ESERCITAZIONI DEL 09/11/2023

1) $\displaystyle\sum_{n=1}^{\infty} \frac{3n \, tg(1/2n)}{n^2+1}$

$\displaystyle\frac{tg x}{x} \xrightarrow{x \to 0} 1$

$$\frac{tg(1/2n)}{1/2n} \xrightarrow{n \to \infty} 1$$

$$tg(1/2n) \sim \frac{1}{2n}$$

$$\frac{3n}{n^2+1} tg(1/2n) \sim \frac{3\cancel{n}}{n^2+1} \cdot \frac{1}{2\cancel{n}} = \frac{3}{2(n^2+1)}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{3}{2(n^2+1)}$? $\qquad \displaystyle\frac{3}{2(n^2+1)} \sim \frac{3}{2n^2} = \frac{3}{2} \cdot \frac{1}{n^2}$

$$\sum \frac{1}{n^2} < \infty$$

$\Rightarrow$ criterio asintotico $\Rightarrow$ la serie converge.

2) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{3n}{4n^2+5}\right)$ CONVERGENZA SEMPLICE E ASSOLUTA.

$$\frac{3n}{4n^2+5} \sim \frac{3}{4n}$$

(*) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{3n}{4n^2+5}\right)$ $\qquad \sin\frac{3n}{4n^2+5} \sim \frac{3n}{4n^2+5} \sim \frac{3}{4n} = \frac{3}{4}\cdot\frac{1}{n}$

$\Rightarrow$ criterio asintotico $\Rightarrow$ la serie (*) diverge.

CRITERIO DI LEIBNIZ : $\quad b_n = \underbrace{\sin}_{\text{decrescente?}}\left(\underbrace{\frac{3n}{4n^2+5}}_{\to 0}\right) \longrightarrow 0$

$a_n = \dfrac{3n}{4n^2+5}$ decrescente ?

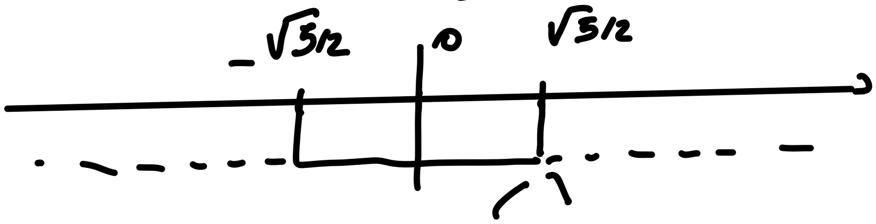
$f(x) = \dfrac{3x}{4x^2+5}$ $\qquad f(0)=0$, $f(x) \xrightarrow{x\to+\infty} 0$

$$f'(x) = \frac{3(4x^2+5) - 3x\cdot 8x}{(4x^2+5)^2}$$

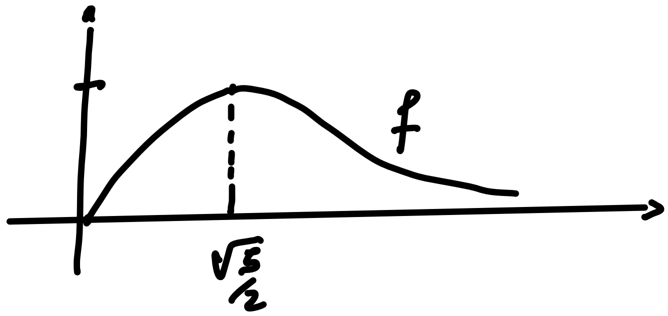
$$= \frac{12x^2+15-24x^2}{(4x^2+5)^2} = \frac{15-12x^2}{(4x^2+5)^2} \geq 0$$

$\Leftrightarrow \quad 5-4x^2 \geq 0 \Leftrightarrow \quad 4x^2-5 \leq 0$

$\Leftrightarrow \quad -\frac{\sqrt{5}}{2} \leq x \leq \frac{\sqrt{5}}{2}$



$$f\left(\frac{\sqrt{5}}{2}\right) = \frac{3\cdot \sqrt{5}/2}{\cancel{4}\cdot\frac{5}{\cancel{4}}+5} = \frac{3\sqrt{5}}{20}$$



CONCLUSIONE: • $b_n = \sin\left(\frac{3n}{4n^2+5}\right)$ è decrescente per $n \geq k \geq \frac{\sqrt{5}}{2}$

$$\sum_{n=1}^{\infty} \frac{n\, x^{n+3}}{3n^2+2n+4} = \sum_{n=1}^{\infty} \frac{n\cdot x^3\, x^n}{3n^2+2n+4} =$$

$$= x^3 \sum_{n=1}^{\infty} \underbrace{\frac{n}{3n^2+2n+4}}_{a_n} x^n$$

CR. RAPPORTO:

$$\frac{a_n}{a_{n+1}} = \frac{n}{3n^2+2n+4}\cdot\frac{3(n+1)^2+2(n+1)+4}{n+1} \longrightarrow 1 = \rho$$

La serie converge (assolutamente) in $(-1,1)$

ESTREMI: $x=1$ $\quad \sum_{n=1}^{\infty} \frac{n}{3n^2+2n+4} \sim \sum \frac{1}{3n} = +\infty$

$x=-1 \quad \sum_{n=1}^{\infty} (-1)^n \frac{n}{\underbrace{3n^2+7n+4}}$ CONVERGE PER LEIBNIZ

$b_n \to 0 \quad , \quad b_n \underset{\neq}{\geq} b_{n+1}$

CONCLUSIONE: la serie iniziale converge in $[-1, 1[$

$$\sum_{n=1}^{\infty} \frac{1}{n(e^x+2)^n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \Big(\overbrace{\frac{1}{e^x+2}}^{y}\Big)^n$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n} \qquad \rho = 1 \qquad \frac{n}{n+1} \xrightarrow{n\to\infty} 1$$

converge quando $-1 < y < 1 \iff \frac{1}{e^x+2} < 1$

$y = \frac{1}{e^x+2}$

SEMPRE!!

CONCLUSIONE: la serie converge $\forall x \in \mathbb{R}$ (assolutamente)

$$\sum \frac{1}{9^n} \left(\frac{6}{\pi} \arccos x\right)^{2n} \qquad x \in [-1,1]$$

$$y = \left(\frac{6}{\pi} \arccos x\right)^2$$

$$\sum_{n=1}^{\infty} \frac{1}{9^n} y^n$$

5

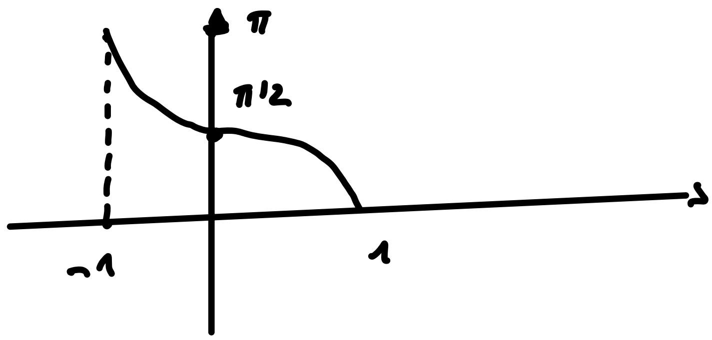
CR. RADICE : $\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{9^n}\right)} = \frac{1}{9}$: $\rho = 9$

PER LA SERIE IN x : $\left(\frac{6}{\pi}\arccos x\right)^2 \underset{\neq}{\leq} 9$

$$\iff \quad -3 < \frac{6}{\pi}\arccos x < 3$$

$$\iff \quad -1 < \frac{2}{\pi}\arccos x < 1$$

$$\iff \quad -\frac{\pi}{2} < \arccos x \underbrace{<}_{?} \frac{\pi}{2}$$



$$\iff \quad 0 \underset{\neq}{\leq} x \leq 1$$

$$\iff \quad 0 \underset{\neq}{<} x \leq 1$$

AGLI ESTREMI ? $y = 9 \iff \sum \frac{1}{\cancel{9^n}} \cdot \cancel{9^n} = +\infty$

CONCLUSIONE : CONVERGE IN]0,1] .

$$\sum_{n=1}^{\infty}\left(\sqrt{2n-\sin n} - \sqrt{2n+4}\right) = \sum_{n=1}^{\infty} \frac{\cancel{2n}-\sin n-\cancel{2n}-4}{\sqrt{2n-\sin n}+\sqrt{2n+4}}$$

$$= -\sum_{n=1}^{\infty} \frac{4+\sin n}{\sqrt{2n-\sin n}+\sqrt{2n+4}}$$

$$\frac{4+\sin n}{\sqrt{2n-\sin n}+\sqrt{2n+4}} > \frac{4-1}{\ldots} = \frac{3}{\sqrt{2n-\sin n}+\sqrt{2n+4}}$$

divege $\leftarrow$ ... UGUALE

$$\sim 2\sqrt{2n}$$

$$\sim \frac{3}{2\sqrt{2}\sqrt{n}} \qquad \sum \frac{1}{n^\alpha} = +\infty \qquad \sqrt{n\left(2-\frac{\sin n}{n}\right)} \sim \sqrt{2n}$$

$\alpha \leq 1$ $\qquad \frac{\sin n}{n} \to 0$

ES. MASSIMO E MINIMO ASSOLUTO DI

$$f(x,y) = x^2 - y^2 + x \qquad y^2 = \frac{1-x^2}{4}$$

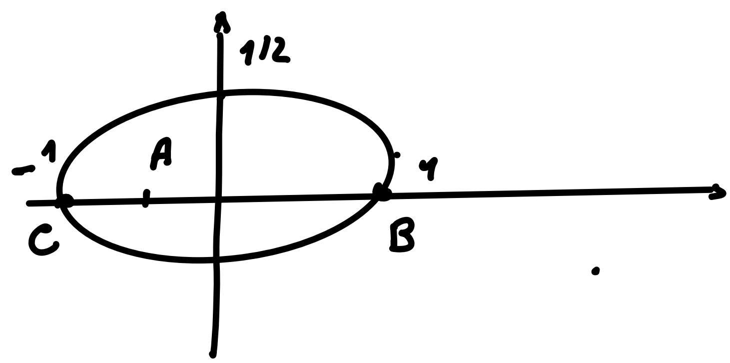
$$x^2 - \frac{1-x^2}{4} + x = g(x)$$

$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 1\}$$

$a = 1$

$$x^2 + \frac{y^2}{\frac{1}{4}} = 1 \qquad b = \frac{1}{2}$$

$$\begin{cases} x = \overset{a}{1}\cos t \\ y = \underset{b}{\frac{1}{2}} \sin t \end{cases} \qquad t \in [0, 2\pi]$$



1 PUNTI CRITICI

$$\nabla f = (2x+1, -2y) = (0,0)$$

$$\Leftrightarrow \begin{cases} x = -\frac{1}{2} \\ y = 0 \end{cases} \qquad A = \left(-\frac{1}{2}, 0\right)$$

PUNTO CRITICO!

$$f(A) = f\left(-\tfrac{1}{2}, 0\right) = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4}$$

$$g_y = 8y \qquad g_x = 2x$$

2 STUDIO SU ∂B

$$x^2 + 4y^2 = 1$$

$$g(x,y) = x^2 + 4y^2 - 1 = 0$$

$$\begin{cases} f_x - \lambda g_x = 0 \\ f_y - \lambda g_y = 0 \\ g(x,y) = 0 \end{cases} \iff \begin{cases} 2x + 1 - 2\lambda x = 0 \\ -2y - 8\lambda y = 0 \\ x^2 + 4y^2 = 1 \end{cases}$$

$$\iff \begin{cases} 2x - 2\lambda x + 1 = 0 \\ y + 4\lambda y = 0 \iff y(1 + 4\lambda) = 0 \\ x^2 + 4y^2 = 1 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ 2x - 2\lambda x + 1 = 0 \\ x^2 = 1 \iff x = \pm 1 \end{cases} \cup \begin{cases} 4\lambda + 1 = 0 \\ 2x - 2\lambda x + 1 = 0 \\ x^2 + 4y^2 = 1 \end{cases}$$

$$\begin{cases} x = -1 \\ y = 0 \\ -2 + 2\lambda + 1 = 0 \end{cases} \cup \begin{cases} x = 1 \\ y = 0 \\ \lambda = -\tfrac{1}{2} \end{cases} \cup \begin{cases} \lambda = -\tfrac{1}{4} \\ 2x + \tfrac{1}{2}x + 1 = 0 \\ x^2 + 4y^2 = 1 \end{cases}$$

$$\iff \lambda = \tfrac{1}{2}$$

$$\begin{cases} \lambda = -\tfrac{1}{4} \\ \tfrac{5}{2}x = -1 \\ x^2 + 4y^2 = 1 \end{cases}$$

$$B = (-1, 0), \qquad C = (1, 0)$$

$$\begin{cases} \lambda = -\frac{1}{4} \\ x = -\frac{2}{5} \\ \frac{4}{25} + 4y^2 = 1 \end{cases} \iff 4y^2 = 1 - \frac{4}{25} = \frac{21}{25}$$

$$y^2 = \frac{21}{100}$$

$$D = \left(-\frac{2}{5}, -\frac{\sqrt{21}}{10} \right) \qquad y_{1,2} = \pm \frac{\sqrt{21}}{10}$$

$$E = \left(-\frac{2}{5}, \frac{\sqrt{21}}{10} \right)$$

SOL. $\quad f(A) = -\frac{1}{4} \quad , \quad f(\overbrace{-1,0}^{B}) = 0$

$f(B) \overset{?}{=} f(-1,0) = 1 - 1 = 0 \quad \ldots\ldots$

CONC. $\quad A$ = p.to di minimo assoluto

B = " " massimo assoluto

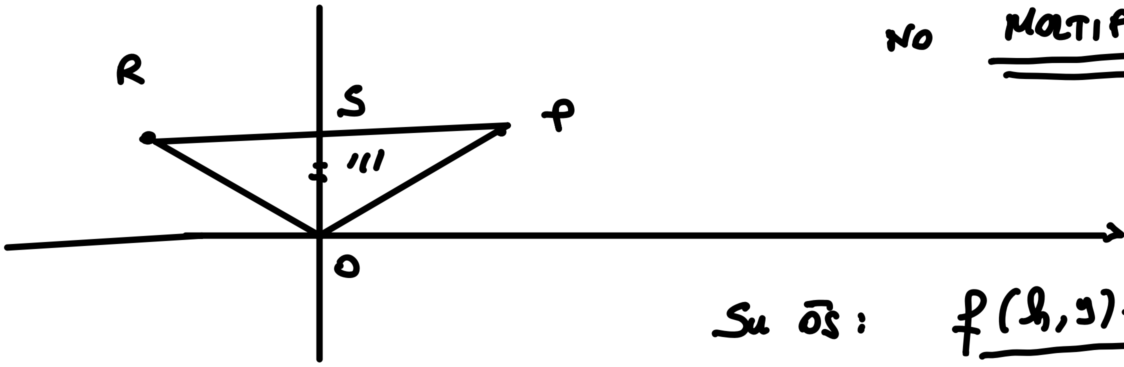
— $f(x,y) = (x - 2y)|x| + 2 \qquad T$ di vertici

$O = (0,0)$

$f(0,0) = 2$

$P = (1,1)$

$R = (-1,1)$



NO <u>MOLTIPLICATORI</u>

Su $\overline{OS}$: $\frac{f(h,y)-f(0,y)}{h} = \frac{(h-2y)|h| + \cancel{2} - \cancel{2}}{h}$

Su $x>0$ $\quad f = (x-2y)x + 2$

$$\begin{cases} f_x = x + x - 2y = 2x - 2y = 0 \\ f_y = -2x = 0 \quad \text{MAI!!} \end{cases}$$

$\lim_{h\to 0^-} \frac{f(h,y)-f(0,y)}{h} = 2y \qquad \lim_{h\to 0^+} \frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0^+} \frac{(h-2y)\cancel{h}}{\cancel{h}} = -2y$

f non è derivabile nei punti $(0,y)$, $y>0$. Invece $\boxed{f_x(0,0) = f_y(0,0) = 0}$

$\frac{f(h,0)-f(0,0)}{h} = \frac{\cancel{h}|h|}{\cancel{h}} \to 0$

$x<0$: $\quad f = -x(x-2y) + 2$

$f_x = -x + 2y - x = -2x + 2y = 0$

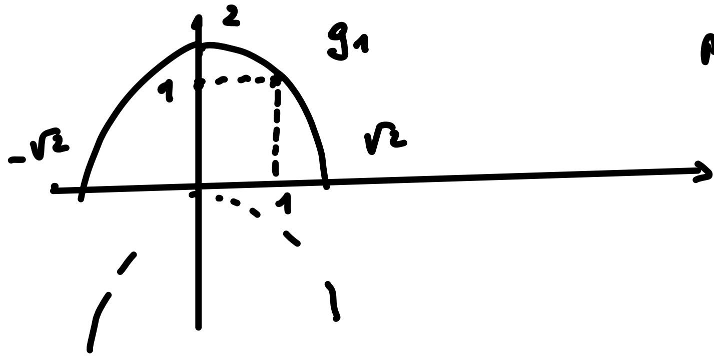
$f_y = 2x = 0 \quad$ MAI!!

CONC. NESSUN PUNTO CRITICO PER $x \neq 0$. L'unico punto critico è $(0,0)$

COSA SUCCEDE PER $x=0$: $\quad f(0,y) = 2$ nei punti $\overline{OS}$, NON DERIVABILE per $y>0$; mentre $f_x(0,0) = f_y(0,0) = 0$

Su $\overline{OP}$: $\quad y = x$, $x \in [0,1]$

$g_1(x) = f(x,x) = -x|x| + 2$, $\quad x \in [0,1]$
$= -x^2 + 2$



$\min g_1 = 1$

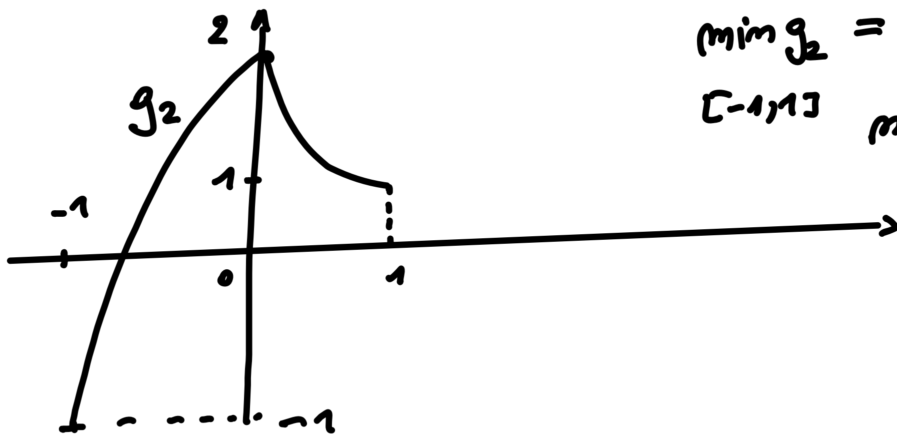
$\max g_1 = 2$

Su $\overline{PR}$: $y = 1$, $x \in [-1, 1]$

$$g_2(x) = f(x,1) = (x-2)|x| + 2 \quad , x \in [-1,1]$$

$$= \begin{cases} (x-2)x + 2 & 0 \leq x \leq 1 \\ -(x-2)x + 2 & -1 \leq x \leq 0 \end{cases}$$

$$= \begin{cases} x^2 - 2x + 2, & 0 \leq x \leq 1 \\ -x^2 + 2x + 2 & -1 \leq x \leq 0 \end{cases}$$



$\min\limits_{[-1,1]} g_2 = -1$

$\max\limits_{[-1,1]} g_2 = 2$

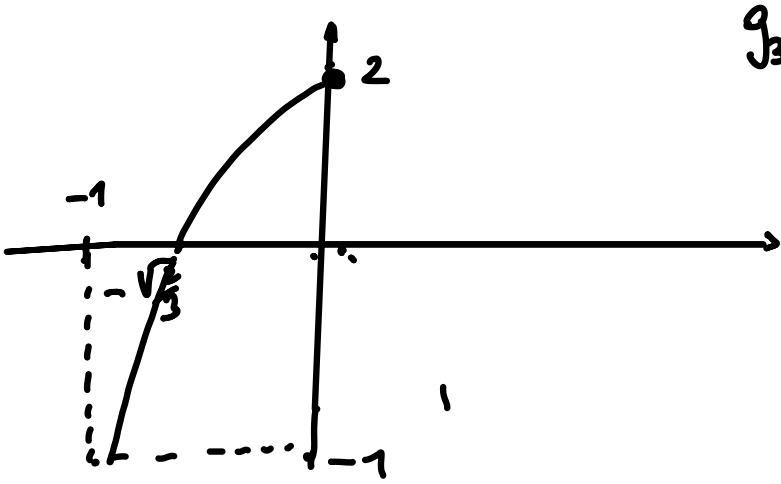
Su $\overline{OR}$: $y = -x$, $x \in [-1, 0]$ → $x \leq 0$

$$g_3(x) = (x + 2x)|x| + 2$$

$$= 3x|x| + 2$$

$= -3x^2+2$

$g_3 = 0 \iff x = \pm\sqrt{\frac{2}{3}}$



$\begin{cases} \min g_3 = -1 \\ \max g_3 = 2 \end{cases}$

$g_3(-1) = -3+2 = -1$

$g_3(0) = 2$

<u>CONCLUSIONE</u>: I punti di massimo sono quelli di $\overline{OS}$, il minimo è $-\underline{1}$.

<u>ES.</u> CLASSIFICARE I PUNTI CRITICI (FARE L'HESSIANO!!)

$f(x,y) = \log x^2 - \frac{x^2}{2} + y - \log y$

R. $(-\sqrt{2},1)$, $(\sqrt{2},1)$ P.TI DI SELLA

$f(x,y) = (-1+y^2)\exp(x^2)$ $\qquad$ $(0,0)$ DI SELLA

e^{x^2}

<u>ASSOLUTI</u> $\quad f(x,y) = 2x^3 + 2y^3 - 6xy$

$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, \ -1 \leq y \leq 2\}$