$$\boxed{\text{Lezione del 28/09/23 Informatica}}$$

Criterio degli infinitesimi

$a_n \to 0$, $n \to \infty$

$\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$. Supponiamo

che $\lim_{n\to\infty} n^\alpha a_n = l \in [0, \infty]$

i) Se $l < +\infty$ e $\alpha > 1 \Rightarrow \sum_{n=1}^{\infty} a_n < +\infty$

ii) Se $l > 0$ e $\alpha \leq 1 \Rightarrow \sum_{n=1}^{\infty} a_n = +\infty$

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

$\bullet$ $\lim_{x\to x_0} f(x) = 0$

$\lim_{n\to\infty} a_n = 0$

$o(x)$ $x \to 0$

$$\lim_{x\to x_0} \frac{|f(x)|}{|x-x_0|^\alpha} = l > 0$$

α ordine di infinitesimo

$f(x) = \sin x$, $x_0 = 0$

$$\lim_{x\to 0} \frac{\sin x}{x^\alpha} = 1$$

$\alpha = 1$

$$\sum_{n=1}^{\infty} \sin \frac{1}{n}$$

$$\lim_{n\to\infty} n^\alpha \sin 1/n = \lim_{n\to\infty} \frac{\sin 1/n}{1/n^\alpha} = 1 \quad \alpha = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \iff$$

$$= \quad \iff \forall \{a_n\},\ a_n \to 0,$$

$$\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$$

$a_n = 1/n$: $\lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$

$$\lim_{x \to +\infty} x^{\alpha} |f(x)| = l > 0$$

$$f(x) = \frac{1}{x^3} \qquad \lim_{x \to +\infty} \cancel{x^3} \cdot \frac{1}{\cancel{x^3}} = 1$$

$$\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n^4}\right) \qquad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{n \to \infty} n^{\alpha} \left(1 - \cos \frac{1}{n^4}\right) = \lim_{n \to \infty} \frac{1 - \cos\left(\frac{1}{n}\right)^4}{\frac{1}{n^{\alpha}}}$$

$$= \lim_{n \to \infty} \frac{1 - \cos\left(\frac{1}{n^4}\right)}{\left(\frac{1}{n^8}\right)^{\frac{\alpha}{8}}} = \qquad \frac{\alpha}{8} = 1 \Leftrightarrow \alpha = 8$$

$$= \lim_{n \to \infty} \frac{1 - \cos \frac{1}{n^4}}{\left(\frac{1}{n^4}\right)^2} = \frac{1}{2}$$

$$\lim_{n \to \infty} \frac{1 - \cos a_n}{a_n^2} = \frac{1}{2} \qquad a_n = \frac{1}{n^4}$$

$$\lim_{n \to \infty} \frac{1 - \cos \frac{1}{n^4}}{\frac{1}{n^8}} = \frac{1}{2}$$

: la serie converge

poiché $\alpha = 8 > 1$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^5 + 4n + 3} \qquad a_n = \frac{2n+1}{n^3 + 4n + 3}$$

$$a_{n+1} = \frac{2n+3}{(n+1)^5 + 4(n+1) + 3}$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{(n+1)^5+4(n+1)+3} \cdot \frac{n^5+4n+3}{2n+1} \xrightarrow{n\to\infty} 1 = \ell$$

$$\lim_{n\to\infty} n^{\alpha} \frac{2n+1}{n^5+4n+3} = \lim_{n\to\infty} \frac{2n^{\alpha+1} + n^{\alpha}}{n^5+4n+3} = 2 > 0$$

$$\alpha+1 = 5 \Leftrightarrow \alpha = 4 > 1$$

$$\frac{2n+1}{n^5+4n+3} \sim \frac{2}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \qquad \frac{1}{\sqrt{n+1}} \sim \frac{1}{\sqrt{n}} \qquad \sum \frac{1}{\sqrt{n}} = +\infty$$

$$\alpha = \tfrac{1}{2}$$

$$n^{\alpha} \frac{1}{\sqrt{n+1}} \xrightarrow{n\to\infty} 1 \quad \alpha = \tfrac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} \qquad a_n = \frac{n^n}{3^n n!}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n \cdot n!}{n^n}$$

$$= \frac{(n+1)^n \cancel{(n+1)}}{\cancel{3^n} \cdot 3 \cancel{(n+1)} \cancel{n!}} \cdot \frac{\cancel{3^n} \cdot \cancel{n!}}{n^n} = \frac{1}{3} \underbrace{\left(\frac{n+1}{n}\right)^n}_{\left(1+\frac{1}{n}\right)^n \to e} \to \frac{e}{3} < 1$$

$\Rightarrow$ la serie converge

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = ?$$

$$\sum_{n=1}^{\infty} e^{\frac{1}{n}} = +\infty \quad \Big| \quad e^{\frac{1}{n}} \xrightarrow{n\to\infty} 1$$

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^{n+2}} = \sum_{n=0}^{\infty} \frac{3\cdot 3^n}{16\cdot 4^n} = \frac{3}{16}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \qquad \frac{1}{1-h}$$

$$= \frac{3}{16}\cdot\frac{1}{1-\frac{3}{4}} = \frac{3}{4} \qquad h = \frac{1}{5^{\frac{2}{3}}} < 1$$

$$\sum_{n=1}^{\infty} 5^{-\frac{2}{3}n} = \sum_{n=1}^{\infty}\frac{1}{5^{\frac{2}{3}n}} = \sum_{n=1}^{\infty}\left(\frac{1}{5^{\frac{2}{3}}}\right)^n \quad \text{converge}$$

$$= \frac{1}{5^{\frac{2}{3}}}\sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{5^{\frac{2}{3}}}\right)^n}_{\frac{1}{1-\frac{1}{5^{\frac{2}{3}}}}} \qquad \sum_{n=1}^{\infty} h^n = h\underbrace{\sum_{n=0}^{\infty} h^n}_{1+h+h^2+\dots}$$

$$\sum_{n=1}^{\infty}\frac{e^n}{n!} < \infty \qquad \lim_{n\to\infty}\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n} =$$

$$= \lim_{n\to\infty}\frac{\cancel{e^n}\cdot e}{\cancel{n!}\,(n+1)}\cdot\frac{\cancel{n!}}{\cancel{e^n}} = 0 < 1$$

$$\sum_{n=1}^{\infty}\frac{e^{\sqrt{n}}}{n!}$$; criterio della radice.

$$\lim_{n\to\infty}\sqrt[n]{\frac{e^{\sqrt{n}}}{n!}} = \lim_{n\to\infty}\frac{e^{\frac{\sqrt{n}}{n}\to 0}}{\sqrt[n]{n!}}$$

Ma $\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$, dunque

$$\lim_{n\to+\infty} \sqrt[n]{\frac{e^{\sqrt{n}}}{n!}} = \frac{1}{\infty} = 0 < 1$$

$\Rightarrow$ la serie converge.

$|\cos n| \leq 1$

$$\sum_{n=1}^{\infty} \frac{n\sqrt{1+\cos^2 n}}{3^n}$$

$$\cos^2 n \leq 1$$

$$\sqrt{1+\underline{\underline{\cos^2 n}}} \leq \sqrt{2}$$

$$\sum_{n=1}^{\infty} \sqrt{2}\,\frac{n}{3^n} < \infty \qquad \frac{n+1}{3^{n+1}}\,\frac{3^n}{n} \to \frac{1}{3} < 1$$

$\Downarrow$

dal criterio del confronto, la serie iniziale converge

$$\sum_{n=1}^{\infty} \left(\frac{n}{2}\sin\frac{1}{n}\right)^{\frac{n^2+1}{n+2}}$$

criterio della radice!