Lezione del 21/03/23

ELEMENTI DI ANALISI MATEMATICA II

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" " "

ESERCIZI DI ANALISI MATEMATICA II
(VOL. I – II)

SERIE NUMERICA

$\{a_n\} \subseteq \mathbb{R}$ $\qquad \lim_{n\to\infty} a_n = \ell \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

$\ell \in \mathbb{R}$: $\forall \varepsilon > 0 \ \exists \nu \in \mathbb{N} : n > \nu$

$$\Downarrow$$

$$|a_n - \ell| < \varepsilon$$

$\ell = +\infty$:

$\forall K > 0 \ \exists \nu \in \mathbb{N} : n > \nu$

$$\Downarrow$$

$$a_n > K$$

$$(\ell = -\infty : \quad a_n < -K)$$

ESEMPI $\qquad a_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$a_n = \frac{n+1}{n} \qquad \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$$b_n = \frac{n}{n+1} \qquad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

$$\lim_{n\to\infty} b_n = 1$$

Succ. crescente: $\quad a_n \leq a_{n+1} \quad : \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n$

$$\neq$$

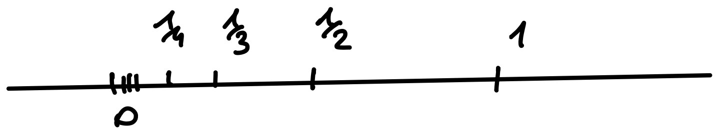
$$a_n < a_{n+1} \qquad \{n\} \quad 1, 2, 3, 4, \dots$$

$$\lim_{n\to\infty} n = +\infty$$

decrescente: $\quad a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$

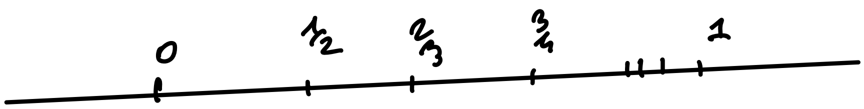
$$a_n > a_{n+1}$$

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n$$



ES. $\lim_{n\to\infty} \frac{1}{n} = 0 = \inf_{n} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 = \sup \frac{n}{n+1}$$

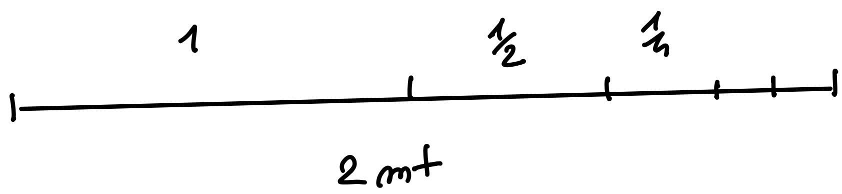


$$a_n = (-1)^n \qquad -1, 1, -1, 1, -1, 1$$

oscillante

$$a_1, a_2, a_3, \dots, a_n, \dots$$

$$a_1 + a_2 + a_3 + \dots + a_n = S_n \in \mathbb{R}$$



$$1 + \frac{1}{2} + \underbrace{\frac{1}{4}}_{\frac{1}{2^2}} + \underbrace{\frac{1}{8}}_{\frac{1}{2^3}} + \dots + \frac{1}{2^n} + \dots = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

$-n$ $\quad -1, -2, -3, -4, \dots$ decrescente

$$\lim_{n\to\infty} (-n) = -\infty = \inf_{n\in\mathbb{N}} (-n)$$

Def

$\{a_n\} \subseteq \mathbb{R}$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$

Somma parziale n-ma $\quad S_n = a_1 + a_2 + \dots + a_n$

$$\vdots$$

$\{S_n\}$ serie numerica di termine generale a_n

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (S)$$

(S) Si dice convergente (risp. divergente, oscillante)

se $\{S_n\}$ è convergente (" " ")

Nel caso in cui (S) sia regolare (convergente o divergente)

scriveremo

$$S = \text{somma della serie } (S)$$

$$= \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$$

$$S = \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

ES.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

$$= 2$$

In generale, possiamo considerare la serie

$$1 + h + h^2 + h^3 + \dots + h^n + \dots = \sum_{n=0}^{\infty} h^n$$

$h \in \mathbb{R}$ serie geometrica di ragione h

$$1 + 2 + 2^2 + 2^3 + \dots + 2^n + \dots$$

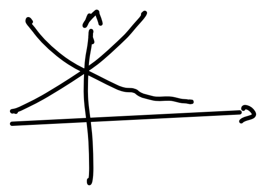
$h=1$: $\quad S_n = \underbrace{1+1+\dots+1}_{n} = n \xrightarrow{n\to\infty} +\infty$

$h=-1$: $\quad 1 \qquad 1-1=0 \qquad 0,1,0,1,0\dots$

$\qquad\qquad 0+1 = 1$

$$\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\dots+(-1)^n \quad \text{oscillante}$$

$\underbrace{-1<h<1}$?



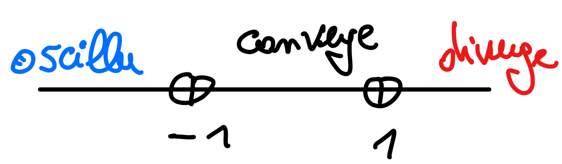
$$S_n = 1+h+h^2+\dots+h^{n-1} = \frac{1-h^n}{1-h} \underset{n\to\infty}{\overset{?}{\longrightarrow}} \frac{1}{1-h}$$

1

$1>h>0 \qquad \lim_{n\to\infty} h^n = 0$

$$\sum_{n=0}^{\infty} h^n = \begin{cases} +\infty & \text{se } h \geq 1 \\ \dfrac{1}{1-h} & \text{se } -1<h<1 \\ \text{oscillante} & \text{se } h \leq -1 \end{cases}$$

$$\sum 2^n = +\infty$$



$$\sum_{n=0}^{\infty} \frac{1}{2^n} \quad \left(h=\tfrac{1}{2}\right) = \frac{1}{\underbrace{1-\frac{1}{2}}_{\frac{1}{2}}} = 2$$

$$\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$$

$$h = -\frac{2}{3} \qquad = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$$

Condizione necessaria per la convergenza

Se $\displaystyle\sum_{n=1}^{\infty} a_n = S \in \mathbb{R}$

$$\Downarrow \quad \cancel{\Uparrow}$$

$$\lim_{n\to\infty} a_n = 0$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \qquad a_n = \frac{1}{2^n} \qquad \lim_{n\to\infty} a_n = 0$$

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n+1}{n}\right)}_{= a_n} \qquad \lim_{n\to\infty} a_n = 1 \neq 0$$

la serie <u>non</u> converge

Dim. $a_{n+1} \to 0$ (TESI)

$$a_{n+1} = S_{n+1} - S_n$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = S - S = 0$$

$$\boxed{\text{Lezione del } 22/09/23}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots = +\infty$$

serie armonica

ma $\lim_{n\to\infty} \frac{1}{n} = 0$!!

MENGOLI

$$\sum_{n=1}^{\infty} \overbrace{\frac{1}{n(n+1)}}^{a_n} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$$

$$= 1$$

$$a_n = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = ? = a_1 + a_2 + \dots + a_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$+ \cancel{\frac{1}{3}} - \frac{1}{4} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1} \to 1 = S = 1$$

$$\frac{1}{n-1} - \cancel{\frac{1}{n}}$$

$$\sum_{n=1}^{\infty} \underbrace{|\sin(n+1)|}_{a_n \geq 0}^{n} \qquad \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right) \to 0$$

Serie a termini non negativi

$$\sum_{n=1}^{\infty} a_n \qquad a_n \geq 0 \qquad \sum_{n=1}^{\infty} \frac{n}{n+1} \text{ non converge}$$

$$\frac{n}{n+1} \to 1 \qquad = (+\infty)$$

Proposizione Se $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$

allora o converge o diverge positivamente

Dim

$$S_n = a_1 + a_2 + \dots + a_n$$

$$S_{n+1} = \underbrace{a_1 + a_2 + \dots + a_n}_{S_n} + \overset{0}{\overset{\wedge|}{a_{n+1}}}$$

$$\geq S_n$$

$$\Rightarrow \lim_{n \to \infty} S_n = S = \sup_{n \in \mathbb{N}} S_n \in [0, +\infty]$$

$$\sum n = 1+2+3+\dots+n+\dots = +\infty$$

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n(n+1)} \qquad |\sin n| \leq 1 \ : \ \frac{|\sin n|}{n(n+1)} \leq \frac{1}{n(n+1)}$$

$$\sum \dots < +\infty$$

$$\Downarrow$$

criterio confronto

$$\Downarrow$$

$$\sum \frac{|\sin n|}{n(n+1)} < +\infty$$

$$b_n \leq a_n \leq c_n$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$l \qquad l \qquad l$$

$$b_n \to +\infty \implies a_n \to +\infty$$

$$c_n \to -\infty \implies a_n \to -\infty$$

<u>Criterio del confronto</u> $\displaystyle\sum_{n=1}^{\infty} a_n \ , \ \sum_{n=1}^{\infty} b_n$

$$a_n, b_n \geq 0$$

e $\qquad a_n \leq b_n \quad \forall n \in \mathbb{N}$.

Allora:

(i) se $\displaystyle\sum_{n=1}^{\infty} b_n < +\infty \implies \sum_{n=1}^{\infty} a_n < +\infty$

(ii) se $\displaystyle\sum_{n=1}^{\infty} a_n = +\infty \implies \sum_{n=1}^{\infty} b_n = +\infty$

Dim.

$$S_n = a_1 + \dots + a_n$$

$$T_n = b_1 + \dots + b_n$$

$$\Rightarrow \quad S_n \le T_n \le \sup_{n\in\mathbb{N}} T_n$$

(i) Se $\displaystyle\sum_{n=1}^{\infty} b_n < +\infty \Rightarrow \sup_{n\in\mathbb{N}} T_n < +\infty \Rightarrow S_n$ limitata $\displaystyle\Rightarrow \sum_{n=1}^{\infty} a_n = \sup_{n\in\mathbb{N}} S_n < +\infty$

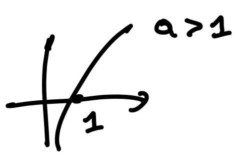
(ii) Se $\displaystyle\sum_{n=1}^{\infty} a_n = +\infty$

$$\underset{+\infty}{\underset{\downarrow}{S_n}} \le T_n$$

$\Rightarrow$ criterio confronto successioni $\displaystyle\Rightarrow \lim_{n\to\infty} T_n = +\infty$

$$\Rightarrow \quad \sum_{n=1}^{\infty} b_n = +\infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots = +\infty$$



$$\left(1 + \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e \in (2,3)$$

$$\left(1 + \frac{1}{n}\right)^n < e \qquad \forall n \in \mathbb{N}$$

$$\log\left(1+\frac{1}{n}\right)^n < \log e = 1$$

$$\log\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

$$\log\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

$$\parallel$$

$$\log(n+1) - \log n < \frac{1}{n} \qquad (1)$$

$$S_n = \cancel{\log 2} - \underbrace{\log 1}_{=0} + \cancel{\log 3} - \cancel{\log 2} + \dots + \log(n+1) - \cancel{\log n} = \log(n+1)$$

Ma da (1): $S_n < 1 + \frac{1}{2} + \dots + \frac{1}{n} = T_n$

$$\log(n+1) < T_n$$

$\downarrow n \to \infty$

$+\infty$

$$\Downarrow$$

$$T_n \to +\infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$$\sum_{n=0}^{\infty} h^n$$

-1 $\quad$ 1

oscilla $\quad$ converge $\quad$ $+\infty$

$h > 1$ $\qquad \lim_{n\to\infty} h^n = +\infty \Rightarrow \sum_{n=0}^{\infty} h^n = +\infty$

$$\sum_{n=1}^{\infty} a_n, \quad a_n \geq 0$$

Criterio della radice

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell \in [0, +\infty]$$

(i) se $l < 1 \Rightarrow$ la serie converge

(ii) se $l > 1 \Rightarrow$ la serie diverge $(+\infty)$

$l = 1$?? NULLA SI PUÒ DIRE A PRIORI

$$\sum_{n=1}^{\infty} 2^n \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{2^n} = 2 > 1$$

$\Rightarrow$ la serie diverge

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{3}{4} < 1 \Rightarrow \text{serie converge}$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n \qquad \text{ESERCIZIO: } a_n \xrightarrow{n\to\infty}$$

$2n + 1$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1 \Rightarrow \text{converge}$$

$$\left(\frac{n}{2n+1}\right)^{n^2} \qquad \lim_{n\to\infty} \left[\left(\frac{n}{2n+1}\right)^{n^{\cancel{2}}}\right]^{1/\cancel{n}} = \lim_{n\to\infty} \left(\frac{n}{2n+1}\right)^n = 0 < 1$$

$\frac{1}{2}$

converge

Criterio del rapporto

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l \in [0,+\infty]$$

$a_n > 0$

$\Rightarrow$ valgono le tesi del criterio della radice!

ES.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} < \infty : \qquad a_n = \frac{n}{2^n}$$

$\searrow 0$

$$a_{n+1} = \frac{n+1}{2^{n+1}} \qquad \frac{n+1}{n} \to 1$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{(n+1)\cancel{2^n}}{\cancel{2^n} \cdot 2 \cdot n} \to \frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$