Integrazione multipla

Def. (Dominio normale)

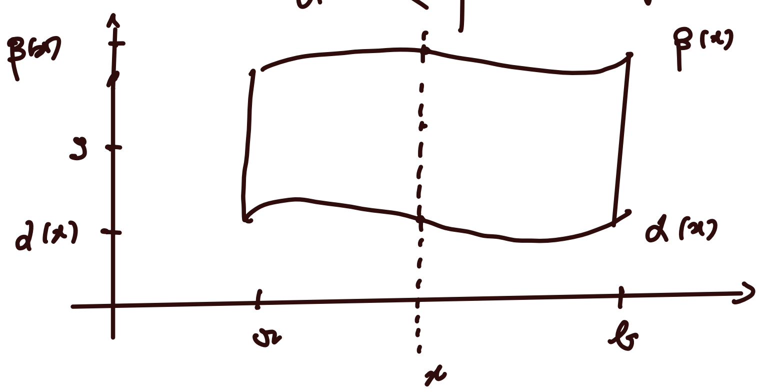
$$\iint_D f(x,y)\,dx\,dy$$

$D \subseteq \mathbb{R}^2$ dominio normale rispetto all'asse x se

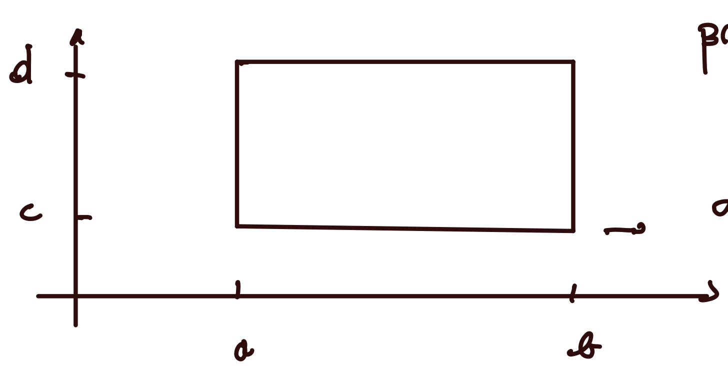
$$D = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b, \ \alpha(x) \leq y \leq \beta(x) \}$$

$\alpha(x)$, $\beta(x)$ funzioni continue in $[a,b]$

tali che $\alpha(x) \leq \beta(x)$, $\forall x \in [a,b]$



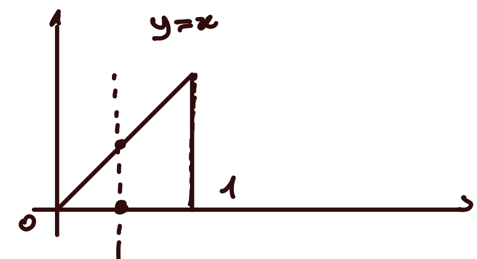
ES. 1



$\beta(x) = d$

$\alpha(x) = c$

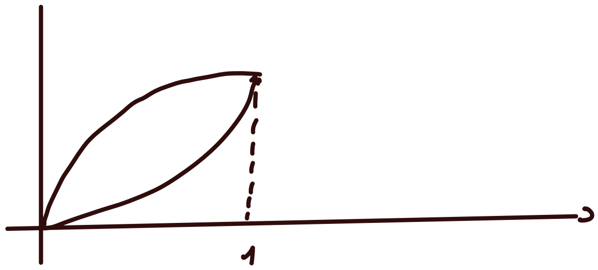
2



$a = 0$, $b = 1$

$\alpha(x) = 0$, $\beta(x) = x$

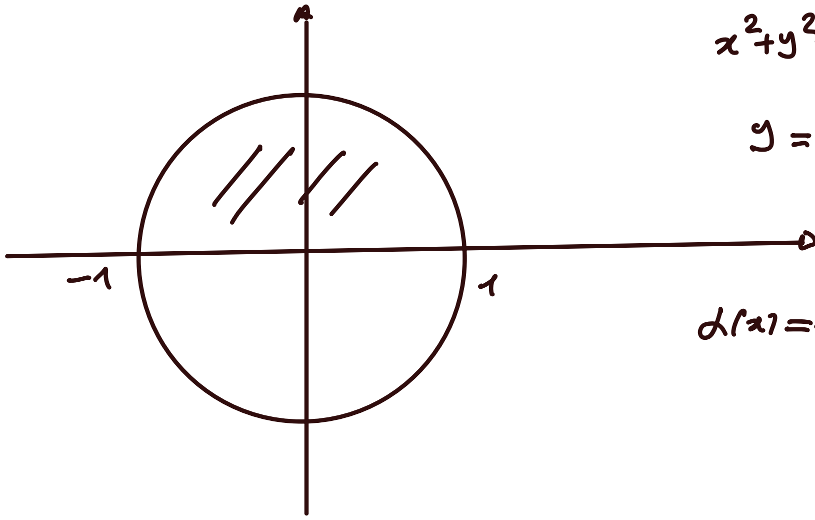
③



$\alpha(x)=x^2$

$\beta(x)=\sqrt{x}$

④



$x^2+y^2=1$

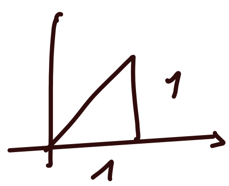
$y=\sqrt{1-x^2}=\beta(x)$

$\alpha(x)=0\ ,\ -1\le x\le 1$

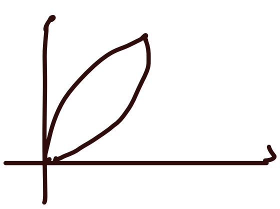
$$\text{Area}(D)=\int_a^b [\beta(x)-\alpha(x)]\,dx = \int_a^b [\beta(x)-\alpha(x)]\,dx$$

① Rettangolo : $\text{Area}(D)=\int_a^b [d-c]\,dx=(d-c)(b-a)$

② Triangolo : " $=\int_0^1 x\,dx=\left(\frac{x^2}{2}\right)_0^1=\frac{1}{2}$



③



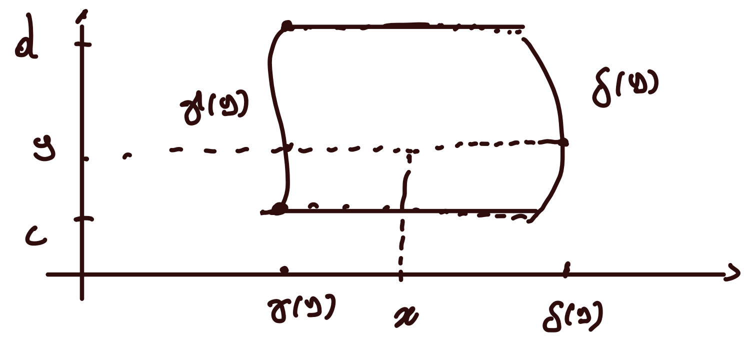
$\text{Area}(D)=\int_0^1\left(\sqrt{x}-x^2\right)dx$

④ Area cerchio $= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx = \pi$.

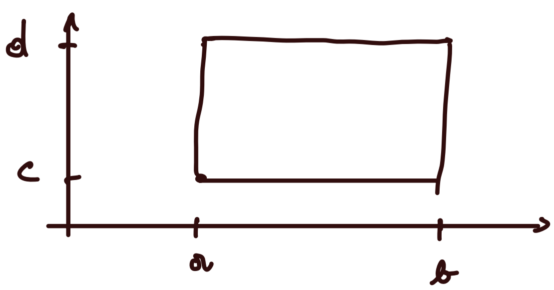
Def. $D \subseteq \mathbb{R}^2$ dominio normale rispetto all'asse y se

$$D = \{ (x,y) \in \mathbb{R}^2 : \ c \le y \le d, \quad \gamma(y) \le x \le \delta(y) \}$$

$$\gamma(y) \le \delta(y) \quad \forall y \in [c,d] \quad \text{continue}$$



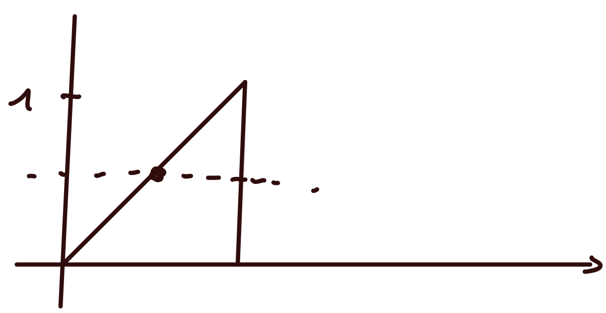
①



$c \le y \le d$

$\gamma(y) = a, \quad \delta(y) = b$

②



$0 \le y \le 1$

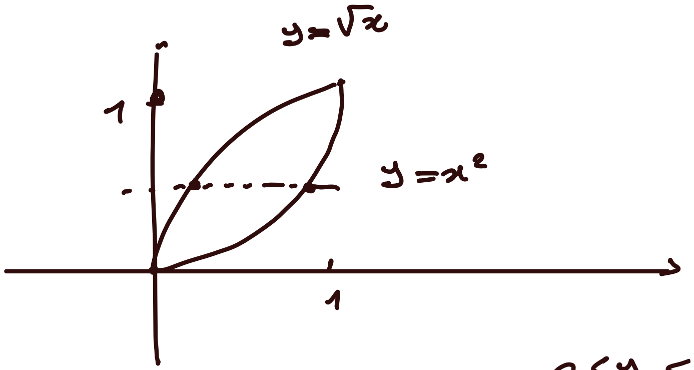
$\gamma(y) = y$

$\delta(y) = 1$

$y \le x \le 1$

$0 \le y \le 1$

③



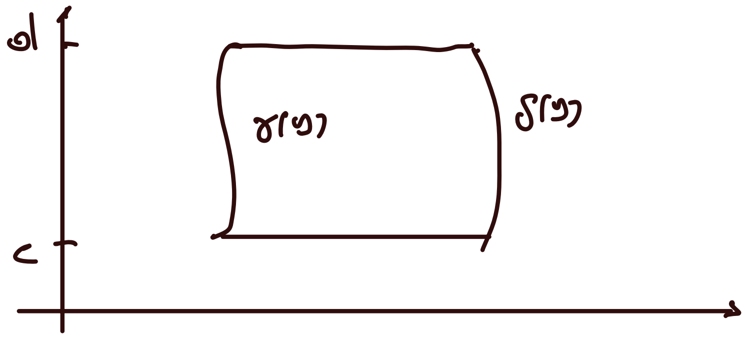
$y=\sqrt{x} \iff x=y^2$

$y=x^2 \iff x=\sqrt{y}$

$0 \le y \le 1$,

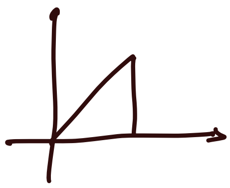
$y^2 \le x \le \sqrt{y}$

Area (D) ?



$$\text{Area}(D) = \int_c^d [\, \delta(y) - \gamma(y) \,]\, dy$$

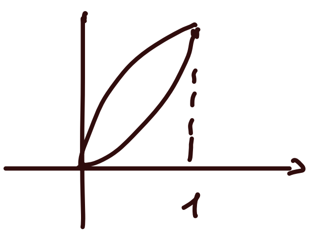
ES. (TRIANGOLO)



$0 \le y \le 1$

$y \le x \le 1$

$$A(D) = \int_0^1 (1-y)\, dy = 1 - \frac{1}{2} = \frac{1}{2}$$



$$A(D) = \int_0^1 [\, y^2 - \sqrt{y} \,]\, dy$$

Def. Riemann

Partizione di un dominio normale.

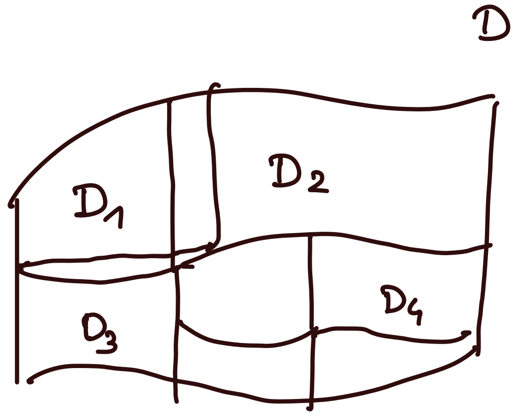
Si dice PARTIZIONE di un dominio normale $D \subseteq \mathbb{R}^2$ un insieme

$$\mathcal{P} = \{ D_1, D_2, \dots, D_N \}$$ dove:

1.) D_i è un dominio normale

2.) $\mathring{D}_i \cap \mathring{D}_j = \emptyset \quad \forall\, i \neq j$

3.) $$\bigcup_{i=1}^{N} D_i = D$$



$f(x,y)$ $\qquad f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ limitata

$\mathcal{P} = \{ D_1, \dots, D_N \}$ partizione di D

$$\inf_{[x_i, x_{i+1}]} f \, (x_{i+1} - x_i)$$

$$s(f, \mathcal{P}) = \sum_{i=1}^{N} \inf_{D_i} f \cdot A(D_i)$$

" SOMMA INTEGRALE INFERIORE DI f CORRISPONDENTE A $\mathcal{P}$ "

$$S(f,P) = \sum_{i=1}^{n} \sup_{D_i} f \cdot A(D_i)$$

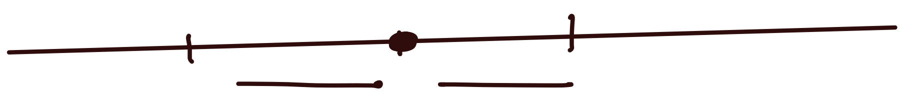
$$\{ s(f,P) : P \text{ partizione di } D \}$$

$$\{ S(f,P) : \text{'' \quad ''} \}$$

<u>SEPARATI</u> : $\forall P_1, P_2$ partizioni di D,

$$s(f,P_1) \leq S(f,P_2)$$

<u>Def</u>- <u>SE</u> i due insiemi sono contigui, si dice che



f è INTEGRABILE secondo Riemann e scriveremo

$$\iint_D f(x,y)\,dx\,dy = \sup_P s(f,P) = \inf_P S(f,P)$$

<u>Prop</u>- Se f è continua, f è integrabile.

FORMULE DI RIDUZIONE

$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ continua su

$D = \{ (x,y) \in \mathbb{R}^2 : \ a \leq x \leq b, \quad \alpha(x) \leq y \leq \beta(x) \}.$

$$\iint_D f(x,y)\,dxdy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

$f : E \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ su

$E = \{ (x,y) \in \mathbb{R}^2 : \ c \leq y \leq d, \quad \gamma(y) \leq x \leq \delta(y) \}$

$$\iint_E f(x,y)\,dxdy = \int_c^d dy \int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx$$

ES. $\displaystyle\iint_D xy\,dxdy \qquad D = \{ (x,y) \in \mathbb{R}^2 : \ \overset{a}{0} \leq x \leq \overset{b}{1}, \ \underset{\alpha(x)}{0} \leq y \leq \underset{\beta(x)}{x} \}$

$$= \int_0^1 dx \int_0^x xy\,dy = \int_0^1 x \left(\int_0^x y\,dy \right) dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{y=0}^{y=x} dx = \frac{1}{2} \int_0^1 x^3\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_{x=0}^{x=1} = \frac{1}{8}$$

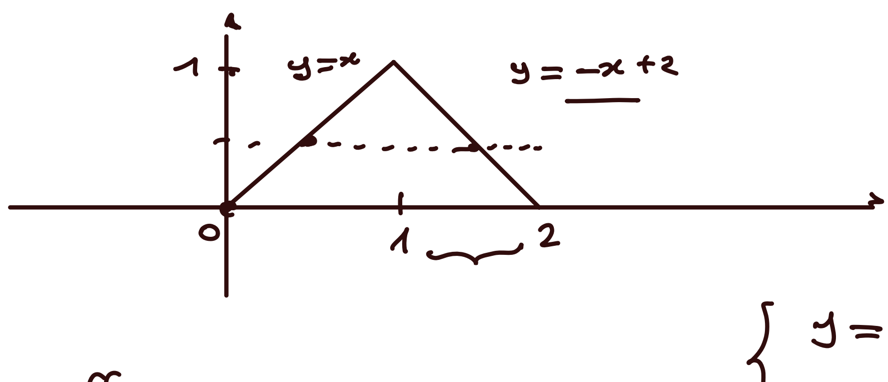
$$D: \quad \overset{c}{0} \leq y \leq \overset{d}{1}, \qquad \overset{\gamma(y)}{y} \leq x \leq \underset{\delta(y)}{1}$$

$$\iint_D xy\,dx\,dy = \int_0^1 dy \int_y^1 xy\,dx = \int_0^1 y\left(\int_y^1 x\,dx\right)dy$$

$$= \frac{1}{2}\int_0^1 y\left(x^2\right)_{x=y}^{x=1} dy = \frac{1}{2}\int_0^1 \left[y - y^3\right]dy$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{2} - \frac{1}{4}}_{\frac{1}{4}}\right] = \frac{1}{8}$$

② $\displaystyle\iint_D xy^2\,dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 : \tilde{y} \leq x,\ y \leq -x+2,\ y \geq 0\}$



$$\begin{cases} y = -x+2 \\ y = x \end{cases}$$

$$\iint_D \cdots = \iint_{D_1} \cdots + \iint_{D_2} \cdots$$

$$\begin{cases} x = 1 \\ y = 1 \end{cases}$$

$D_1: \quad 0 \leq x \leq 1, \qquad 0 \leq y \leq x$

$D_2: \quad 1 \leq x \leq 2, \qquad 0 \leq y \leq -x+2$

$$y = -x+2$$
$$x = 2 - y$$

NORMALE RISPETTO AD y: $\quad 0 \le y \le 1$

$$y \le x \le 2-y$$

2ª FORMULA

$$\iint_D x y^2 \,dx\,dy = \qquad = \int_0^1 dy \int_y^{2-y} x y^2 \,dx =$$

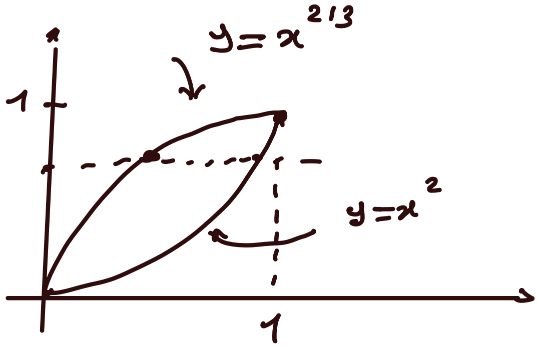
$$= \int_0^1 y^2 \left(\int_y^{2-y} x \,dx \right) dy =$$

$$= \frac{1}{2} \int_0^1 y^2 \left(x^2 \right)_{x=y}^{x=2-y} dy =$$

$$= \frac{1}{2} \int_0^1 y^2 \left[(2-y)^2 - y^2 \right] dy = 2 \int_0^1 y^2 (1-y)\,dy$$

$$4 - 4y + \cancel{y^2} - \cancel{y^2}$$

③ $\displaystyle \iint_D x e^{y^2} \,dx\,dy \qquad D = \{ (x,y) : 0 \le x \le 1, \; x^2 \le y \le x^{2/3} \}$



$$\begin{cases} 0 \le y \le 1 \\ y\sqrt{y} \le x \le \sqrt{y} \end{cases}$$

$$= \int_0^1 dx \int_{x^2}^{x^{2/3}} x e^{y^2} \,dy$$

$$\iint_D x e^{y^2}\,dx\,dy = \int_0^1 dy \int_{y\sqrt{y}}^{\sqrt{y}} x e^{y^2}\,dx =$$

$$= \int_0^1 e^{y^2}\left(\int_{y\sqrt{y}}^{\sqrt{y}} x\,dx\right)dy =$$

$$= \frac{1}{2}\int_0^1 e^{y^2}\left(x^2\right)_{x=y\sqrt{y}}^{x=\sqrt{y}}dy = \frac{1}{2}\int_0^1 e^{y^2}\left(y - y^3\right)dy$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{2}\int_0^1 e^{y^2}(2y)\,dy}_{\frac{1}{2}\left(e^{y^2}\right)_{y=0}^{y=1}} - \int_0^1 e^{y^2}\cdot y^3\,dy\right]$$

$f = y^2 \qquad g' = 2y e^{y^2}$

$f' = 2y \qquad g = e^{y^2}$

$$\int y^3 e^{y^2}dy = \frac{1}{2}\int y^2\left(2y\,e^{y^2}\right)dy$$

$$= \text{PARTI} = \frac{1}{2}\left[y^2 e^{y^2} - 2\int y e^{y^2}dy\right] =$$

$$= \frac{1}{2}\left[y^2 e^{y^2} - e^{y^2}\right] = \frac{1}{2}e^{y^2}\left(y^2-1\right)$$

Esercizi

(1) $\iint_D (x^2+y^2)\,dx\,dy$

$D = \{(x,y) : 1 \le x \le 3,\ 1 \le y \le 2\}$

(2) $\iint_D (2x+3y)\,dx\,dy$

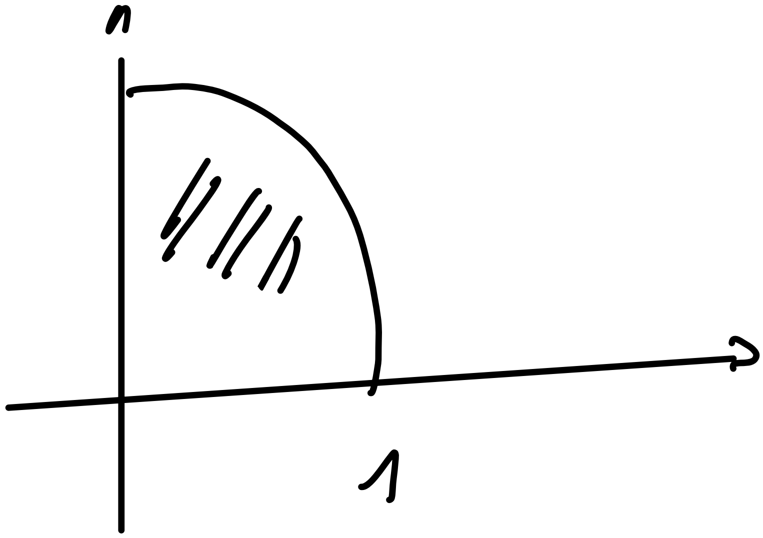
D = triangolo di vertici $(0,0)$, $(1,0)$, $(0,1)$

(3) $\iint_D xy\,dx\,dy$

D dominio delimitato da

$y = x^2$ e da $y = x$

(4) $\iint_D \sqrt{1-x^2}\, dx\, dy$



(5) $\iint_D xy \, dx\, dy$

D delimitato dalle parabole

$$\begin{cases} y = 2x^2 - 4x \quad e \\ y = 2x - x^2 \end{cases}$$