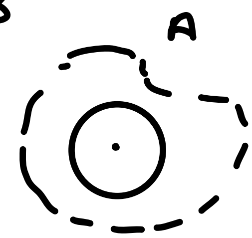
Lezione del 24/10/2023



$$f(x,y) \qquad f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$x \qquad (x_0, y_0) \in A$$

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0)$$

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0)$$

$$f(x,y) = y \sin(xy)$$

$$f_x = y \cos(xy) \cdot y = y^2 \cos(xy)$$

$$f_y = \sin(xy) + y x \cos(xy)$$

$$f(x,y) = y^2 \qquad f_x = \frac{\partial}{\partial x}(y^2) = 0$$

$$f_y = 2y$$

$$f = \frac{x}{y} \qquad f_x = \frac{1}{y} \frac{\partial}{\partial x}(x) = 1/y$$

$$f_y = x \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -x/y^2$$

$$f = \frac{\arcsin(x+y)}{x^2+y^2}$$

$$f_x = \frac{\frac{1}{\sqrt{1-(x+y)^2}} \cdot 1 \cdot (x^2+y^2) - \arcsin(x+y) \cdot 2x}{(x^2+y^2)^2}$$

$f = f(x)$ $\qquad f'$ $\qquad f'' = (f')'$

DERIVATE SECONDE

$f : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\qquad f = f(x,y)$ derivabile in A

: $\forall (x,y) \in A \quad \exists \; f_x(x,y) \,,\; f_y(x,y)$

$$f_x : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad f_y : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Se f_x, f_y sono a loro volta derivabili :

$$\frac{\partial}{\partial x} f_x = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y} f_x = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

DERIVATE SECONDE

$$\frac{\partial}{\partial x} f_y = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \qquad \frac{\partial}{\partial y} f_y = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$f = x^2 y$ $\qquad f_x = 2xy$ $\qquad f_y = x^2$

$$\frac{\partial}{\partial x} f_x = 2y \qquad \frac{\partial}{\partial y} f_x = 2x$$

$$\frac{\partial}{\partial x} f_y = 2x \qquad \frac{\partial}{\partial y} f_y = 0$$

DERIVATE PURE

$$\begin{array}{cc} f_x & \\ f_y & \end{array} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = D^2 f(x,y)$$

DERIVATE MISTE

matrice Hessiana di $f(x,y)$

f_{xx}, f_{yy} DER. PURE; f_{xy}, f_{yx} DER. MISTE

$$f(x,y) = \sin(x^2y^2)$$

$$f_x = 2xy^2 \cos(x^2y^2)$$

$$f_y = 2yx^2 \cos(x^2y^2)$$

$$f_{xx} = 2y^2 \left[\cos(x^2y^2) + x\,(-\sin(x^2y^2)) \cdot 2xy^2 \right]$$

$$= 2y^2 \left[\cos(x^2y^2) - 2x^2y^2 \sin(x^2y^2) \right]$$

$$f_{xy} = 2x \left[2y \cos(x^2y^2) + y^2 (-\sin(x^2y^2)) \cdot 2yx^2 \right]$$

$$f_{xy} = 2x \frac{\partial}{\partial y} \left(y^2 \cos(x^2y^2) \right)$$

$$= 2x \left[2y \cos(x^2y^2) - 2y^3x^2 \sin(x^2y^2) \right].$$

$$f_{yx} = 2y \left[2x \cos(x^2y^2) + x^2 (-\sin(x^2y^2))\, 2xy^2 \right]$$

$$= 2y \left[2x \cos(x^2y^2) - 2x^3y^2 \sin(x^2y^2) \right]$$

$$f_{yy} = 2x^2 \left[\cos(x^2y^2) + y\left(-\sin(x^2y^2)\right) \cdot 2yx^2 \right]$$
$$= 2x^2 \left[\cos(x^2y^2) - 2y^2x^2 \sin(x^2y^2) \right]$$

TEOREMA DI SCHWARZ

$f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

derivabile due volte in A ($\exists$ f_x, f_y e tutte le derivate seconde)

Allora, se $f_{xy}, f_{yx}: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ sono continue

si ha $$f_{xy} = f_{yx} \quad \text{in } A.$$

$$f(x,y,z) = x^2 y z^2$$

$$f_x = 2xyz^2 \qquad f_y = x^2z^2 \qquad f_z = 2x^2yz$$

$$D^2f = \begin{pmatrix} f_{xx} = 2yz^2 & f_{xy} = 2xz^2 & f_{xz} = 4xyz \\ f_{yx} = 2xz^2 & f_{yy} = 0 & f_{yz} = 2x^2z \\ f_{zx} = 4xyz & f_{zy} = 2x^2z & f_{zz} = 2x^2y \end{pmatrix}$$

$f(x,y,z)$ derivabile due volte

$f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}$ continue

$f_{xy} = f_{yx}$; $f_{xz} = f_{zx}$; $f_{yz} = f_{zy}$

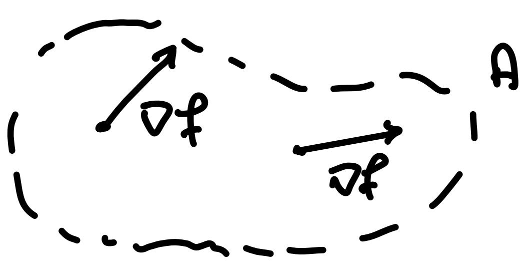
GRADIENTE $f(x,y)$, $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$ derivabile

$\forall (x,y) \in A \quad \exists \ f_x(x,y), f_y(x,y) \in \mathbb{R}$

gradiente di f in $(x,y) \in A$

$$\nabla f(x,y) = Df(x,y) = (f_x(x,y), f_y(x,y)) \in \mathbb{R}^2$$

"NABLA DI f"

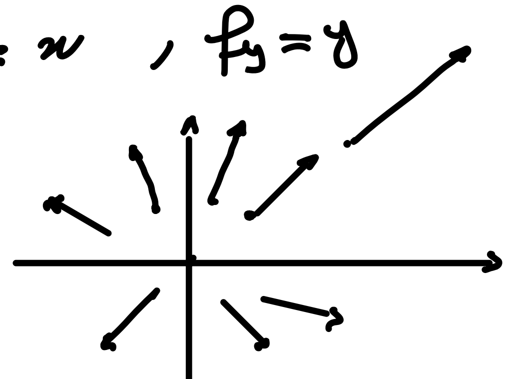


$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

$f_x = x$, $f_y = y$

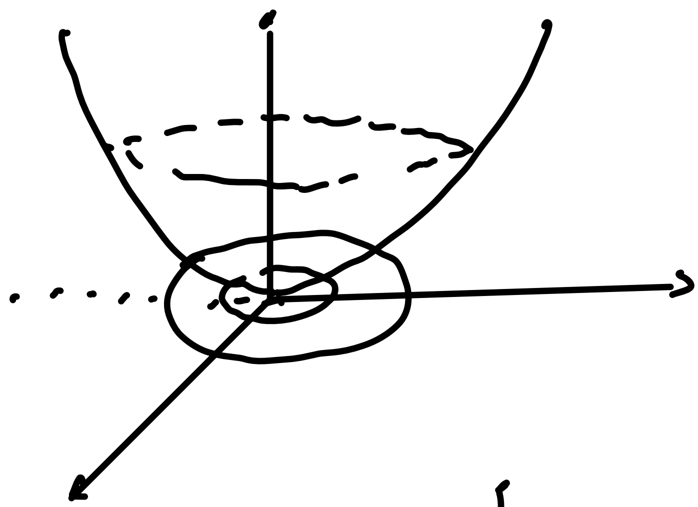
$\nabla f(x,y) = (x,y)$

$\|\nabla f(x,y)\| = \sqrt{x^2+y^2}$



$f = -\frac{x^2}{2} - \frac{y^2}{2}$ $\qquad \nabla f = -(x,y)$

$z = \frac{x^2}{2} + \frac{y^2}{2}$ paraboloide ellittico



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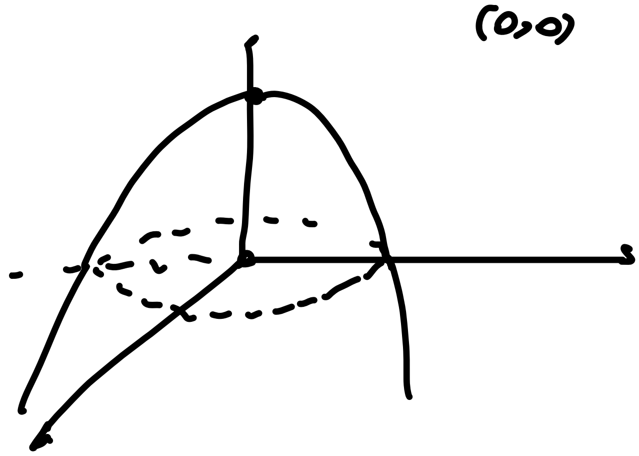

$z = 1 - x^2 - y^2$

$\nabla f = -(x,y)$