SIS Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- The 2D DFT.
- The 2D Fourier Transform.
- Some properties of 2D FT.

2D Discrete Fourier Transform (2D DFT and IDFT)

Let **f** be a matrix $(M\times N)$ $\mathbf{f}=(\mathbf{f}_{h,k})\in\mathbb{C}^{M\times N}$: by definition, the **2D Discrete Fourier Transform of f** is the matrix $\mathbf{F}\in\mathbb{C}^{M\times N}$ whose elements are:

$$\mathbf{F}_{u,v}=\sum_{h=0}^{M-1}\sum_{k=0}^{N-1}\mathbf{f}_{h,k}\,e^{-2\pi i\left(\frac{uh}{M}+\frac{vk}{N}\right)},\quad u=0,1,\dots,M-1,\ v=0,1,\dots,N-1$$

and, conversely, the **2D Inverse Discrete Fourier Transform of F** is the matrix **f** whose elements are:

$$\mathbf{f}_{h,k}=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\mathbf{F}_{u,v}\,e^{+2\pi i\left(\frac{uh}{M}+\frac{vk}{N}\right)},\quad h=0,1,\dots,M-1,\ \ k=0,1,\dots,N-1$$

in *MATLAB* `fft2()`, `ifft2()` and `fftshift()`

2D Discrete Fourier Transform (2D DFT)

$$\mathbf{F}_{u,v} = \sum_{h=0}^{M-1}\sum_{k=0}^{N-1}\mathbf{f}_{h,k}\, e^{-2\pi i\left(\frac{uh}{M}+\frac{vk}{N}\right)}, \quad u=0,1,\ldots,M-1,\ v=0,1,\ldots,N-1$$

split

$$\mathbf{F}_{u,v} = \sum_{h=0}^{M-1}\left[\sum_{k=0}^{N-1}\mathbf{f}_{h,k}\, e^{-\frac{2\pi i}{N}vk}\right] e^{-\frac{2\pi i}{M}uh}, \quad u=0,1,\ldots,M-1,\ v=0,1,\ldots,N-1$$

Φ_v

for each h, it is a **DFT** ...

A **2D DFT** can be computed by means of **1D DFTs**, each successively computed along a dimension.

A **2D DFT** (**fft2()** in MATLAB) can be computed by means of **1D DFTs**, each of them along a dimension.

Example: M=2, N=3

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

1. 3 DFTs(M=2) along columns
2. 2 DFTs(N=3) along rows

```
f=[1 2 3;4 5 6]; disp( fft( fft(f) .') .')
   21.0000              -3.0000 + 1.7321i     -3.0000 - 1.7321i
   -9.0000               0                     0
```

1. 2 DFTs(N=3) along rows
2. 3 DFTs(M=2) along cols

```
f=[1 2 3;4 5 6]; disp( fft( fft(f.') .'))
   21.0000              -3.0000 + 1.7321i     -3.0000 - 1.7321i
   -9.0000               0                     0
```

```
f=[1 2 3;4 5 6];  disp(fft2(f))
   21.0000              -3.0000 + 1.7321i     -3.0000 - 1.7321i
   -9.0000               0                     0
```

Exercise

From the scalar form of a **2D DFT** explain why it can also be obtained by the following:

```
f=[1 2 3;4 5 6];  disp(fft2(f))
  21.0000 + 0.0000i  -3.0000 + 1.7321i  -3.0000 - 1.7321i
  -9.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.0000i
[M,N]=size(f);
wM=exp(-2i*pi/M); k=0:M-1; WM=wM.^(k'*k);
wN=exp(-2i*pi/N);  k=0:N-1; WN=wN.^(k'*k);
WM * f * WN
ans =
  21.0000 + 0.0000i  -3.0000 + 1.7321i  -3.0000 - 1.7321i
  -9.0000 - 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.0000i
```

matrix form of a **2D DFT**

$$\Omega_M = \left(\omega_M^{kj}\right)_{k,j=0,1,\ldots,M-1}, \quad \omega_M = e^{-i\frac{2\pi}{M}}$$

$$\Omega_N = \left(\omega_N^{kj}\right)_{k,j=0,1,\ldots,N-1}, \quad \omega_N = e^{-i\frac{2\pi}{N}}$$

$$\Rightarrow \quad \Omega_M \cdot \mathbf{f} \cdot \Omega_N$$

2D Fourier Transform (2D FT)

When the integrals exist, the **2D Fourier Transform** (**2D FT**) is defined as

$$F(\omega_x,\omega_y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-i(x\omega_x+y\omega_y)}dxdy$$

ω_x, ω_y are angular frequencies

$$F(\nu_x,\nu_y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-2\pi i(x\nu_x+y\nu_y)}dxdy$$

ν_x, ν_y are angular frequencies

The **2D Inverse Fourier Transform** (**2D IFT**) is defined as

$$f(x,y)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega_x,\omega_y)e^{+i(x\omega_x+y\omega_y)}d\omega_x d\omega_y$$

ω_x, ω_y are angular frequencies

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\nu_x,\nu_y)e^{+2\pi i(x\nu_x+y\nu_y)}d\nu_x d\nu_y$$

ν_x, ν_y are angular frequencies

Like a **2D DFT**, a **2D FT** can be seen as a combination of two **1D FT**s:

$F_x(\omega_x,y)=\int_{-\infty}^{+\infty} f(x,y)e^{-i\omega_x x}dx$ ⇓ $F(\omega_x,\omega_y)=\int_{-\infty}^{+\infty} F_x(\omega_x,y)e^{-i\omega_y y}dy$	$F_x(\nu_x,y)=\int_{-\infty}^{+\infty} f(x,y)e^{-2\pi i x\nu_x}dx$ ⇓ $F(\nu_x,\nu_y)=\int_{-\infty}^{+\infty} F_x(\nu_x,y)e^{-2\pi i y\nu_y}dy$
w.r.t. the angular frequency ω	w.r.t. the circular frequency ν

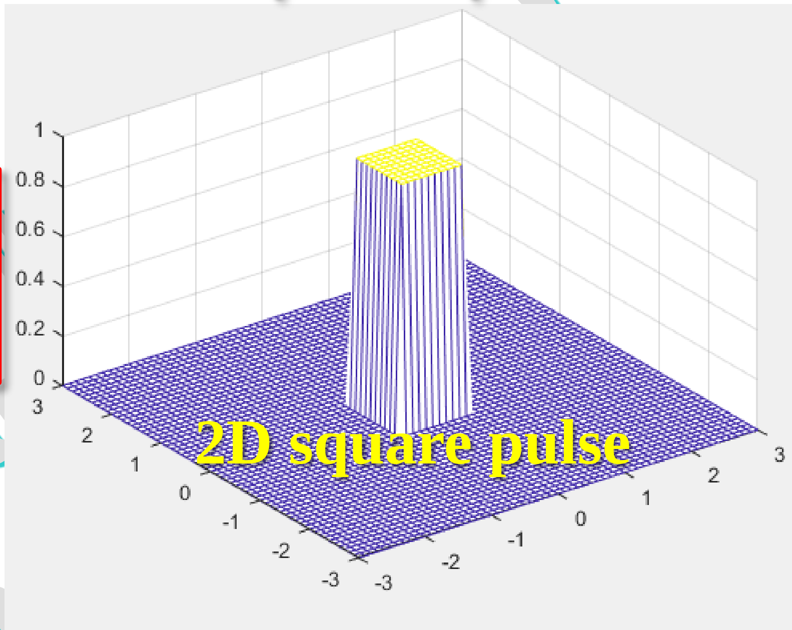
All the properties of 1D FT apply to 2D FT

Example of 2D FT: 2D square pulse

```
T=6; N=60;
[x,y]=meshgrid(linspace(-T/2,T/2,N+1));
f=zeros(size(x));
L=0.5;[h,k]=find(abs(x)<L & abs(y)<L);
f(h,k)=ones(size(f(h,k))); mesh(x,y,f)
```

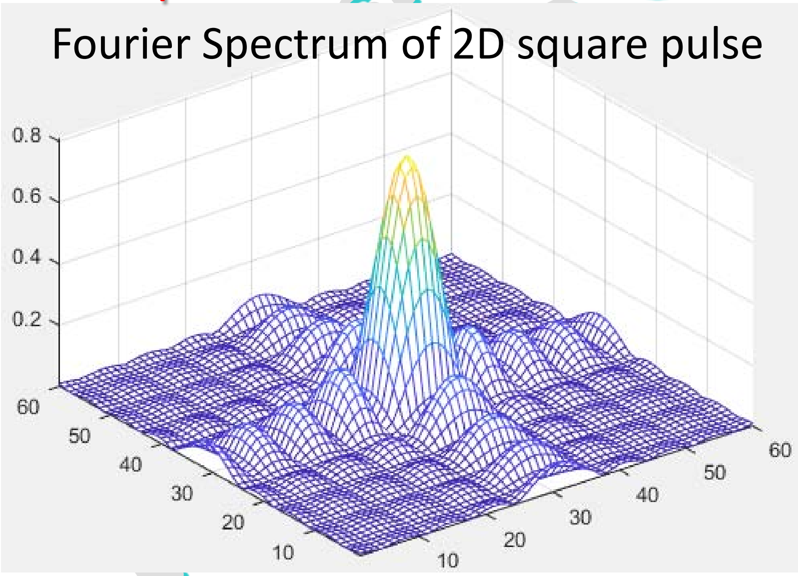
the same

```
f=rectpuls(x,2*L).*rectpuls(y,2*L);
mesh(x,y,f); axis tight
```



f is the product of two **rect pulse** functions

⇒ The **2D FT** of **f** is the convolution product of two **sinc** functions



Not very efficient code!

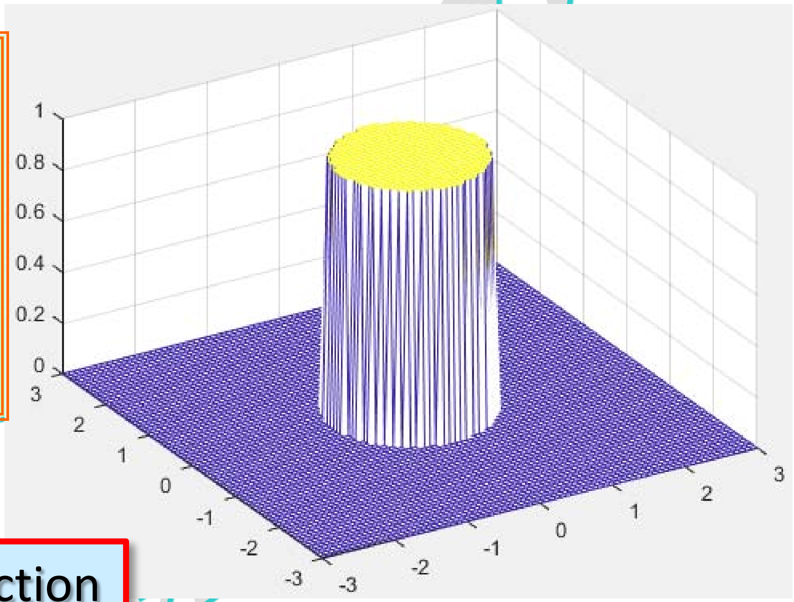
```
fn=f(1:end-1,1:end-1);
[h,k]=meshgrid(0:N-1, 0:N-1);
F=fftshift(fft2(fn)).*(-1).^(h+k)*(T/N)^2;
mesh(abs(F));
```

we can see two **sinc**s: each along an axis direction

Exercise: Do the same (in MATLAB) for a 2D rectangular pulse.

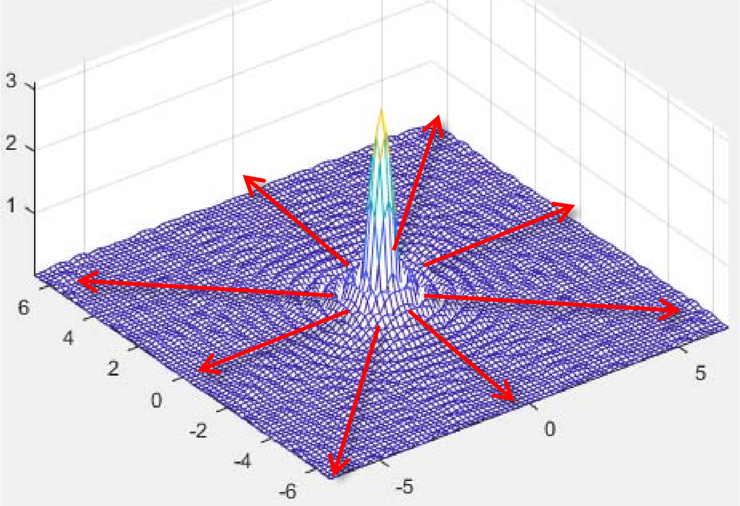
Example of 2D FT: 2D circ pulse

```
T=6; N=80;
[x,y]=meshgrid(linspace(-T/2,T/2,N)); z=x+i*y;
k=find(abs(z)<1);
f=zeros(size(z)); f(k)=ones(size(k));
mesh(x,y,f); axis tight
[h,k]=meshgrid(0:N-1, 0:N-1);
F=fftshift(fft2(f)).*(-1).^(h+k)*(T/N)^2;
surf(real(F)); axis tight
```



Exercise: write a better code for a 2D FT function

Fourier Spectrum of 2D circ pulse



we can see infinitely many **sinc** functions along any direction in the horizontal plane

Example of 2D FT: 2D gaussian $F(\omega_x,\omega_y) = e^{-\frac{\omega_x^2+\omega_y^2}{4\pi}}$

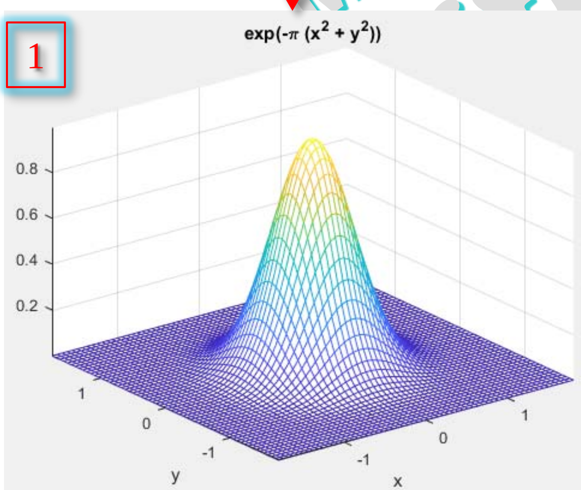
$$F_x(\omega_x, y) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-2\pi i x \omega_x}\, dx \;\Rightarrow\; F(\omega_x,\omega_y) = \int_{-\infty}^{+\infty} F_x(\omega_x, y)\, e^{-2\pi i y \omega_y}\, dy$$

↑ 1D FT gaussian ↑ 1D FT gaussian

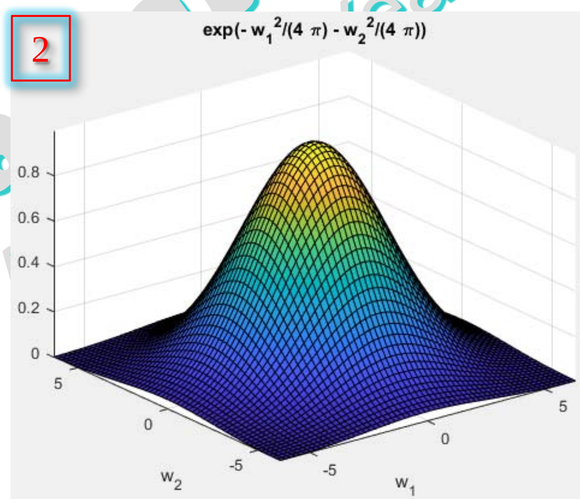
The **FT of a 2D gaussian** is still a gaussian, computed by composing the **FT**s of the two 1D gaussians.

symbolic

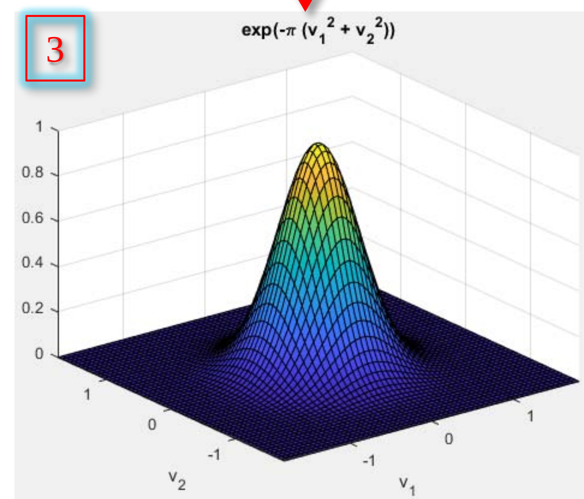
```
syms x y w1 w2 real; f=exp(-pi*(x^2+y^2)); ezmesh(f)  [1]
F1=fourier(f,x,w1);  F=fourier(F1,y,w2);   ezsurf(F)  [2]
syms v1 v2 real; Fv=subs(F,{w1,w2},{2*pi*v1,2*pi*v2}); Fv=simplify(Fv);
ezsurf(Fv)  [3]
```



f(x,y)

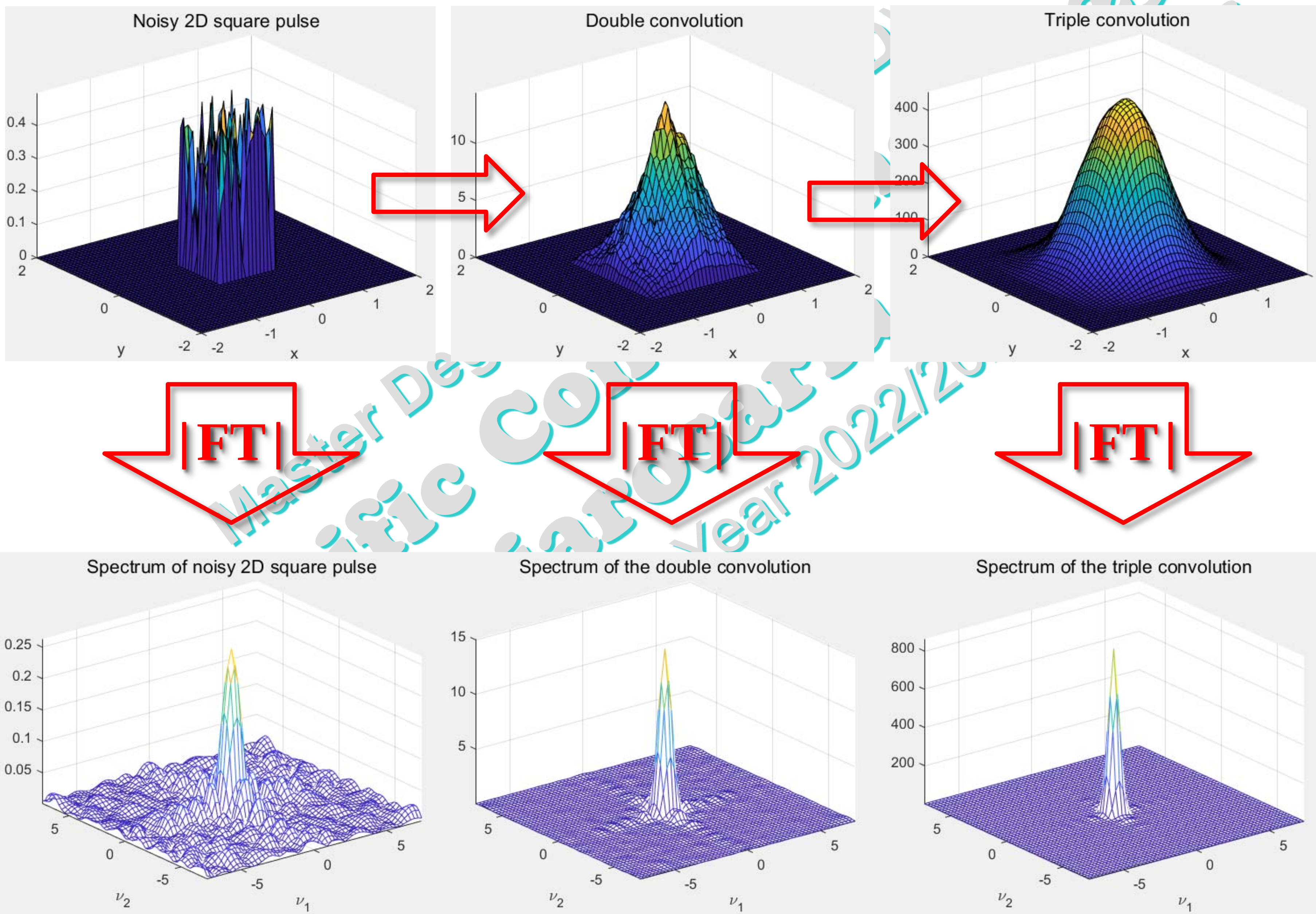


$F(\omega_1,\omega_2)$ w.r.t. angular frequencies



$F(\nu_1,\nu_2)$ w.r.t. circular frequencies

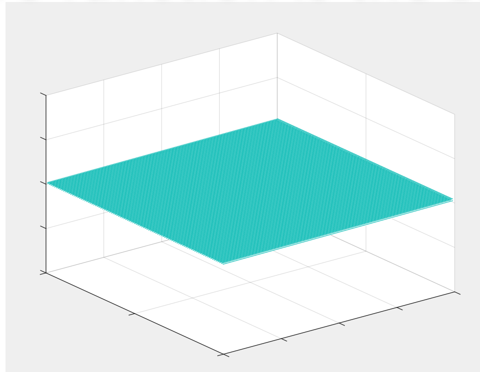
Esempio: 2D convolution (conv2( ))



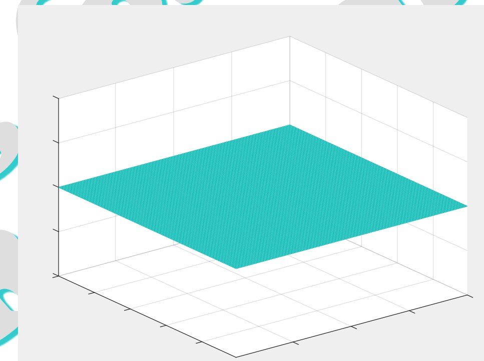
Examples of some 2D FT

The FT of the zero function is the zero function

```
Z=zeros(100,200);
FZ=fftshift(fft2(Z));
figure; mesh(Z)
title('zero matrix')
figure; mesh(abs(FZ))
title('FT of zeros')
```

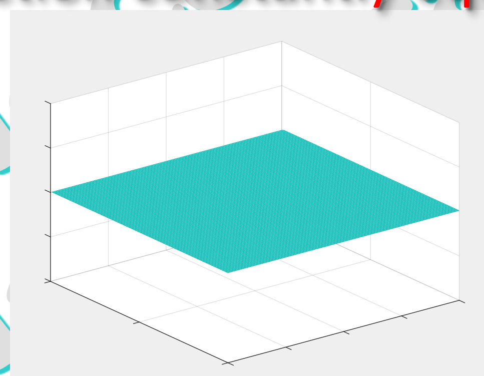


|FT|

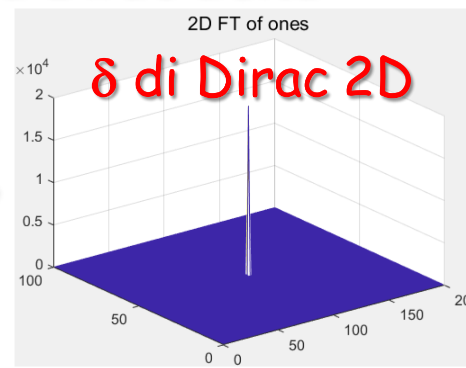


The FT of the function constantly equal to 1 is a Dirac delta

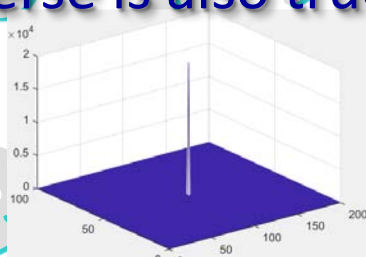
```
O=ones(100,200);
FO=fftshift(fft2(O));
figure; mesh(O)
title('matrix of ones')
figure; mesh(abs(FO))
title('FT of ones')
```



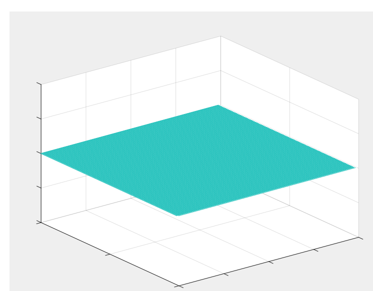
|FT|



the reverse is also true



|IFT|

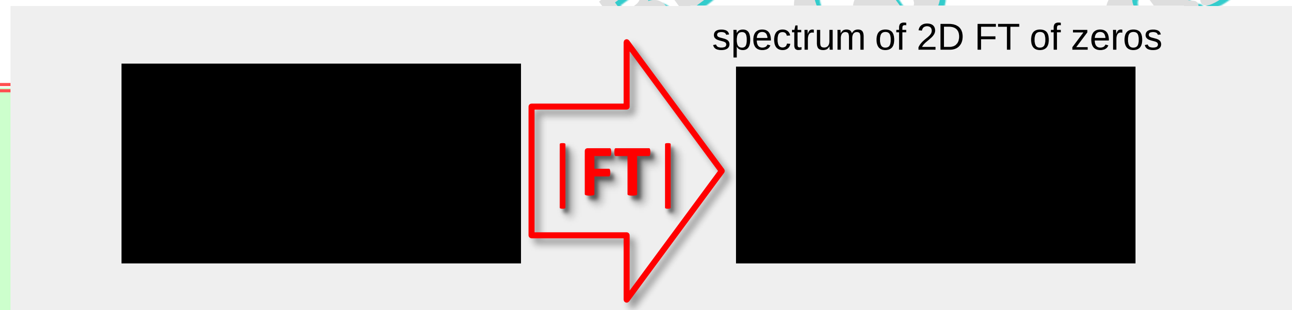


```
O=ones(100,200);
iFO=fftshift(ifft2(O));
figure; mesh(iFO)
title('matrix of ones')
figure; mesh(abs(iFO))
title('IFT of ones')
```

… from image point of view [RGB]

The **FT** of the zero function is the zero function

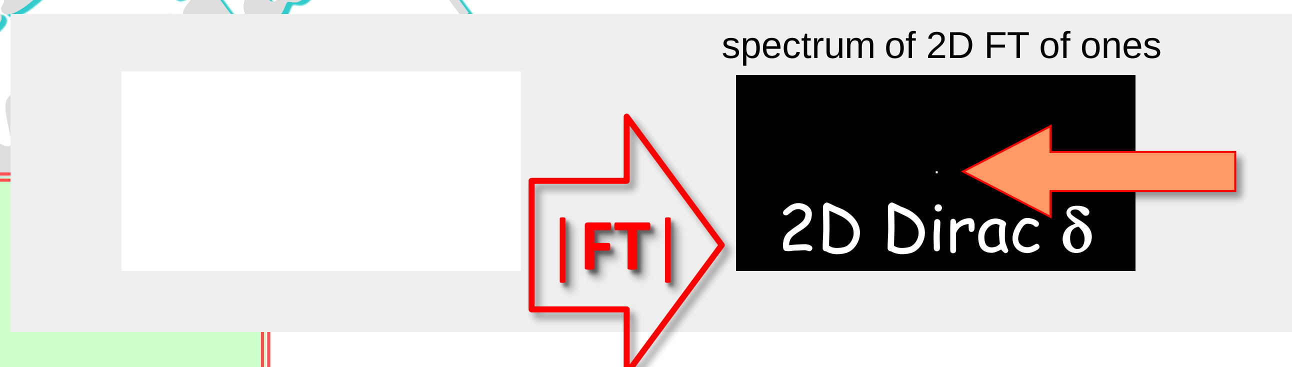
```
Z=zeros(100,200);
FZ=fftshift(fft2(Z));
figure; imshow(Z)
title('zero matrix')
figure; imshow(abs(FZ))
title('spectrum of 2D FT of zeros')
```



in MATLAB [RGB] 0 corresponds to black and 1 to white

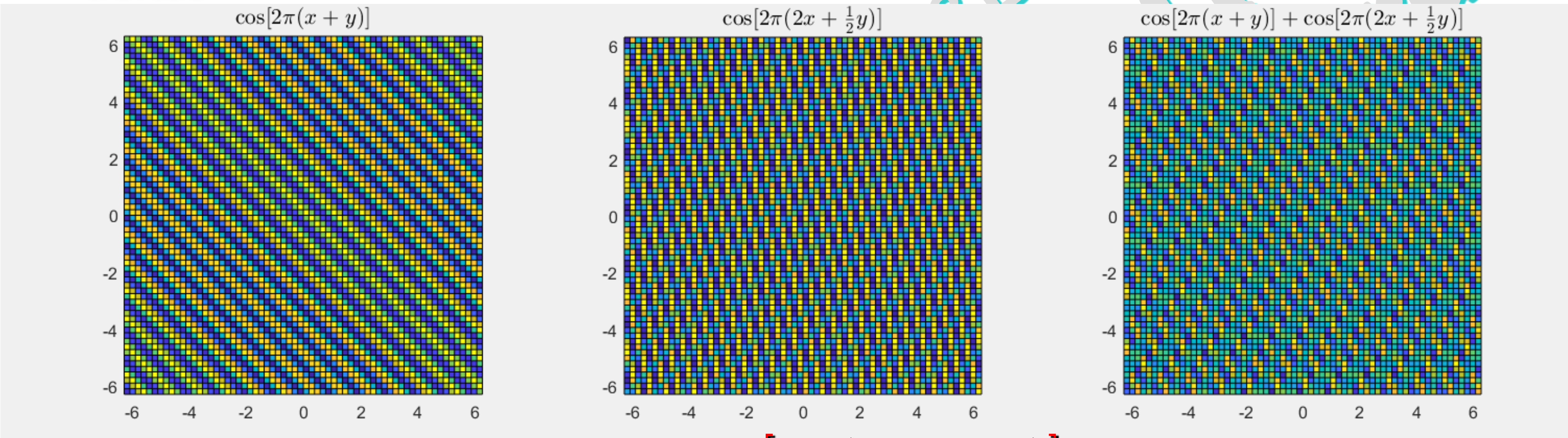
The **FT** of the function constantly equal to 1 is a Dirac delta

```
O=ones(100,200);
FO=fftshift(fft2(O));
figure; imshow(O)
title('matrix of ones')
figure; imshow(abs(FO))
title('spectrum of 2D FT of ones')
```



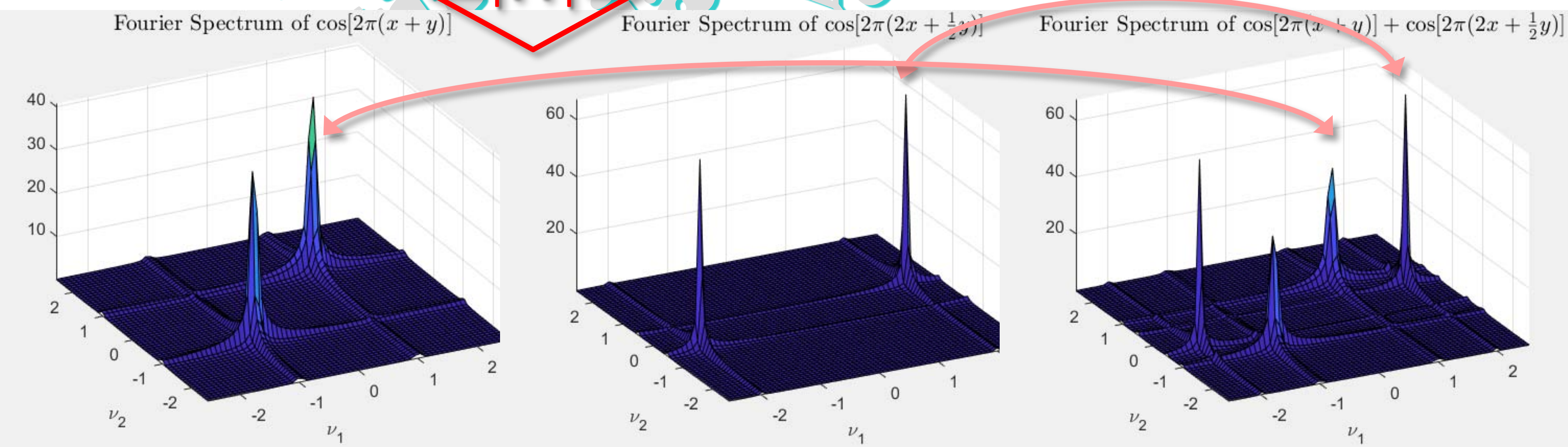
Examples of 2D FT properties

The **FT** of trigonometric functions consists of two symmetric Dirac deltas



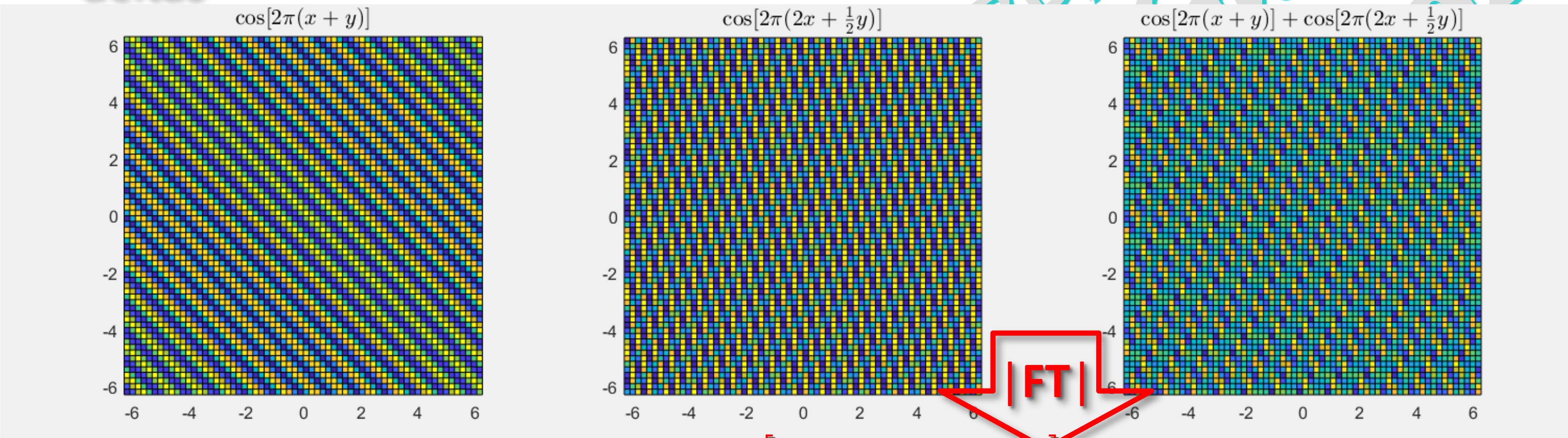
$\mathbf{a}=\cos[2\pi(x+y)]$ $\mathbf{b}=\cos\left[2\pi\left(2x+\frac{1}{2}y\right)\right]$ **a+b**

|FT|



Examples of 2D FT properties

The **FT** of trigonometric functions consists of two symmetric Dirac deltas



$\mathbf{a} = \cos[2\pi(x+y)]$ $\mathbf{b} = \cos\left[2\pi\left(2x+\frac{1}{2}y\right)\right]$ **a+b**

$\cos(2\pi\nu)$: periodic function with circular frequency ν