Lezione del 28/11/23

$$y'' + ay' + by = f(x)$$

$$f(x) = \sin x$$
$$= \cos x - \sin x$$
$$= \left(x\cos x + (x-1)\sin x\right)$$
$$= e^{x}\sin(2x) \quad \dots$$

$$y'' + y' = \sin x$$

OM. ASS.: $y'' + y' = 0$

EQ. CARATT. $\lambda^2 + \lambda = 0 \longrightarrow \lambda = -1,\ \lambda = 0$

$$y = c_1 e^{-x} + c_2$$

$$\bar{y}(x)\ ??$$

$x \qquad x^2$

$$f(x) = e^{\bar{\lambda} x}\left[\underline{p(x)}\cos(\mu x) + \underline{q(x)}\sin(\mu x)\right]$$

ES. $f(x) = \sin x = e^{0\cdot x}\left[0\cdot\cos(1\cdot x) + 1\cdot\sin(1\cdot x)\right]$

$\bar{\lambda} = 0 \qquad p(x) = 0 \qquad q(x) = 1 \qquad \underline{\mu = 1}$

$$f(x) = e^{-x}\left[x\cos(2x) - x^2 \sin(2x)\right]$$

$\bar{\lambda} = -1,$ $\quad p(x) = x \quad q(x) = -x^2 \quad \mu = 2$

a) $\bar{\lambda} \pm i\mu$ non è radice della caratteristica:

$$p(\bar{\lambda} \pm i\mu) \neq 0$$

$$y'' + y' = e^{0\cdot x} \sin x \qquad \mu = 1 \;, \; \bar{\lambda} = 0$$

$\bar{\lambda} \pm i\mu = \pm i$ NO RADICE CARATT.

Allora un integrale particolare sarà della forma

$$\bar{y}(x) = e^{\bar{\lambda}x}\left[z(x)\cos(\mu x) + s(x)\sin(\mu x)\right]$$

da determinarsi

di grado $= \max\{gr(p), gr(q)\}$

$$y'' + y' = e^{0\cdot x} \sin x \qquad p(x) = 0,\; q(x) = 1$$

$$\bar{y}(x) = 1\cdot\left[\underbrace{a\cdot\cos x + b\cdot \sin x}_{}\right]$$

a, b da determinare!!

$$\bar{y}' = -a\sin x + b\cos x$$

$$\bar{y}'' = -a\cos x - b\sin x$$

$$-a\cos x - b\sin x - a\sin x + b\cos x = \sin x$$

$$(\underbrace{-a+b}_{\sim})\cos x + (\overbrace{-b-a}^{\sim})\sin x = \overset{\sim}{1}\cdot\sin x + \underset{\sim}{0}\cdot\cos x$$

$$\begin{cases} -a+b=0 \\ -b-a=1 \end{cases} \iff \begin{cases} a=b = -\frac{1}{2} \\ b=-a-1 = -b-1 \end{cases}$$

$$b = -\frac{1}{2}$$

$$\bar{y}(x) = -\frac{1}{2}(\cos x + \sin x)$$ INT. EQ. COMPLETA $\boxed{\text{PART.}}$

Int. generale completa

$$y(x) = c_1 + c_2 e^{-x} - \frac{1}{2}(\cos x + \sin x)$$

$$y'' + y' = \overbrace{[x\cos(2x) - \sin(2x)]}^{f(x)}$$

$p(x)$ $\quad q(x) = -1$

$\bar{\lambda} = 0 \quad \mu = 2$

$\bar{\lambda} \pm i\mu = \pm 2i$ NO RADICE

$$\bar{y}(x) = \left[(ax+b)\cos(2x) + (cx+d)\sin(2x) \right]$$

$$\bar{y}'(x) = a\cos(2x) - 2(ax+b)\sin(2x)$$
$$+ c\sin(2x) + 2(cx+d)\cos(2x)$$

$$\bar{y}'' = -2a\sin(2x) - 2a\sin(2x) - 4(ax+b)\cos(2x)$$
$$+ 2c\cos(2x) + 2c\cos(2x) - 4(cx+d)\sin(2x)$$

$$= -4a\sin(2x) + 4c\cos(2x) +$$
$$- 4(ax+b)\cos(2x) - 4(cx+d)\sin(2x)$$

$$-4a\sin(2x) + 4c\cos(2x) - 4(ax+b)\cos(2x)$$
$$-4(cx+d)\sin(2x) + a\cos(2x) - 2(ax+b)\sin(2x)$$
$$+ c\sin(2x) + 2(cx+d)\cos(2x)$$

$$= x\cos(2x) - \sin(2x)$$

$$\left[-4a - 4(cx+d) - 2(ax+b) + c \right] \sin(2x)$$

$$+ \left[4c - 4(ax+b) + a + 2(cx+d) \right] \cos(2x)$$

$$= x\cos(2x) - \sin(2x)$$

$$\begin{cases} -4a - \tilde{4}(cx+d) - 2(ax+b) + c = -1 \\ 4c - 4(ax+b) + a + 2(cx+d) = x \end{cases}$$

$$\begin{cases} (-4c - 2a)x - 4a - 4d - 2b + c \overset{0\cdot x}{=} -1 \\ (-4a + 2c)x + 4c - 4b + a + 2d = x \end{cases}$$

$$\begin{cases} -4c - 2a = 0 \\ -4a - 4d - 2b + c = -1 \\ \vdots \end{cases} \quad \Big\|$$

b) $\bar{\lambda} \pm i\mu$ è <u>radice</u> : $P(\bar{\lambda} \pm i\mu) = 0$

$$\bar{y}(x) = x\, e^{\bar{\lambda} x} \left[R(x)\cos(\mu x) + S(x)\sin(\mu x) \right]$$

$$y'' + y = \cos x = e^{\overset{\bar\lambda}{\widetilde{0}}\cdot x}\left[\overset{p(x)=1}{1}\cdot\cos(\underset{\mu=1}{1}\cdot x) + \overset{q(x)=0}{0}\cdot\sin(1\cdot x)\right]$$

$$y'' + y = 0 \quad : \; y = C_1\cos x + C_2 \sin x$$

$$\lambda^2 + 1 = 0 \longrightarrow \lambda = \pm i$$

$\bar\lambda \pm i\mu = \pm i$ È RADICE!

$$\bar y(x) = x\,[a\cos x + b\sin x]$$

$\bar y'$, y'' ... INSERIRE IN

$$y'' + y = \cos x$$

E DETERMINARE a, b COME PRIMA!!

$$y'' + 5y' + 4y = e^{-x} + \cos x$$

$$y'' + 5y' + 4y = 0 \qquad y = C_1 e^{-4x} + C_2 e^{-x}$$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{-5 \pm 3}{2} \begin{cases} -4 \\ -1 \end{cases}$$

$$y'' + 5y' + 4y = e^{-x} \leftarrow \bar{y}(x)$$

$$= \cos x \leftarrow \bar{\bar{y}}(x)$$

$y_1 = \bar{y}(x) + \bar{\bar{y}}(x)$ integrale particolare completo

$$f(x) = \cos x + \sin(3x)$$

$$y'' + 5y' + 4y = \cos x$$

$$y'' + 5y' + 4y = \sin(3x)$$

ES. $y'' + y = \dfrac{1}{\sin x}$?

$$= \log x$$

METODO DELLA VARIAZIONE DELLE COSTANTI DI LAGRANGE

$$y'' + a(x)y' + b(x)y = f(x)$$

$$\underbrace{y'' + a(x)y' + b(x)y = 0}$$

$y_1(x), y_2(x)$ LIN. INDIPENDENTI

$$y = c_1 y_1(x) + c_2 y_2(x)$$

Allora un integrale particolare della completa è

$$y = c_1(x) y_1(x) + c_2(x) y_2(x)$$

dove $c_1(x)$, $c_2(x)$ sono tali che le loro derivate $c_1'(x)$, $c_2'(x)$ risolvono il

SISTEMA

$$\begin{cases} y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0 \\ y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = f(x) \end{cases}$$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

DA CRAMER:

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix}}{W(x)}$$

$$c_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix}}{W(x)}$$

$$y'' + y = \frac{1}{\sin x}$$

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$y(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$$

$$\bar{y}(x) = c_1(x)\cos x + c_2(x)\sin x$$

$$\begin{cases} (\cos x)\, c_1'(x) + (\sin x)\, c_2'(x) = 0 \\[2em] -\sin x \; c_1'(x) + (\cos x)\, c_2'(x) = \dfrac{1}{\sin x} \end{cases}$$

$$W(x) = 1$$

$$c_1'(x) = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix}$$

$$= -1$$

$$c_1(x) = -x$$

$$c_2'(x) = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin x} \end{vmatrix}$$

$$= \frac{\cos x}{\sin x}$$

$$c_2(x) = \int \frac{\cos x}{\sin x}\,dx = \log|\sin x| + C$$

$$\bar{y}(x) = c_1(x)\cos x + c_2(x)\sin x =$$

$$= -x\cos x + \left(\log|\sin x|\right)\sin x$$

Int. generale completo: $y = c_1 \cos x + c_2 \sin x + \bar{y}(x)$

<u>ES.</u> $y'' + 2y' + y = \dfrac{\log x}{e^x}$

EQ. OM. $y'' + 2y' + y = 0$

$\lambda^2 + 2\lambda + 1 = 0 \qquad \lambda = -1$ RADICE DOPPIA

INT. GEN. OMOG.: $y = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{x e^{-x}}_{y_2}$

$\bar{y}(x) = c_1(x) e^{-x} + c_2(x)\, x e^{-x}$

$$\begin{cases} e^{-x} c_1'(x) + x e^{-x} c_2'(x) = 0 \\[2ex] -e^{-x} c_1'(x) + \left(e^{-x} - x e^{-x}\right) c_2'(x) = \dfrac{\log x}{e^x} \end{cases}$$

$$W(x) = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix}$$

$$= e^{-2x}(1-x) + x e^{-2x} = e^{-2x}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & x e^{-x} \\ \frac{\log x}{e^x} & (1-x) e^{-x} \end{vmatrix}}{e^{-2x}}$$

$$= e^{2x} \left(- x (\log x) e^{-2x} \right)$$

$$= -x \log x$$

$\int x \log x \, dx$ = PER PARTI.....

$f(x) = \log x \qquad g'(x) = x$

$$f'(x) = \frac{1}{x} \qquad g(x) = \frac{x^2}{2}$$

$$= \frac{x^2 \log x}{2} - \frac{1}{2} \int \frac{1}{\cancel{x}} \cdot x^{\cancel{2}} \, dx$$

$$c'_2(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{\log x}{e^x} \end{vmatrix}}{e^{-2x}}$$

$$= \log x$$

$$y'' + y = \sin x$$