Lezione del 14/12/2023

Formula del cambiamento di variabili

$f = f(x)$, $\quad f : [a,b] \longrightarrow \mathbb{R}$ continua

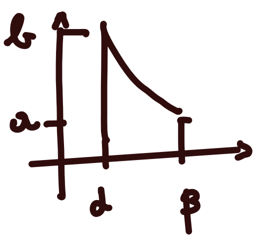
$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

(φ st. crescente)

$x = \varphi(t)$, $\quad \varphi : [\alpha, \beta] \longrightarrow [a,b]$

invertibile (strett. crescente o decrescente)

$dx = \varphi'(t)\,dt$ $\quad \varphi \in C^1$



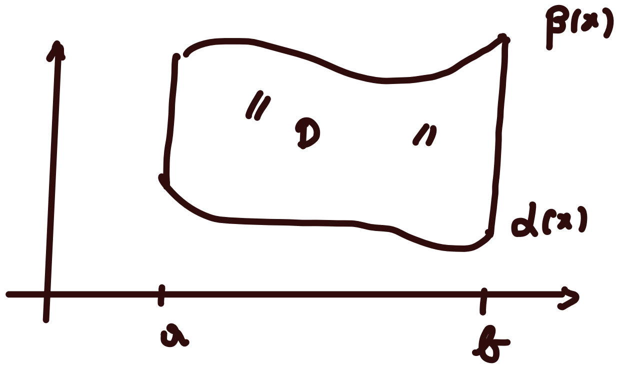
$$= \int_\beta^\alpha f(\varphi(t))\,\varphi'(t)\,dt = -\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$$= \int_\alpha^\beta f(\varphi(t))\,|\varphi'(t)|\,dt$$

$$\boxed{\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\underbrace{|\varphi'(t)|}\,dt}$$

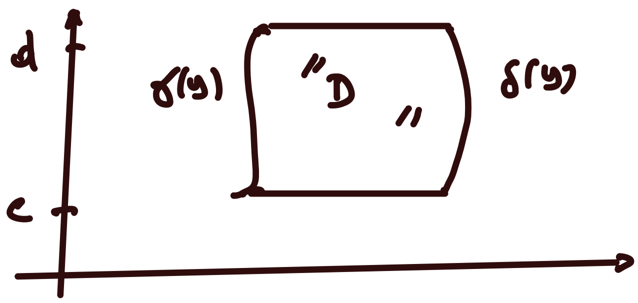
$f(x,y) \longleftrightarrow u,v$ $\qquad$ $x,y \longleftrightarrow u,v$

Def. $D = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b, \quad \alpha(x) \leq y \leq \beta(x) \}$



regolare se $\alpha, \beta \in C^1([a,b])$

$D = \{ (x,y) \in \mathbb{R}^2 : c \leq y \leq d, \quad \gamma(y) \leq x \leq \delta(y) \}$



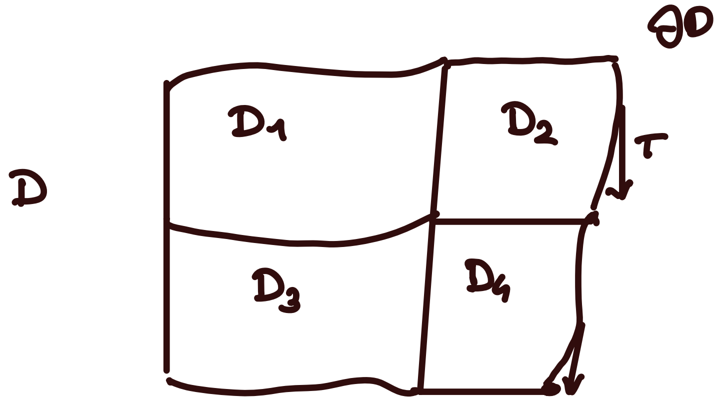
regolare se $\gamma, \delta \in C^1([c,d])$

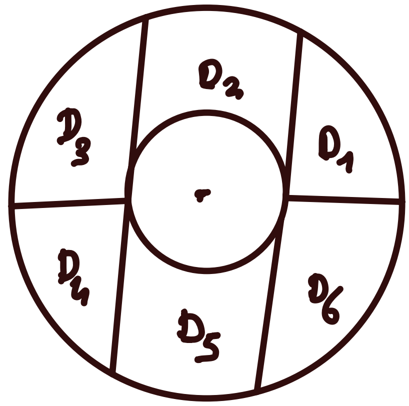
Def. $D \subseteq \mathbb{R}^2$ dominio regolare se D è UNIONE di un numero finito $D_1, \dots, D_N$ di domini normali regolari a due a due privi di punti interni in comune

$$D = \bigcup_{i=1}^{N} D_i$$

D_i normale regolare $\forall i = 1, \dots, N$

$\mathring{D}_i \cap \mathring{D}_j = \emptyset$





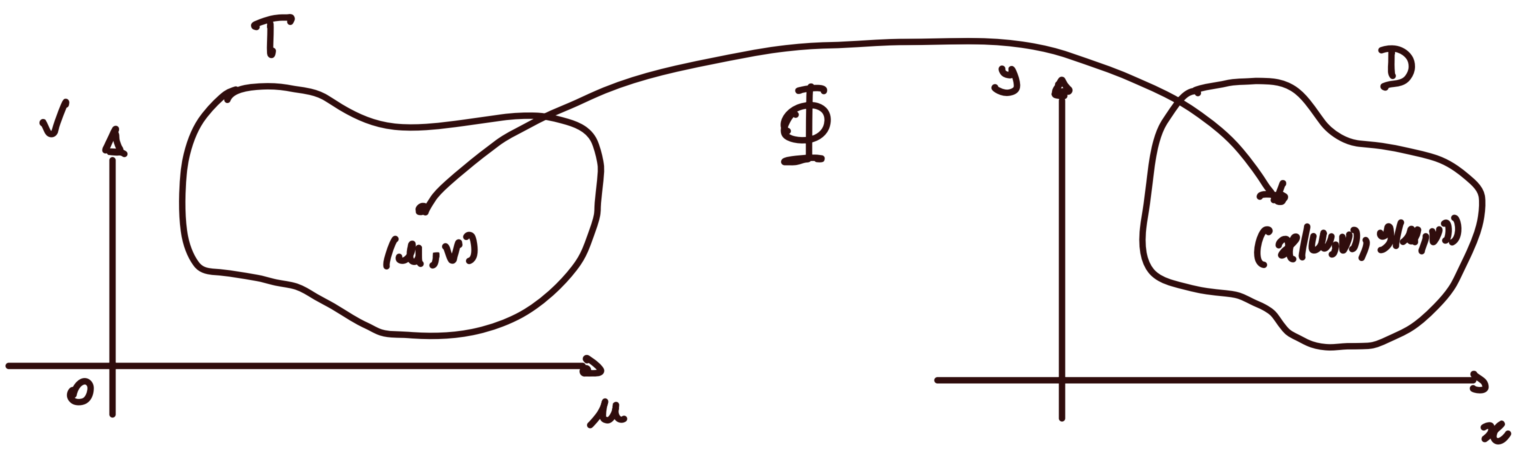
∂D curva generalmente regolare (o regolare a tratti): in quasi ogni punto esiste il versore tangente $\underline{T}$, tranne che in un numero finito di punti.

DETERMINANTE JACOBIANO

$T, D \subseteq \mathbb{R}^2$ domini regolari

$$\Phi(T) = D$$

$$\Phi : T \longrightarrow D$$
$$(u,v) \longrightarrow (x(u,v), y(u,v))$$



$$\Phi \equiv \begin{cases} x = \underline{\underline{x}}(u,v) \\ y = \underline{\underline{y}}(u,v) \end{cases} \qquad \Phi \in C^1(T)$$

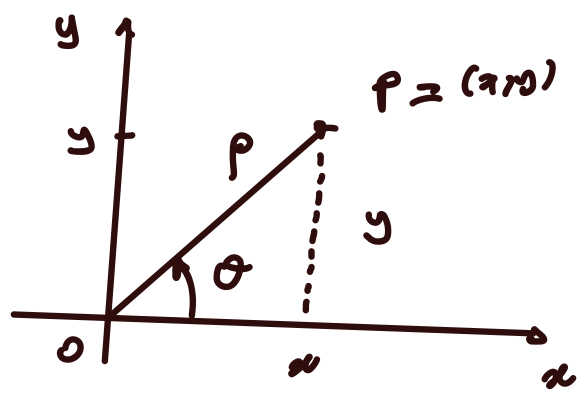
Determinante jacobiano di Φ

$$J_\Phi(u,v) = \frac{\partial(x,y)}{\partial(u,v)}(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

TRASFORMAZIONE ALLE COORDINATE POLARI

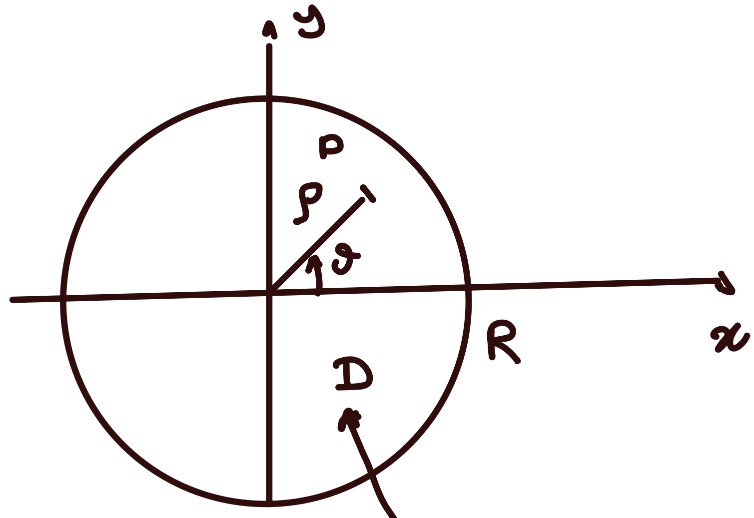
ρ, θ

$$\begin{array}{c} \quad\quad u \quad\quad v \\ \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \end{array}$$

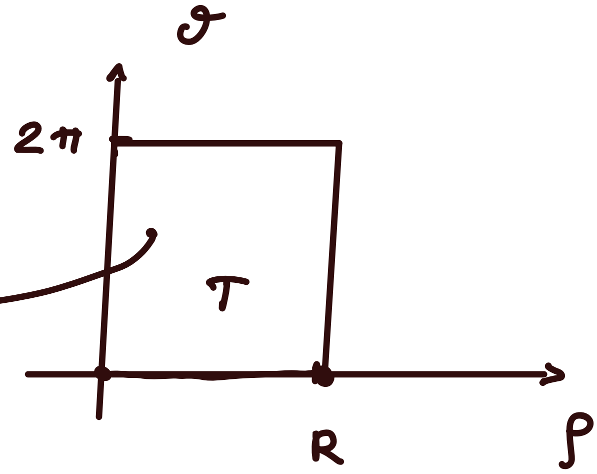


$$\Phi : (\rho,\theta) \in \mathbb{R}^2 \longrightarrow (\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2$$

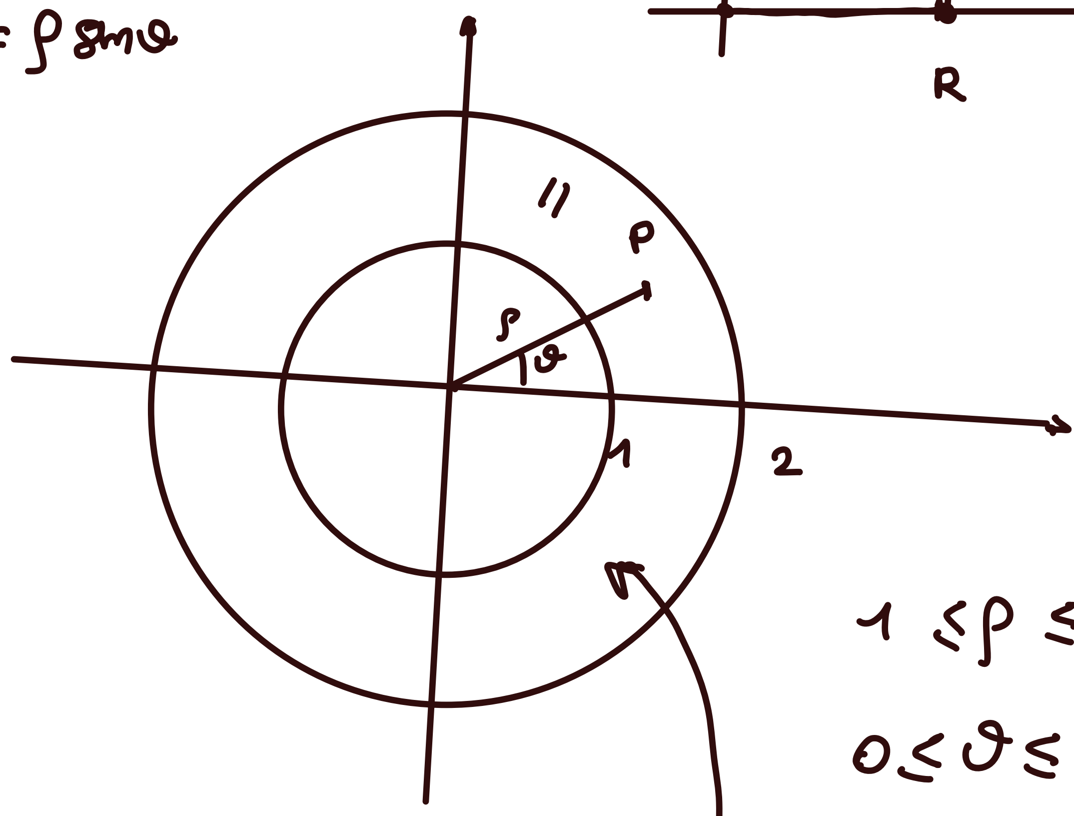
$$J_\Phi = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$$



$$T = \begin{cases} 0 \le \rho \le R \\ 0 \le \vartheta \le 2\pi \end{cases}$$

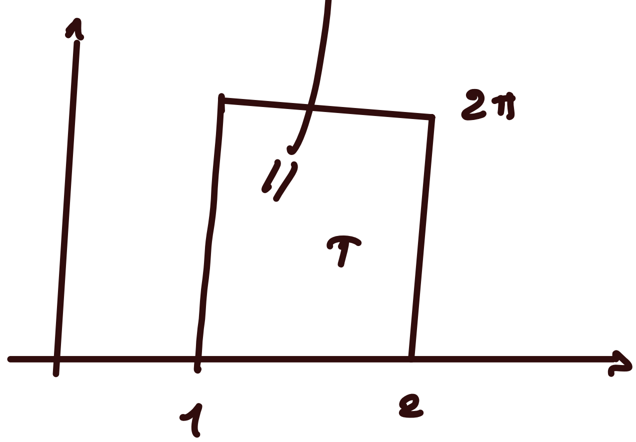


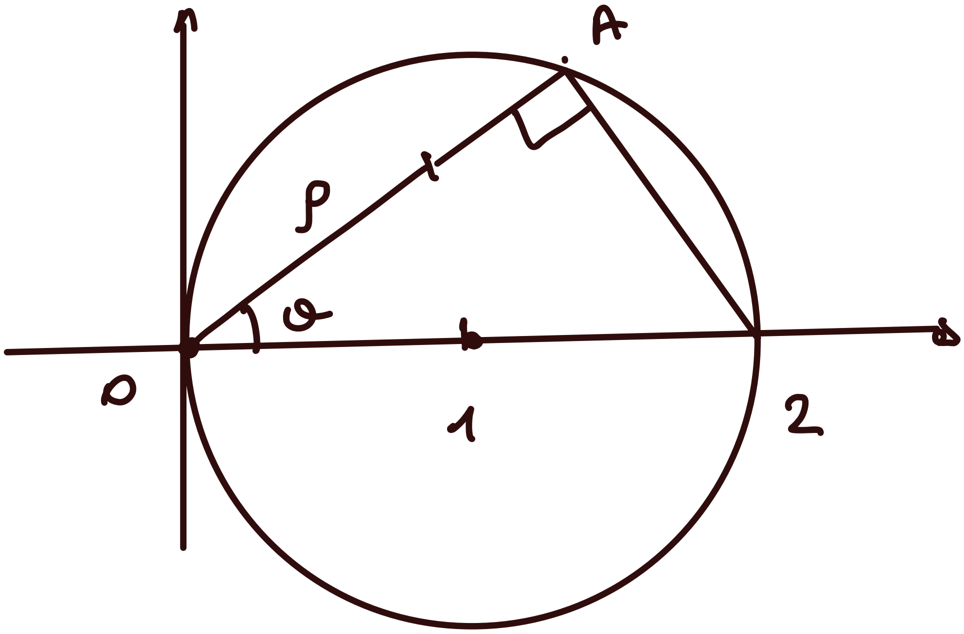
$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \end{cases}$$



$$1 \le \rho \le 2$$

$$0 \le \vartheta \le 2\pi$$





$$\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$

$$\boxed{0 \le \rho \le 2\cos\theta}$$

$\overline{OA} = 2\cos\theta$

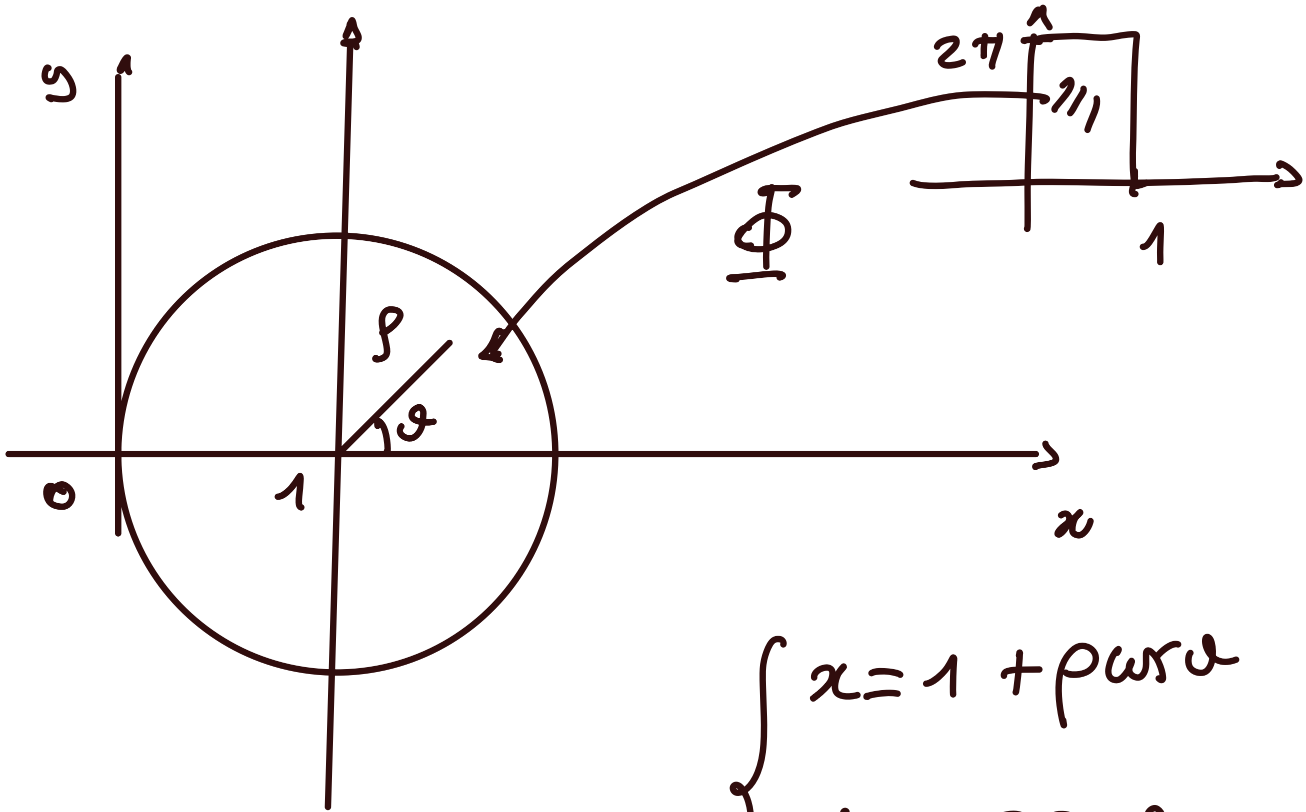
$$(x-1)^2 + y^2 = 1$$

$$x^2 + y^2 - 2x = 0 \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\rho^2 - 2\rho\cos\theta = 0 \iff \boxed{\rho = 2\cos\theta}$$

equazione polare della circonferenza in (1,0) e raggio 1

$$T = \left\{ (\rho,\theta) \in \mathbb{R}^2 : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le \rho \le 2\cos\theta \right\}$$



$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$0 \le \theta \le 2\pi$$

$$0 \le \rho \le 1$$

$$\det \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

Formula del cambiamento di variabili negli integrali doppi

$T,\ D \subseteq \mathbb{R}^2$ domini regolari

$T \subseteq \mathbb{R}^2_{u,v}\,, \quad D \subseteq \mathbb{R}^2_{x,y}$

$\Phi : T \longrightarrow D$ biunivoca, di classe C^1 e tale che $\det \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$ in T.

Se $f(x,y)$, $f : D \longrightarrow \mathbb{R}$ continua

si ha:

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_T f(x(u,v),y(u,v)) \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \qquad D = \Phi(T)}$$

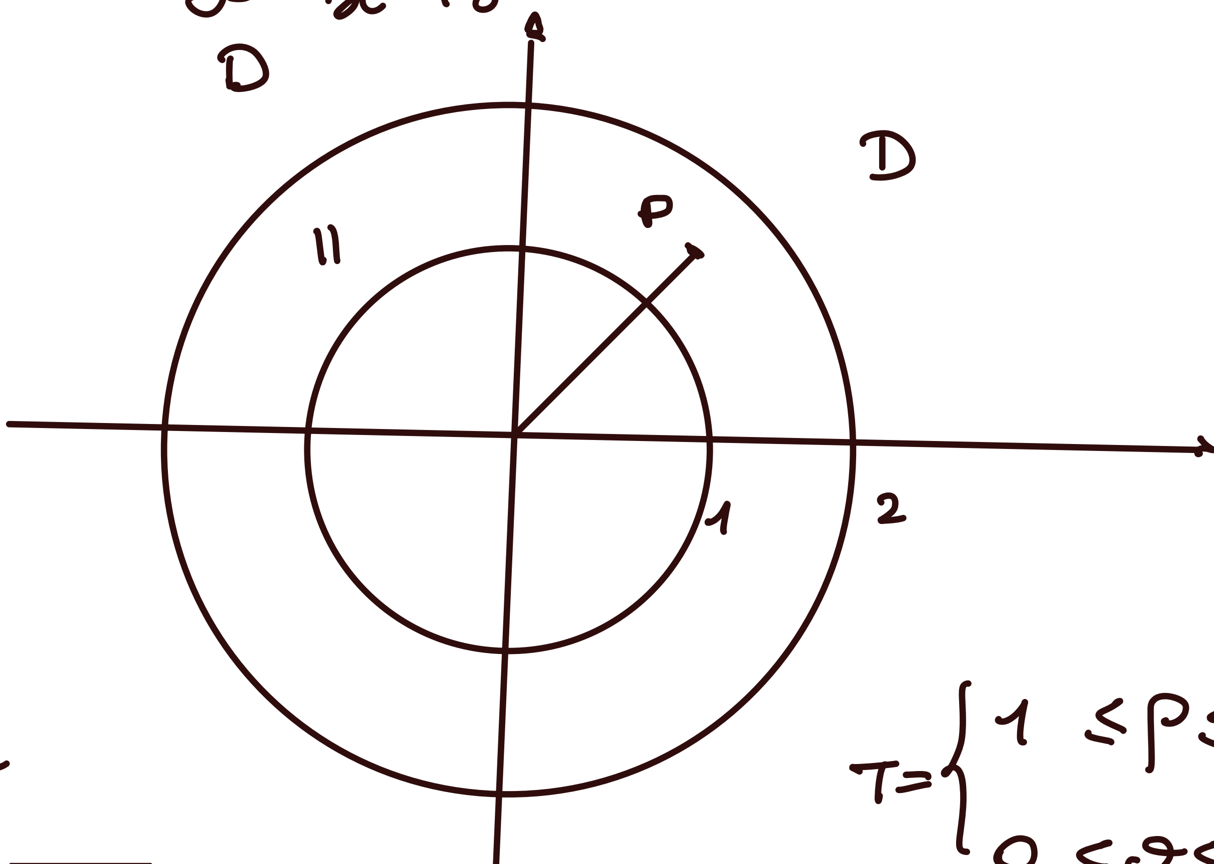
OSS. Φ alle coordinate polari:

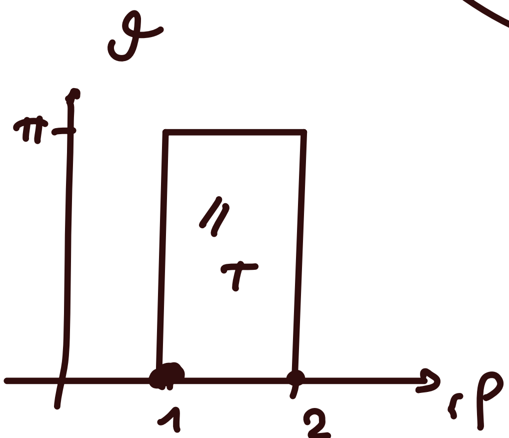
1. NON È INVERTIBILE
2. $J_\Phi = \rho = 0$ in qualche punto!

NON È VERO CHE $J_\Phi \neq 0$

Però si può dimostrare che la formula vale lo stesso!

<u>ES.</u> 1 $$\iint_D \frac{y}{x^2+y^2}\,dx\,dy$$





$$T = \begin{cases} 1 \le \rho \le 2 \\ 0 \le \theta \le \pi \end{cases}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\iint_D \frac{y}{x^2+y^2}\,dx\,dy =$$

$$= \iint_T \frac{\cancel{\rho}\sin\theta}{\cancel{\rho^2}} \cdot \cancel{\rho}\,d\rho\,d\theta$$

$$= \iint_T \sin\theta\,d\rho\,d\theta = \text{(f.le di riduzione)}$$

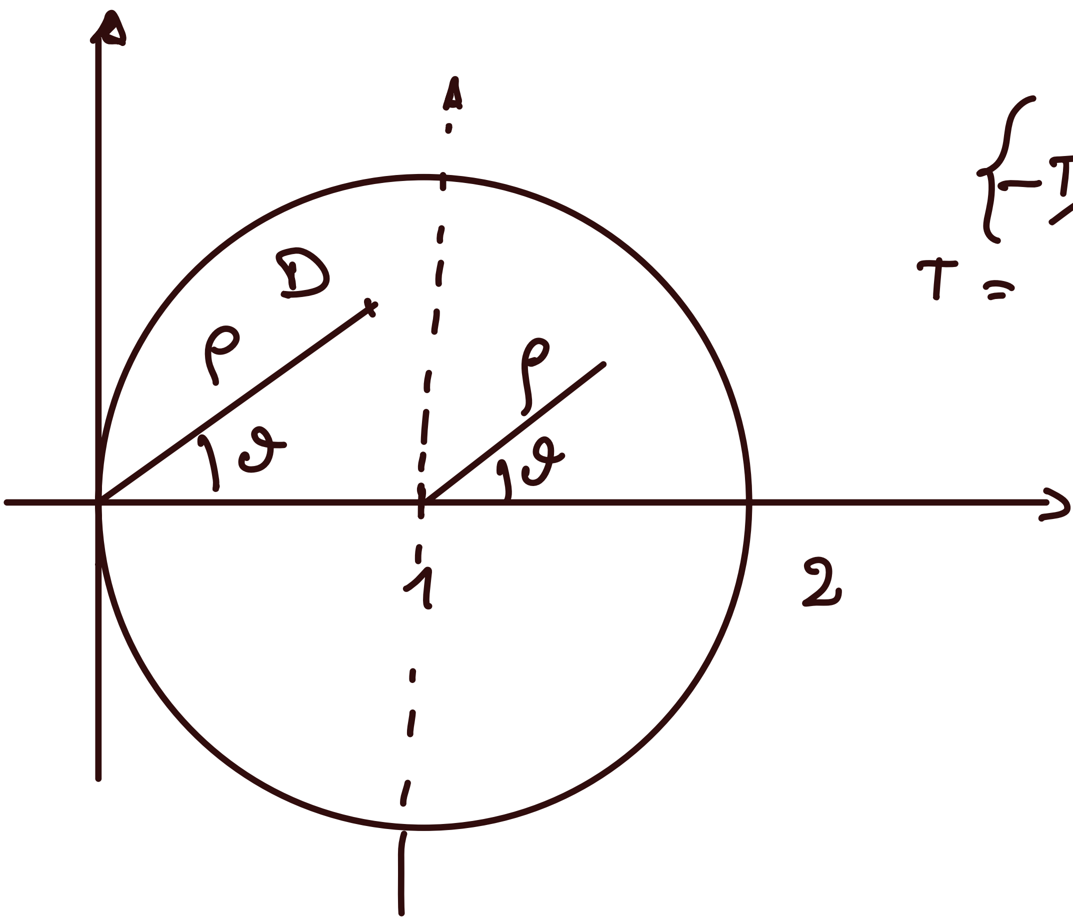
$$= \left(\int_1^2 d\rho\right)\left(\int_0^\pi \underbrace{\sin\theta\,d\theta}_{=}\right) =$$

$$= -1 \cdot \left(\cos\theta\right)_0^\pi = -(-1-1) =$$

$$= 2$$

② $\iint_D \underbrace{\sqrt{x^2+y^2}}_{\rho}$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$



$$T = \begin{cases} -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2} \\ 0 \le \rho \le 2\cos\vartheta \end{cases}$$

$$= \iint_T \rho \cdot \rho \, d\rho \, d\vartheta = \iint_T \rho^2 \, d\rho \, d\vartheta$$

$$= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} \rho^2 \, d\rho =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{\rho^3}{3}\right)\Bigg|_{\rho=0}^{\rho=2\cos\theta} d\theta =$$

$$= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta \quad \ldots\ldots$$

$$\int \cos^3\theta \, d\theta = \int \cos^2\theta \cdot \cos\theta =$$

$$= \int (1-\sin^2\theta)\cos\theta = \ldots$$

$$\iint_D (x^2+y^2)\,dx\,dy = \iint_T \rho^3\,d\rho\,d\theta =$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta$$

$$0 \le \rho \le 1, \qquad 0 \le \theta \le 2\pi$$

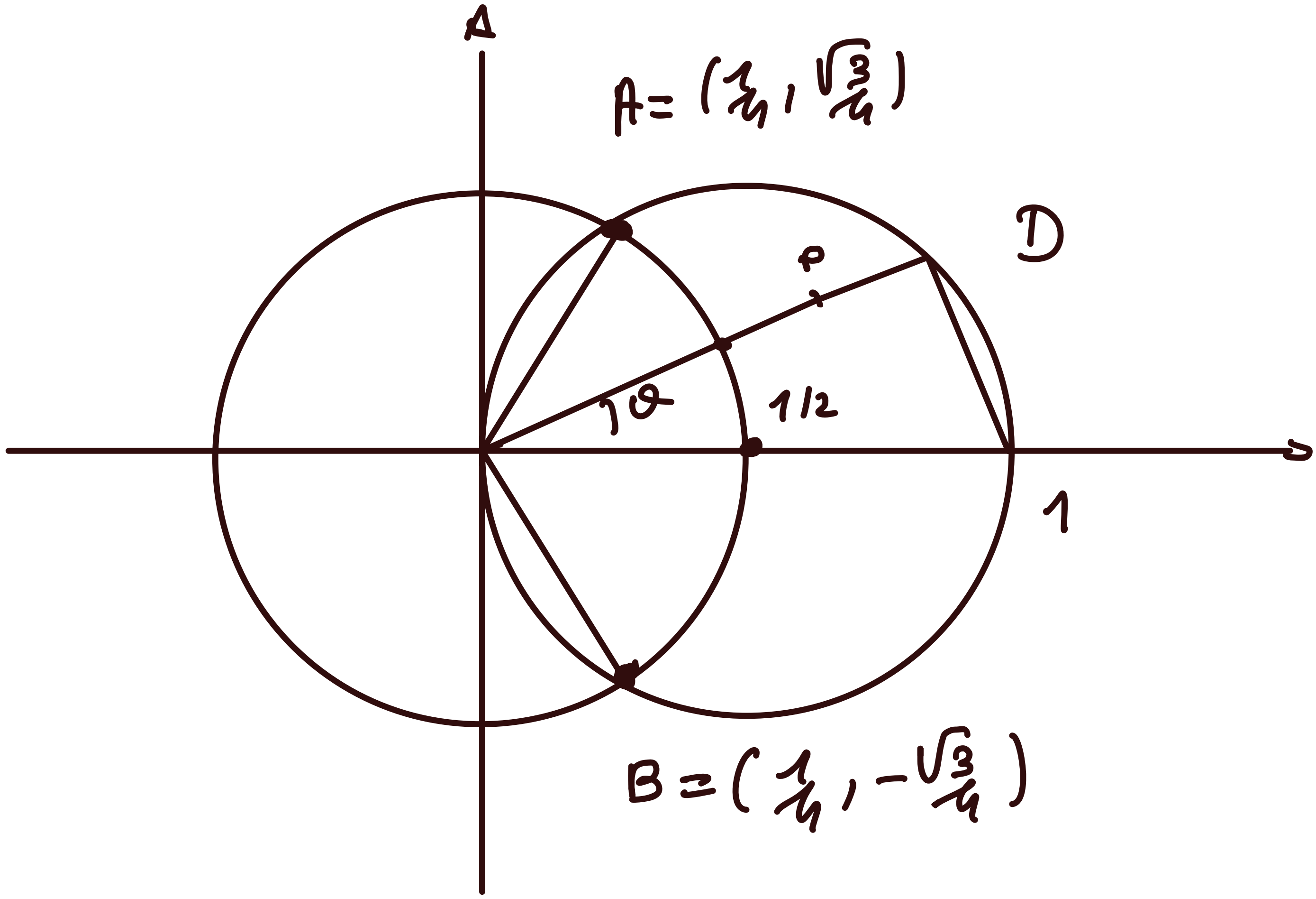
$$\begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\iint_D (x^2+y^2)\,dx\,dy = \iint_T \left[1 + \rho^2\cos^2\theta + 2\rho\cos\theta + \rho^2\sin^2\theta\right]\rho\,d\rho\,d\theta$$

$$= \iint_T \left[1 + \rho^2 + 2\rho\cos\theta\right] \rho \, d\rho \, d\theta$$

$$= \iint_T \rho \, d\rho \, d\theta + \iint_T \rho^3 \, d\rho \, d\theta + 2\underbrace{\iint_T \rho^2 \cos\theta \, d\rho \, d\theta}_{= 0}$$

3)



$$\iint_D \frac{dx\,dy}{\sqrt{1 - x^2 - y^2}} \qquad -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$$

$$x^2+y^2=1/4$$

$$\left(x-\frac{1}{2}\right)^2+y^2=1/4$$

$$x^2-x+y^2=0$$

$$\begin{cases} x^2+y^2=1/4 \\ x^2-x+y^2=0 \end{cases}$$

$$\begin{cases} x^2+y^2=1/4 \\ x=1/4 \end{cases} \iff \begin{cases} y^2=\frac{1}{4}-\frac{1}{16} \\ x=1/4 \end{cases}$$

$$\iff \begin{cases} y=\pm\sqrt{3}/4 \\ x=1/4 \end{cases}$$

$A:$

$$x = \frac{1}{2}\cos\theta = \frac{1}{4}$$

$$y = \frac{1}{2}\sin\theta = \frac{\sqrt{3}}{4}$$

$$\Leftrightarrow \begin{cases} \cos\theta = \frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$$

$$\theta = \frac{\pi}{3}$$

$$\frac{1}{2} \leq \rho \leq \cos\theta$$

$$T = \left\{ (\rho, \theta) : -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, \ \frac{1}{2} \leq \rho \leq \cos\theta \right\}$$

$$\iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \iint_T \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho\, d\theta =$$

$$= \int_{-\pi/3}^{\pi/3} d\theta \int_{1/2}^{\cos\theta} \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho$$

$$= -\frac{1}{2}\int_{-\pi/3}^{\pi/3} d\theta \int_{1/2}^{\cos\theta} (-2\rho)(1-\rho^2)^{-1/2}\, d\rho$$

$$= -\cancel{\frac{1}{2}}\cdot 2\int_{-\pi/3}^{\pi/3} \left(\sqrt{1-\rho^2}\right)_{\rho=1/2}^{\rho=\cos\theta} d\theta$$

$$= -\int_{-\pi/3}^{\pi/3} \left(|\sin\theta| - \frac{1}{2}\right) d\theta$$

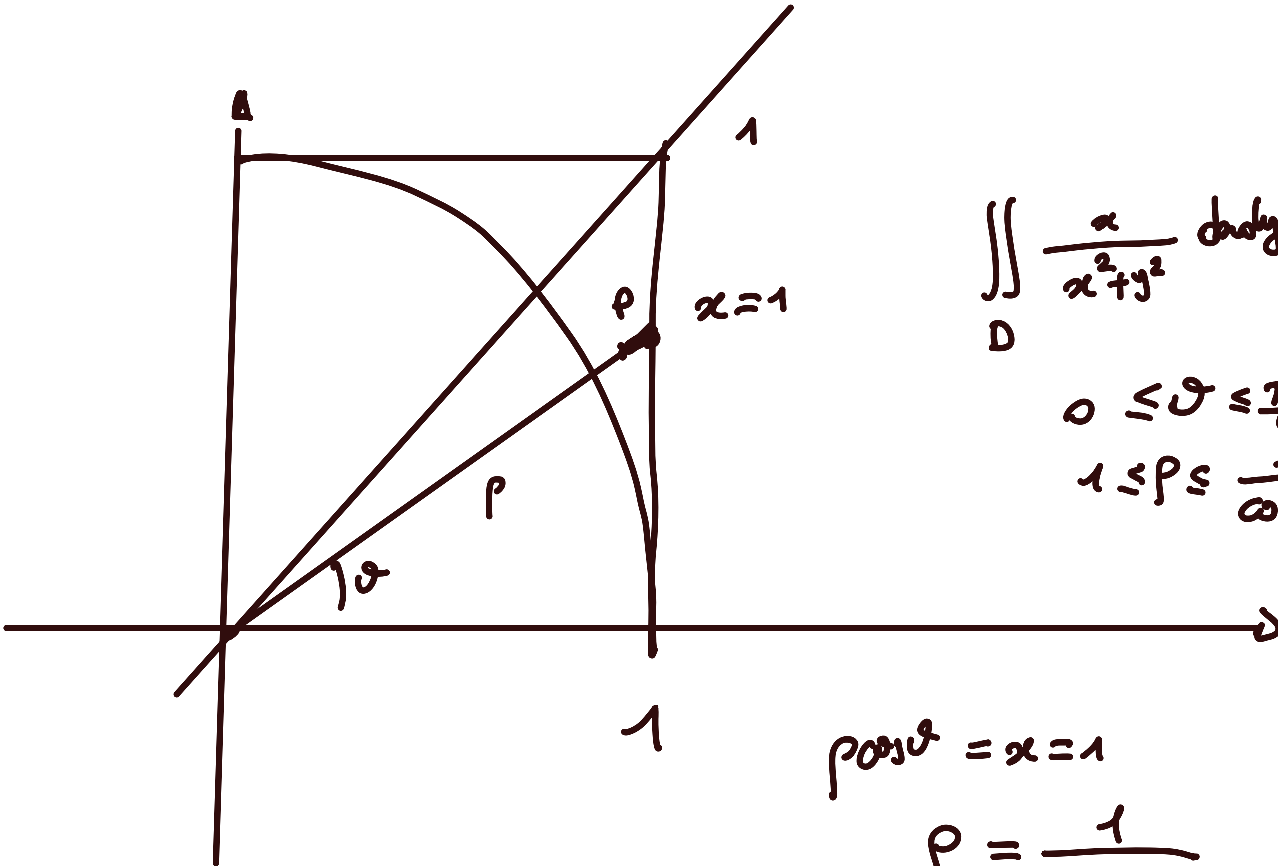
$$= -\left[2\int_{0}^{\pi/3} \sin\theta\, d\theta - \cancel{\frac{1}{2}}\cdot\frac{\cancel{2}}{3}\pi\right]$$

$$= -\left[-2\left(\cos\theta\right)_{0}^{\pi/3} - \frac{\pi}{3}\right]$$

$$= 2\cdot\left(\frac{1}{2}-1\right) + \frac{\pi}{3} =$$

$$= \frac{\pi}{3} - 1$$

4)



$$\iint_D \frac{x}{x^2+y^2}\,dx\,dy$$

$$0 \leq \vartheta \leq \frac{\pi}{4}$$

$$1 \leq \rho \leq \frac{1}{\cos\vartheta}$$

$$\rho\cos\vartheta = x = 1$$

$$\rho = \frac{1}{\cos\vartheta}$$

$$T = \left\{ (\rho,\vartheta) \in \mathbb{R}^2 : \ 0 \leq \vartheta \leq \frac{\pi}{4}, \ 1 \leq \rho \leq \frac{1}{\cos\vartheta} \right\}$$

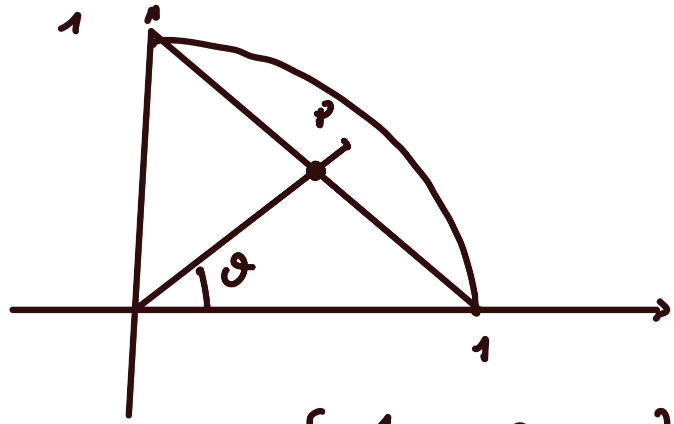
$$\iint_D \frac{x}{\underbrace{x^2+y^2}_{\rho^2}}\,dx\,dy = \iint_T \frac{\cancel{\rho}\cos\vartheta}{\cancel{\rho^2}} \cdot \cancel{\rho}\, d\rho\, d\vartheta =$$

$$= \int_0^{\pi/4} d\vartheta \int_1^{1/\cos\vartheta} \underbrace{\cos\vartheta}\, d\rho =$$

$$= \int_0^{\pi/4} \cos\vartheta \left(1 - \frac{1}{\cos\vartheta}\right) d\vartheta = \text{(FACILE)}$$

5) $$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \iint_T \frac{1}{\cancel{\rho}}\cdot\cancel{\rho}\,d\rho\,d\theta$$



$$y = -x+1 \qquad \left\{\begin{array}{l} \frac{1}{\sin\theta+\cos\theta} \le \rho \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right\} \equiv T$$

$$\rho\sin\theta = -\rho\cos\theta + 1$$

$$\rho\,(\sin\theta+\cos\theta) = 1$$

$$\rho = \frac{1}{\sin\theta+\cos\theta}$$

$$= \int_0^{\pi/2} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} d\rho$$

$$= \int_0^{\pi/2}\left(1-\frac{1}{\sin\theta+\cos\theta}\right)d\theta = \frac{\pi}{2} - \int_0^{\pi/2}\frac{d\theta}{\underbrace{\sin\theta+\cos\theta}_{\text{FORMULE PARAMETRICHE!}}}$$