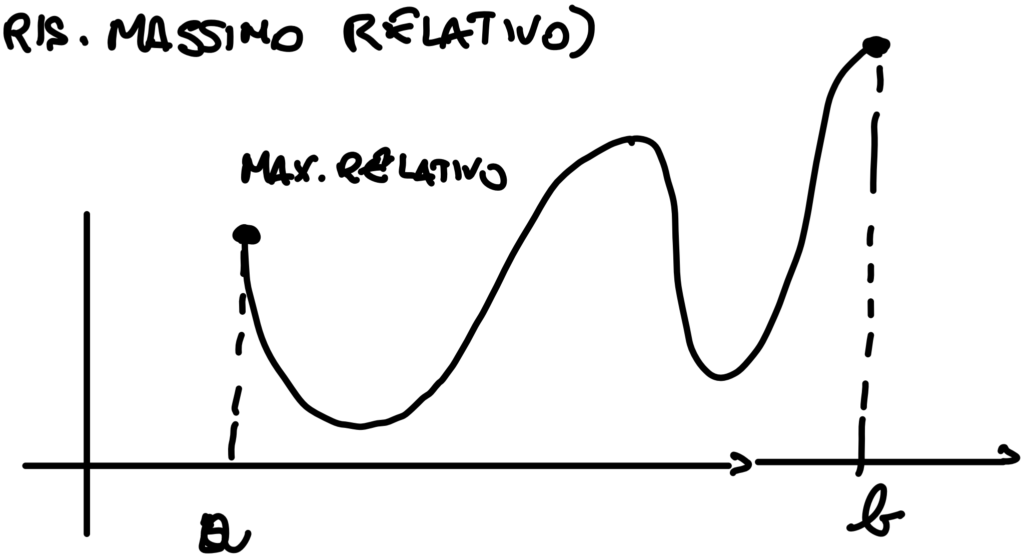


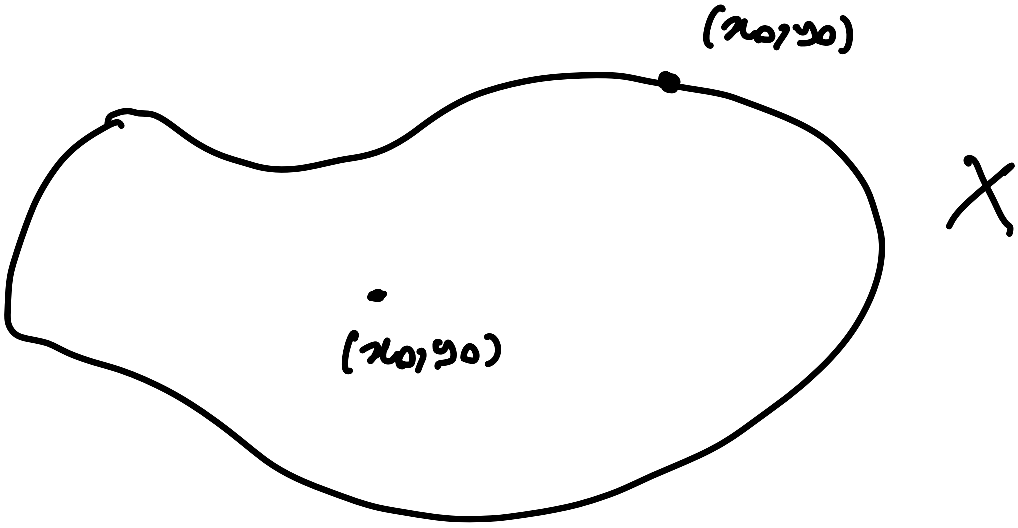
(RIS. MASSIMO)

OSS. SE ESISTONO, I PUNTI DI MINIMO ASSOLUTO SONO DI MINIMO RELATIVO

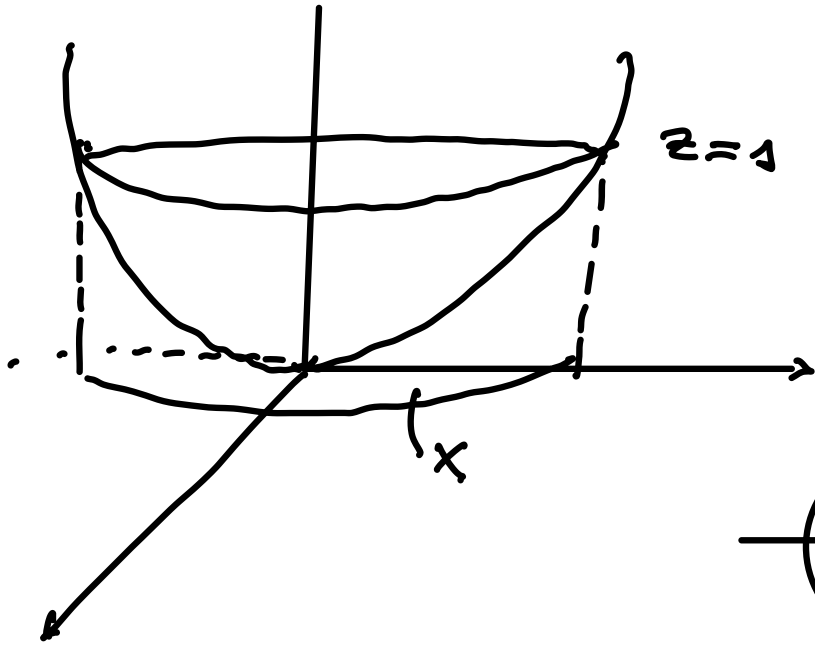
(RIS. MASSIMO RELATIVO)

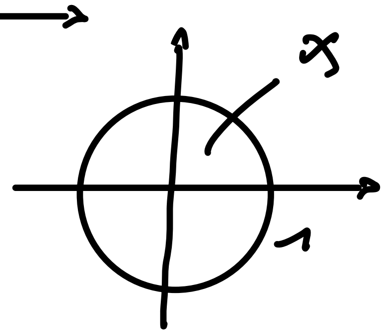
OSS.





$f(x,y) = x^2+y^2 \qquad X = \{(x,y) : x^2+y^2 \leq 1\}$





$(0,0)$ minimo assoluto

Su ∂X : $x^2+y^2=1 \Rightarrow$

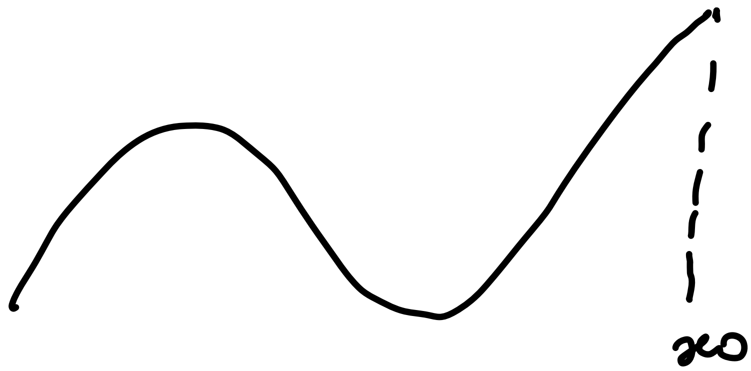
Se $(x_0,y_0) \in \partial X$, $f(x_0,y_0) = x_0^2+y_0^2 = 1$

$\geq f(x,y)$

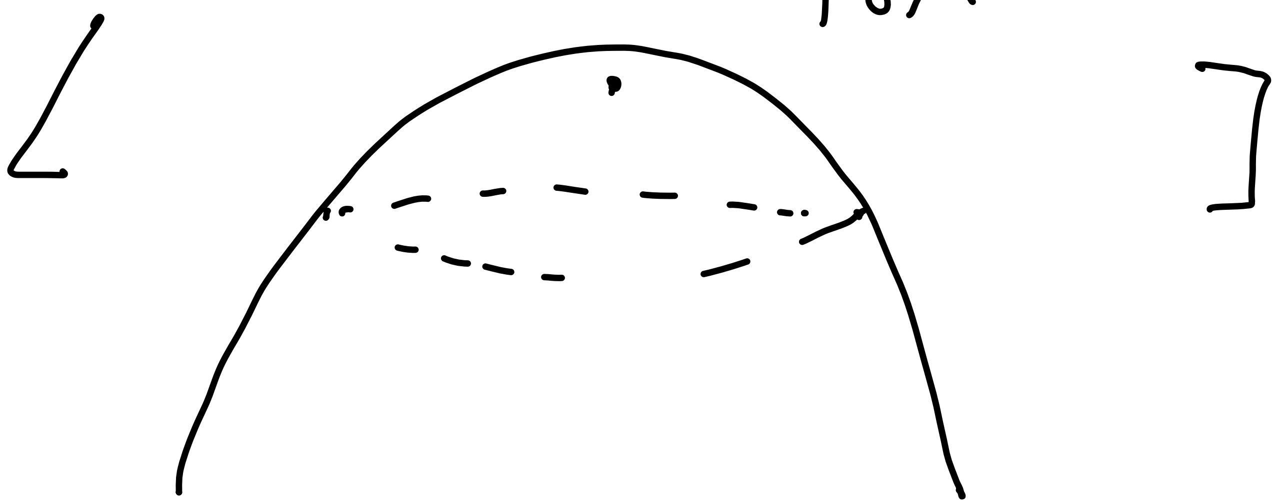
<u>ANALISI I</u> : x_0 ESTREMO LOCALE

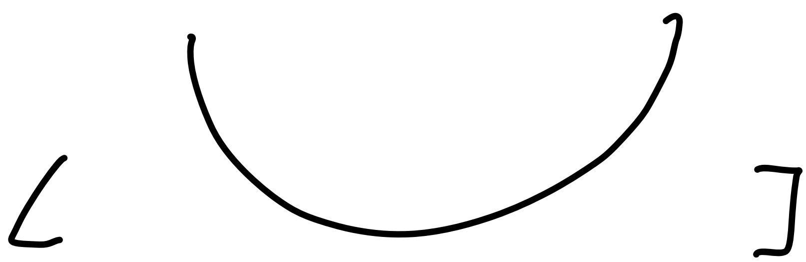
INTERNO AD $(a,b) = I$

$\Longrightarrow f'(x_0) = 0$ (T. FERMAT)



$$z = f(x_0, y_0) + f_x(\cancel{x_0}, y_0)(x - x_0) +$$
$$f_y(\cancel{x_0}, y_0)(y - y_0)$$





$$\nabla f(x_0, y_0) = (0, 0)$$

CONDIZIONE NECESSARIA AL I° ORDINE

$f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ derivabile in (x_0, y_0) **interno** ad X. Allora, se (x_0, y_0) è un estremo locale per f, si ha

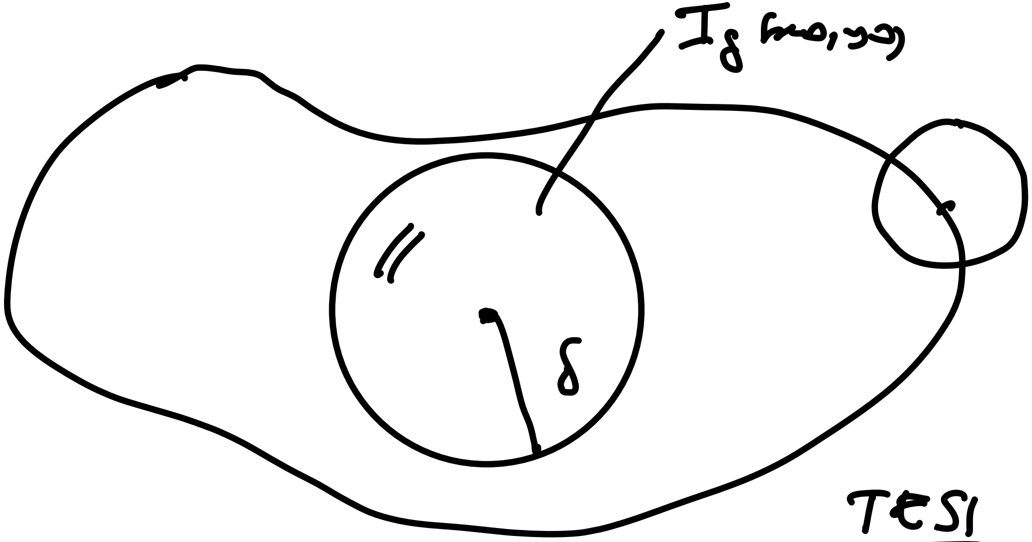
$$\nabla f(x_0, y_0) = \underline{0} = (0,0)$$

Dim. (x_0, y_0) MAX. RELATIVO: poiché (x_0, y_0) è INTERNO AD X,

$\exists \delta > 0 : \quad I_\delta(x_0, y_0) \subseteq X \quad$ e

$$f(x,y) \leq f(x_0, y_0)$$

$$\forall (x,y) \in I_\delta(x_0, y_0)$$

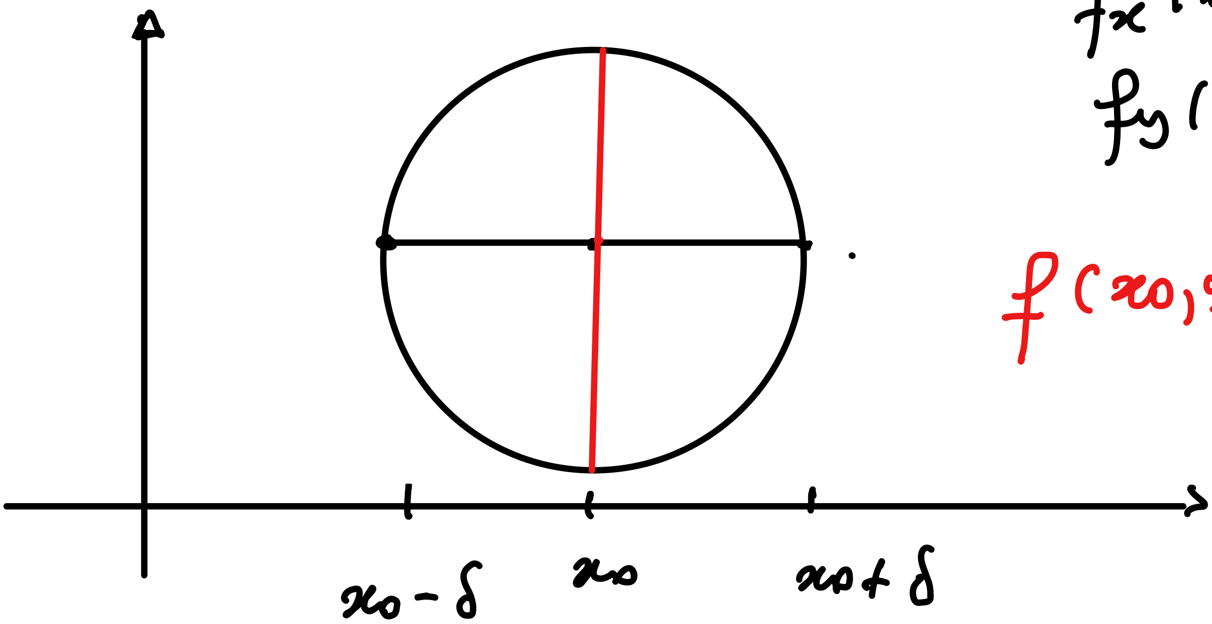


TESI

$f_x(x_0,y_0)=0$

$f_y(x_0,y_0)=0$

$f(x_0,y) := F(y)$



POICHÉ (x_0,y_0) è di MAX,

$$f(x,y_0) \leq f(x_0,y_0) \quad (1)$$

$$\forall x \in \left]x_0-\delta, x_0+\delta\right[$$

$F(x) := f(x,y_0)$. Dalla (1),

$$F(x) \leq F(x_0)$$
$$\forall x \in \,]x_0 - \delta, x_0 + \delta[$$

$\Rightarrow$ x_0 è di massimo per $F(x)$

$$\Rightarrow \quad \underbrace{F'(x_0)}_{\substack{\parallel \\ f_x(x_0,y_0)}} = 0 \quad \text{(TEOR. FERMAT)}$$

$$f_x(x_0,y_0) = 0$$

ANALOGAMENTE, $f_y(x_0,y_0) = 0$.

C.V.D.

Se $(x_0, y_0) \in \partial X$ È VERO

CHE $\nabla f(x_0,y_0) = \underline{0}$??

ALCUNI ESERCIZI SULLE SERIE DI FUNZIONI

Determinare l'insieme di convergenza delle seguenti serie:

1) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(2-x^2)^n}$; 2) $\displaystyle\sum_{n=1}^{\infty} e^{nx+\sin n}$

3) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{3^n(n-1)} \left(\log x - 1\right)^n$;

4) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)^{2n}}{(4n-1)^{2n}} (x^2-1)^n$.