Criterio asintotico

Lezione del 26/09/2023/ Informatica

$$\sum_{n=1}^{\infty} a_n \,, \quad \sum_{n=1}^{\infty} b_n \qquad a_n, b_n \geqslant 0 \quad \forall n \in \mathbb{N}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \qquad a_n \sim b_n \quad (a_n \text{ "va" come } b_n \text{ per } n \to \infty)$$

Allora, le due serie hanno lo stesso carattere

(se $\sum a_n$ converge / diverge anche $\sum b_n$ converge / diverge e viceversa

NOTA Non sto dicendo che le serie hanno la stessa somma

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad (\text{serie di Mengoli}) = 1$$

$$\lim_{n\to\infty} \frac{1/n^2}{1/n(n+1)} = \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1$$

$$a_n = 1/n^2 \sim \overbrace{\frac{1}{n(n+1)}}^{b_n} \Rightarrow \text{poiché } \sum 1/n(n+1) \text{ converge,}$$

dal criterio asintotico si ha che $\sum 1/n^2 < +\infty$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \pi^2/6$$

$$\sum_{m=1}^{\infty} \frac{1}{m^\alpha} \quad \text{(serie armonica generalizzata)}$$

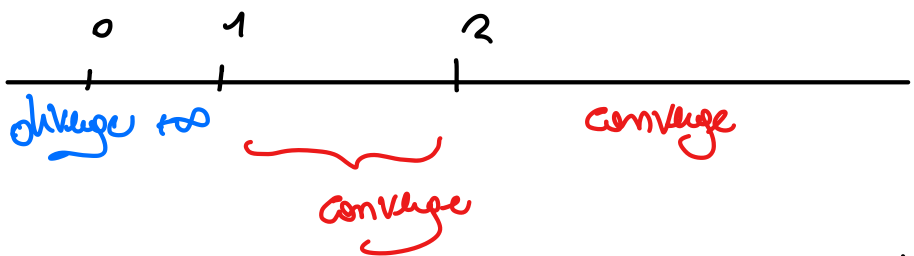
$\alpha = 1$ (diverge) ; $\alpha = 2$ (converge)

$\underline{\alpha > 2}$ $\quad m^\alpha \geq m^2 \Rightarrow \frac{1}{m^\alpha} \leq \frac{1}{m^2}$, $\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty$

$\Rightarrow$ criterio confronto $\Rightarrow \sum_{m=1}^{\infty} \frac{1}{m^\alpha} < \infty$

$\underline{0 < \alpha < 1}$ $\quad m^\alpha \leq m \Rightarrow \frac{1}{m^\alpha} \geq \frac{1}{m}$ $\quad \sum_{m=1}^{\infty} \frac{1}{m} = +\infty$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{m^\alpha} = +\infty$$



$\alpha > 0$

$$\sum_{M=1}^{\infty} \frac{1}{m^\alpha} \equiv \begin{cases} \text{converge per } \alpha > 1 \\ \text{diverge per } \alpha \leq 1 \end{cases}$$

$\underline{\alpha < 0}$ $\quad \frac{1}{m^\alpha} = m^{\overset{\overset{0}{\wedge}}{-\alpha}} \xrightarrow[m \to \infty]{} +\infty$

$$\frac{1}{n^\alpha} = n^{-\alpha}$$

$\frac{1}{n} \to 0, \ n \to \infty$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \longrightarrow \frac{\sin \frac{1}{n}}{\frac{1}{n}} \longrightarrow 1$$

$$a_n \to 0 \qquad \frac{\sin a_n}{a_n} \longrightarrow 1$$

$\sin \frac{1}{n} \sim \frac{1}{n} \Rightarrow$ criterio asintotico $\sum \frac{1}{n} = +\infty$

$$\sum \sin \frac{1}{n} = +\infty$$

$$\sum \sin \frac{1}{n^\pi} \sim \sum \frac{1}{n^\pi} < \infty$$

$\Downarrow$

converge

$a_n \to 0$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad \frac{1-\cos a_n}{a_n^2} \longrightarrow \frac{1}{2}$$

$$1-\cos a_n \sim \frac{1}{2} a_n^2 \qquad \frac{1-\cos a_n}{\frac{1}{2} a_n^2} \longrightarrow 1$$

$$\sum_{n=1}^{\infty} \left(1 - \underbrace{\cos \overset{\nearrow 0}{\frac{1}{n}}}_{\substack{\vee\!| \\ 0}}\right) \qquad 1 - \cos \frac{1}{n} \qquad a_n = \frac{1}{n}$$

$$\frac{1 - \cos 1/n}{\frac{1}{2} \cdot \frac{1}{n^2}} \longrightarrow 1$$

$$1 - \cos(1/n) \sim \frac{1}{2n^2} \qquad \sum \frac{1}{2n^2} = \frac{1}{2} \sum \frac{1}{n^2} < +\infty$$

$\Rightarrow$ dal criterio asintotico, $\sum \left(1 - \cos \frac{1}{n}\right) < \infty$

$$\sum_{n=1}^{\infty} \underbrace{\frac{2n+1}{n^5+4n+3}}_{\to 0} \qquad \frac{2n+1}{n^5+4n+3} \sim \frac{2}{n^4}$$

Poiché $\sum_{n=1}^{\infty} \frac{2}{n^4} < \infty \Rightarrow$ per il criterio asintotico, la serie iniziale converge

ES $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$

$0! = 1$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$
$$+ \frac{f''(x_0)}{2}(x-x_0)^2 + o((x-x_0)^2)$$
$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + o((x-x_0)^3)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + \underbrace{o((x-x_0)^m)}_{\text{resto di Peano}}$$

f. la di Taylor di ordine m

$\underline{x_0 = 0}$ $\quad f(x) = e^x \quad f'(x) = f''(x) = \dots = f^{(m)}(x) = e^x$

Mac Laurin $\quad f^{(m)}(0) = 1$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^m}{m!} + o(x^m)$$

$$\sum_{m=0}^{\infty} \underbrace{\frac{x^m}{m!}}_{a_m\,?} = e^x \qquad x > 0$$

$$a_m = \frac{x^m}{m!} \qquad a_{m+1} = \frac{x^{m+1}}{(m+1)!} \qquad (m+1)! = (m+1)\,m!$$

$$\frac{a_{m+1}}{a_m} = \frac{x^{m+1}}{(m+1)!} \cdot \frac{m!}{x^m} = \frac{\cancel{x^m} \cdot x}{(m+1)\cancel{m!}} \cdot \frac{\cancel{m!}}{\cancel{x^m}} = \frac{x}{m+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

$\Rightarrow$ la serie converge per il criterio del rapporto!!

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n}{n+1}\right)^{n^2}}_{a_n} \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^{\cancel{2}}}\right]^{\cancel{1/n}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{(n+1)-1}{n+1}\right)^n \overset{(*)}{=}$$

$$\left(1+\frac{1}{b_n}\right)^{b_n} \longrightarrow e \qquad \left(1+\frac{1}{x}\right)^x \xrightarrow[x\to\pm\infty]{} e$$

$b_n \to \pm\infty$

$$\overset{(*)}{=} \lim_{n\to\infty} \left(1-\frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{-(n+1)}\right)^{n\cdot\frac{(-(n+1))}{-(n+1)}}$$

$$= \lim_{n\to\infty} \left[\underbrace{\left(1+\frac{1}{-(n+1)}\right)^{-(n+1)}}_{\downarrow e}\right]^{-\frac{n}{n+1}} = e^{-1}$$

$$= \frac{1}{e} < 1$$

Quindi la serie converge!

$$\sum \frac{n^n}{3^n n!} \qquad ?$$