Lezione del 29/09/2023

Serie a segni alterni $\qquad \sum_{n=1}^{\infty} a_n \quad a_n \geq 0$

ES $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1}\frac{1}{n} + \dots$

serie armonica a segni alterni

$$\sum (-1)^{n-1} \overbrace{|\sin n|}^{\geq 0} = \sin 1 - |\sin 2| + |\sin 3| + \dots$$

Def (1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + \dots + (-1)^{n-1} b_n + \dots$

$b_n \geq 0$

serie a segni alterni

Criterio di Leibniz

Supponiamo che $\{b_n\}$ sia decrescente ($b_n \geq b_{n+1}$) ed infinitesima per $n \geq k \in \mathbb{N}$

$$\lim_{n\to\infty} b_n = 0.$$

Allora la (1) converge. Inoltre, detta S la sua somma, si ha $|S_n - S| \leq b_{n+1} \quad \forall n \in \mathbb{N}$

ES $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\left(\frac{1}{n}\right)}_{\text{t.ne generale}} = b_n$

$b_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\left\{\frac{1}{n}\right\}$ succ. decrescente <u>st.</u>

$\frac{1}{n} \not\geq \frac{1}{n+1}$

$\Rightarrow$ criterio di Leibniz, la serie converge $\sum (-1)^{n-1} \frac{1}{n} = S \in \mathbb{R}$

$= \log 2$

"Quanti termini dobbiamo sommare in modo che la somma parziale S_n differisca da S per meno di $1/100$" ?

$$|S_n - S| \leq \frac{1}{100}$$

$|S_n - S| \leq b_{n+1} = \frac{1}{n+1}$: devo prendere n in maniera che $\frac{1}{n+1} \leq \frac{1}{100} \Leftrightarrow n+1 \geq 100$

$\Leftrightarrow n \geq 99$

2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ $\quad b_n = \frac{n-1}{n^2+n} \xrightarrow{n\to\infty} 0$

$b_n \geq b_{n+1} \iff \frac{n-1}{n^2+n} \geq \frac{n+\cancel{1}-\cancel{1}}{(n+1)^2+(n+1)}$

$0 \,, \quad \frac{1}{4+2} = 1/6$

$n=1$

$0 < \frac{1}{6}$

$\frac{n-1}{n\cancel{(n+1)}} \geq \frac{n}{\cancel{(n+1)}(n+2)}$

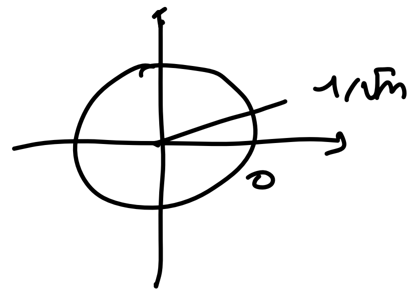
$\iff (n-1)(n+2) \geq n^2$

$\cancel{n^2} + 2n - n - 2 \geq \cancel{n^2}$

$n - 2 \geq 0$

$\iff \boxed{n \geq 2}$

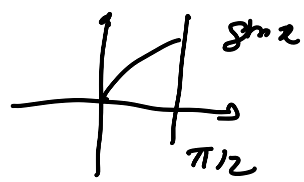
$$\sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{\sqrt{n}}$$



$b_n = \sin \frac{1}{\sqrt{n}} > 0 \quad n \geq 1 \Rightarrow \frac{1}{\sqrt{n}} \leq 1 < \frac{\pi}{2}$

$\neq$

$b_n \longrightarrow 0$ pu $n \to \infty$



$n < n+1 \Rightarrow \sqrt{n} < \sqrt{n+1} \Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

$\Rightarrow b_n = \sin \frac{1}{\sqrt{n}} > \sin \frac{1}{\sqrt{n+1}} = b_{n+1}$

$\Rightarrow$ criterio di Leibniz $\Rightarrow$ la serie converge

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converge}$$

$$|(-1)^n| = 1 \quad \underline{\underline{\text{SEMPRE}}}$$

$$b_n = 1/n^3 \longrightarrow 0$$

$$n < n+1 \Rightarrow n^3 < (n+1)^3$$

$$\Rightarrow \frac{1}{n^3} > \frac{1}{(n+1)^3}$$

$$\sum_{n=1}^{\infty} (-1)^n \arcsin 1/n$$

$$\sum_{n=1}^{\infty} \overbrace{\left(\frac{n}{2} \sin 1/n\right)^{\frac{n^2+1}{n+2}}}^{a_n}$$

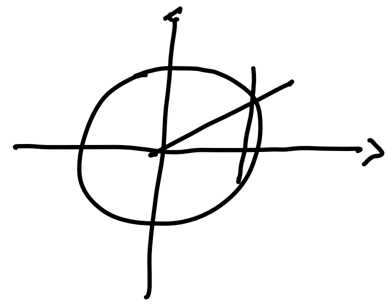
$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n}{2} \sin \frac{1}{n}\right)^{\frac{n^2+1}{n+2}}\right]^{1/n}$$

$$\lim_{n\to\infty}\left(\frac{n}{2}\underbrace{\sin\frac{1}{n}}\right)^{\overset{\to 1}{\frac{n^2+1}{n(n+2)}}} = \frac{1}{2} < 1$$

$\Rightarrow$ la serie converge

$$\frac{1}{2}\underbrace{\left(\frac{\sin 1/n}{1/n}\right)}_{\to 1} \longrightarrow \frac{1}{2}$$

$$\sum_{n=1}^{\infty}(-1)^n\left[\underbrace{1-n\sin\frac{1}{n}}_{b_n}\right]$$



$$b_n = 1 - n\sin\frac{1}{n} > 0 \iff n\sin\frac{1}{n} < 1$$

$$\iff \sin 1/n < 1/n < \frac{\pi}{2}$$

vero

b_n decrescente?

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}(\underset{\downarrow\, 1}{1} - \underset{\downarrow\, 1}{n\sin 1/n}) = 0$$

$$b_n \geq b_{n+1} \iff \cancel{1} - n\sin 1/n \geq \cancel{1} - (n+1)\sin\frac{1}{n+1}$$

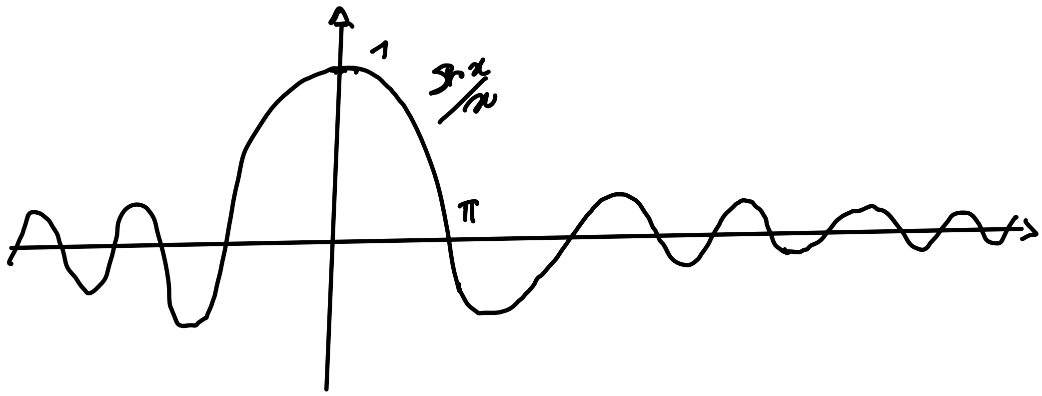
$\Longleftrightarrow \quad n \sin \frac{1}{n} \le (n+1) \sin \frac{1}{n+1}$

$\Longleftrightarrow \quad \dfrac{\sin \frac{1}{n}}{\frac{1}{n}} \le \dfrac{\sin \frac{1}{n+1}}{\frac{1}{n+1}}$

$f(x) = \dfrac{\sin x}{x}$ decrescente? $\quad n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$

$\lim_{x \to 0} f(x) = 1 \qquad \dfrac{\sin \frac{1}{n}}{\frac{1}{n}} \le \dfrac{\sin \frac{1}{n+1}}{\frac{1}{n+1}}$

$\lim_{x \to +\infty} \dfrac{\sin x}{x} = 0 \qquad \sin x = 0 \; : \; x = \pm\pi, \pm 2\pi, \pm 3\pi$



$\Rightarrow$ la serie converge!

$$\sum_{n=1}^{\infty} (-1)^n \underbrace{\log\left(1 + \sin \tfrac{1}{n}\right)}_{b_n > 0} \qquad b_n = \log\left(1 + \sin \tfrac{1}{n}\right) \to 0$$

$b_n > b_{n+1} \Longleftrightarrow \log\left(1 + \sin \frac{1}{n}\right) > \log\left(1 + \sin \frac{1}{n+1}\right)$

$\Longleftrightarrow \cancel{1} + \sin \frac{1}{n} > \cancel{1} + \sin \frac{1}{n+1}$

$\sin \frac{1}{n} > \sin \frac{1}{n+1}$ **VERA!**

$\sum (-1)^n = -1 + 1 - 1$

Convergenza assoluta

$$\sum_{n=1}^{\infty} a_n \qquad a_n \in \mathbb{R}$$

(può essere anche ≤ 0)

converge assolutamente se

$$|a_1| + |a_2| + \dots + |a_n| + \dots = \sum_{n=1}^{\infty} |a_n|$$

converge.

ES $\sum (-1)^{n-1} \frac{1}{n^2}$ $\qquad \sum \frac{1}{n^2} < +\infty$

Prop. Se $\sum_{n=1}^{\infty} a_n$ convergente assolutamente

$\Downarrow$ $\not\Uparrow$

allora $\sum_{n=1}^{\infty} a_n$ converge

ES $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converge

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

ESERCIZI

$$\sum_{n=2}^{\infty} \frac{\log n}{n}$$

$$\sum_{n=1}^{\infty} n^{3/2}\left(\frac{1}{n} - \sin\frac{1}{n}\right) \quad]]$$