Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DIPARTIMENTO DI SCIENZE E TECNOLOGIE

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- Windowing and aliasing.
- The Nyquist-Shannon Sampling Theorem.

Windowing effect

When performing spectral analysis of a sample from a physical phenomenon $f(t)$, we are actually using an observation of the phenomenon obtained over a finite interval of time.

From a mathematical point of view, this process is equivalent to the multiplication of the function $f(t)$, which describes the phenomenon, by a rectangular function $w(t)$ (rectangular window function of width L); thus the “observed” function is:

$$g(t) = f(t) \cdot w(t)$$

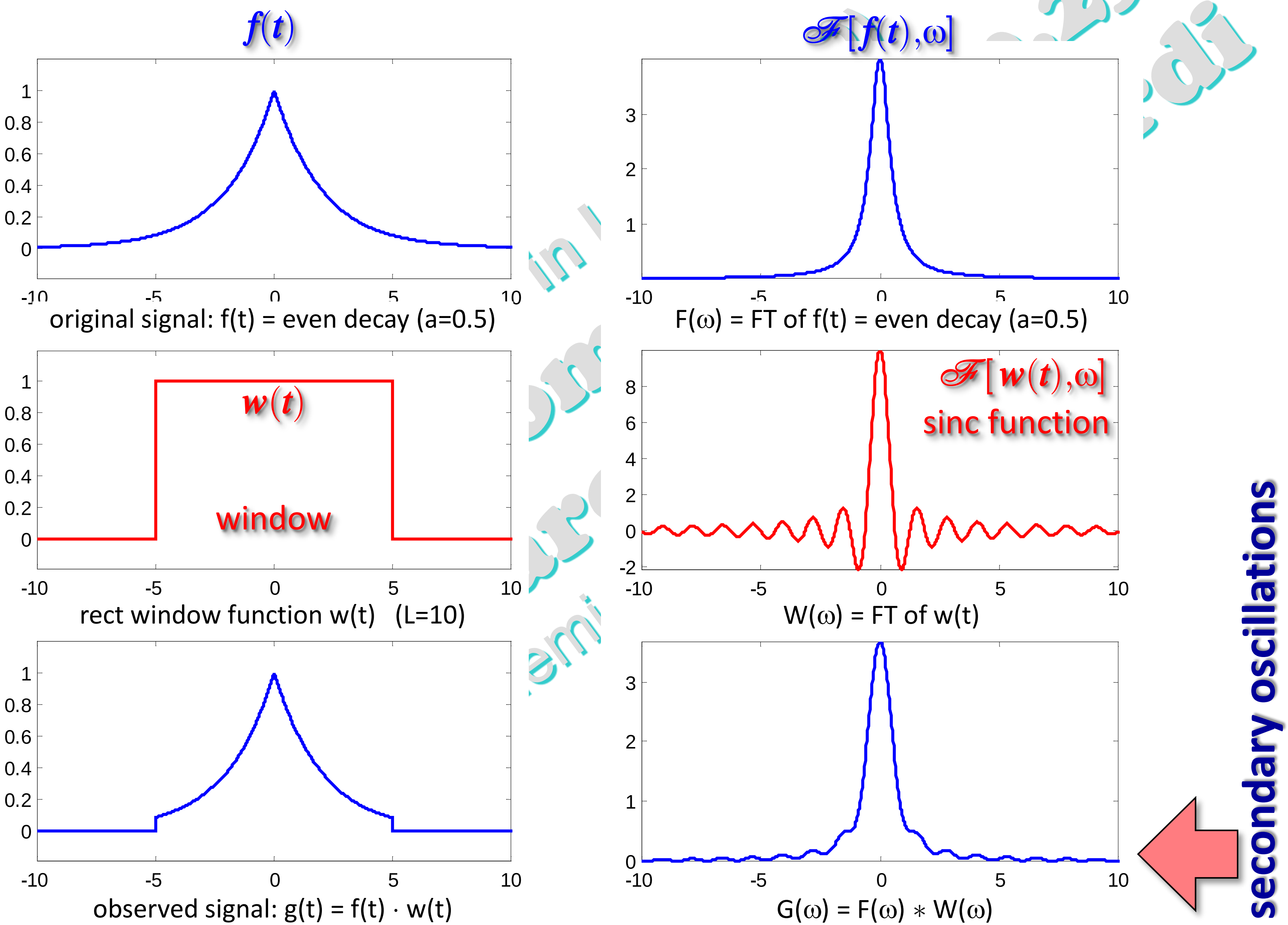
$$F(\nu) = \mathscr{F}[f(t),\nu]$$
$$W(\nu) = \mathscr{F}[w(t),\nu]$$

so that the ***Fourier Transform*** $G(\nu)$ of $g(t)$ is then given by:

$$G(\nu) = \mathscr{F}[g(t),\nu] = \mathscr{F}[f(t),\nu] * \mathscr{F}[w(t),\nu] = F(\nu) * W(\nu)$$

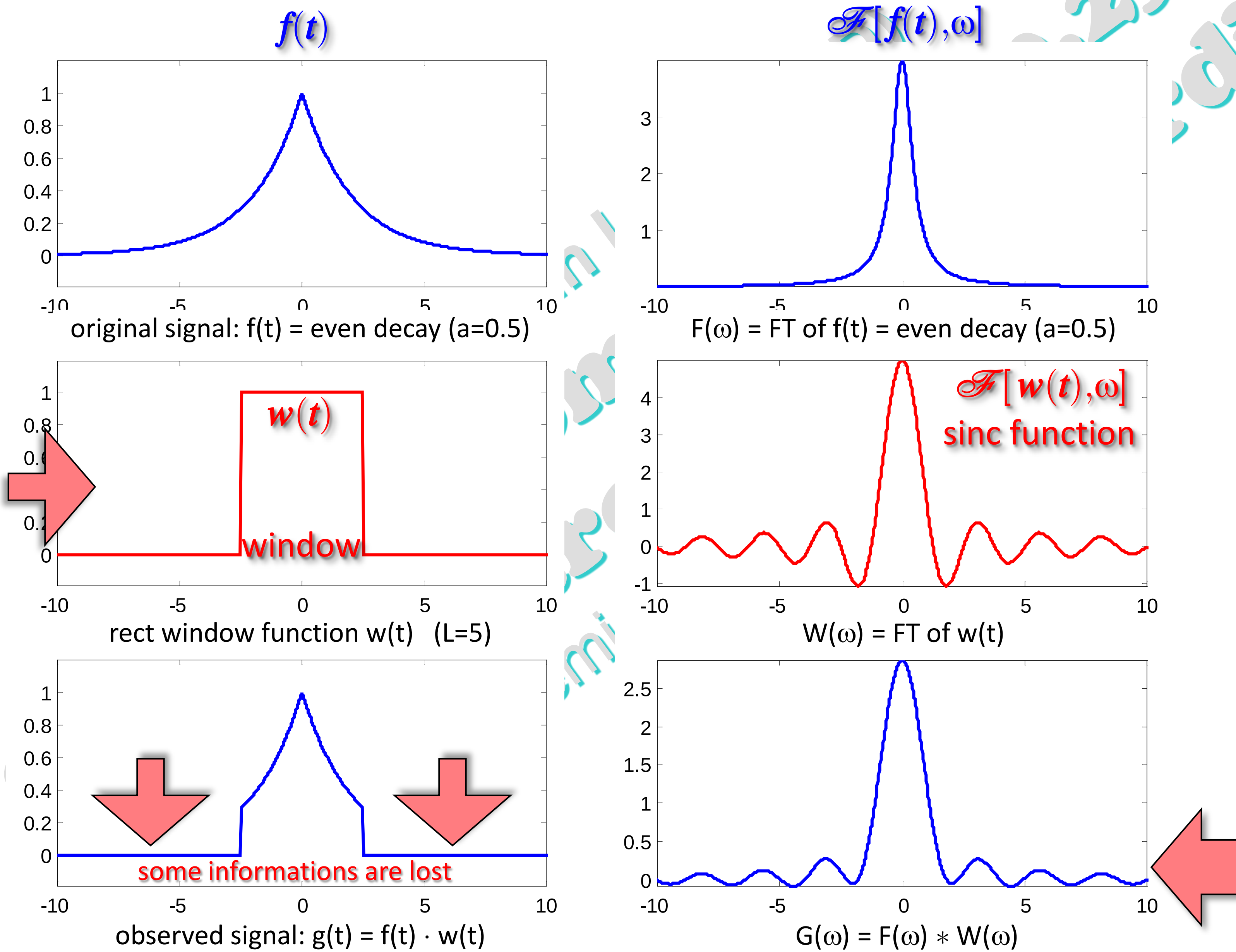
where $F(\nu) * W(\nu)$ represents the convolution product of F and W. Remember that W is a `sinc()`.

Windowing effect: example



(prof. M. Rizzardi)

Windowing effect: example (cont.)

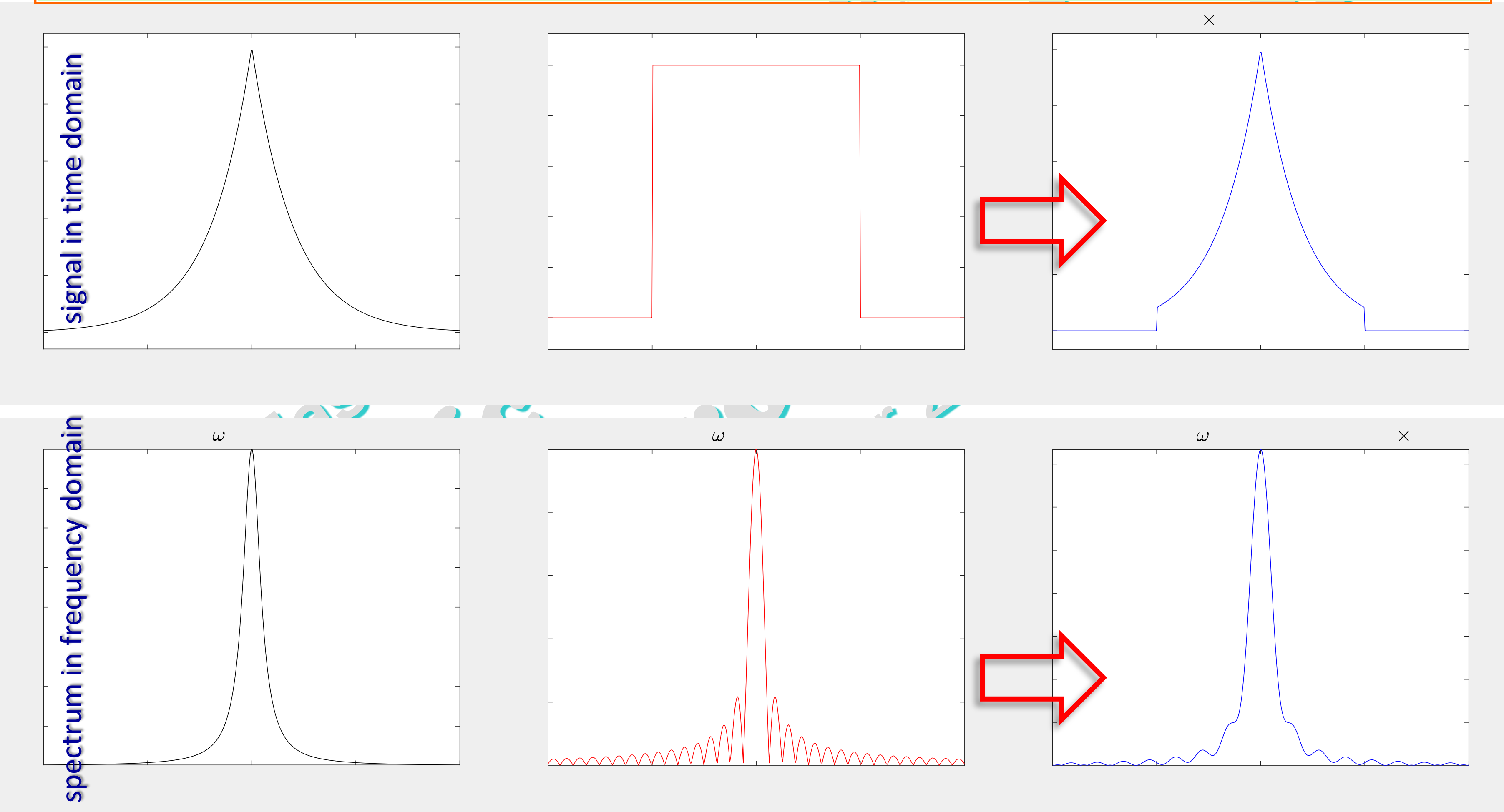


the window has been shrunk

secondary oscillations increase in amplitude

Example by means of Symbolic Math Toolbox

```
syms t real;  f=exp(-abs(t)/2);            figure; ezplot(f,[-10 10])
L=10;  w=rectangularPulse(-L/2,+L/2,t);  figure; ezplot(w,[-10 10])
figure; ezplot( f*w, [-10 10])
F=fourier(f);   figure; h=ezplot(abs(F),[-10 10]);  set(h,'Color','k')
W=fourier(w);   figure; h=ezplot(abs(W),[-10 10]); set(h,'Color','r')
G=fourier(f*w); figure; h=ezplot(abs(G),[-10 10]);  set(h,'Color','b')
```



Exercise: What happens for L=5?

Aliasing effect

When performing spectral analysis of a sample from a physical phenomenon $f(t)$, we are actually using a discrete set of observations of the phenomenon obtained by equispaced time series.

From a mathematical point of view, this process is equivalent to the multiplication of the function $f(t)$, which describes the phenomenon, by a Comb function $\delta_T(t)$ (of step T); thus the "observed" function is:

$$h(t) = f(t) \cdot \delta_T(t)$$

$$F(\nu) = \mathscr{F}[f(t),\nu]$$
$$\delta_{1/T}(\nu) = \mathscr{F}[\delta_T(t),\nu]$$

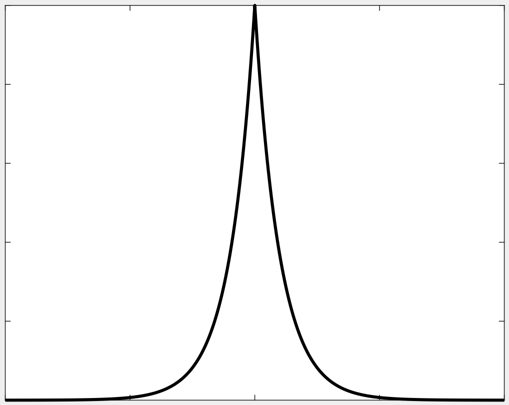
so that the ***Fourier Transform*** $H(\nu)$ of $h(t)$ is then given by:

$$H(\nu) = \mathscr{F}[h(t),\nu] = \mathscr{F}[f(t),\nu] * \mathscr{F}[\delta_T(t),\nu] = F(\nu) * \delta_{1/T}(\nu)/T$$

where $F(\nu) * \delta_{1/T}(\nu)/T$ represents the convolution product of F and $\delta_{1/T}/T$. Remember that the Fourier Transform of a comb function is still a comb function, but with a period equal to the reciprocal of the period of the original comb function.

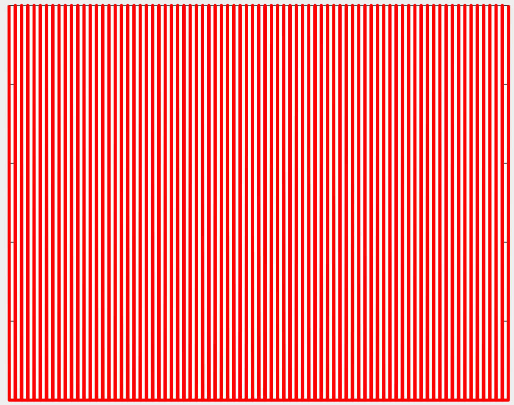
Aliasing effect: example

$f(t)$



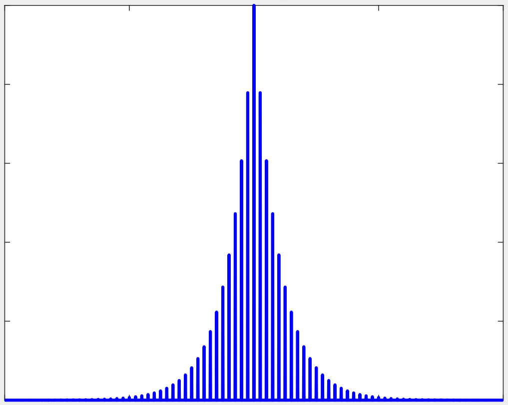
original signal: f(t)=even decay $e^{-|t|}$

$\delta_T(t)$



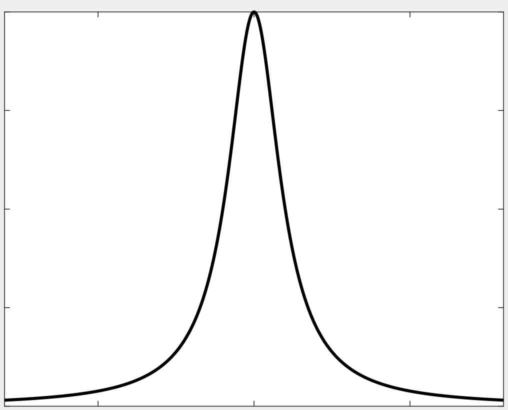
pulse train $\delta_T(t)$,   T = 0.25

$f(t)\cdot\delta_T(t)$



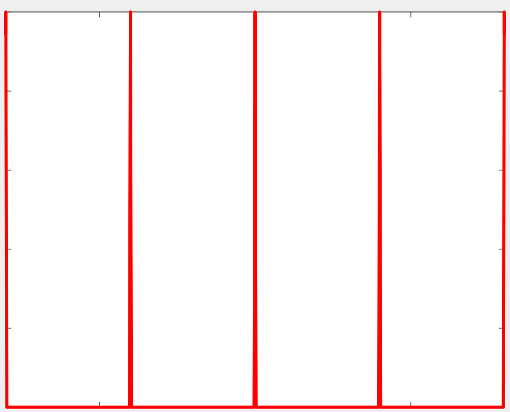
observed signal: $h(t) = f(t) \cdot \delta_T(t)$

$\mathscr{F}[f(t),\omega]$



$F(\omega) = 2/(1+\omega^2)$ = FT of $f(t) = e^{-|t|}$

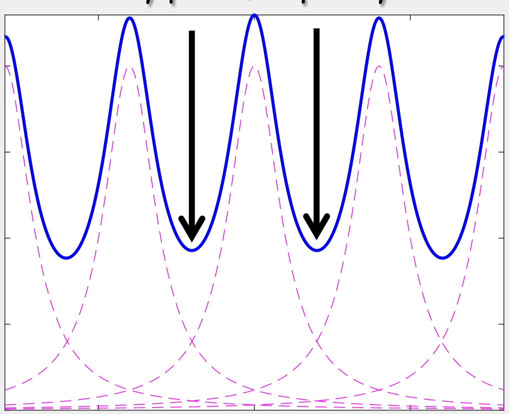
$\mathscr{F}[\delta_T(t),\omega]$



$\delta_P(\omega)$ = FT of $\delta_T(t)$, P = 1/T = 4

$\mathscr{F}[f(t),\omega] * \mathscr{F}[\delta_T(t),\omega]$

Nyquist frequency



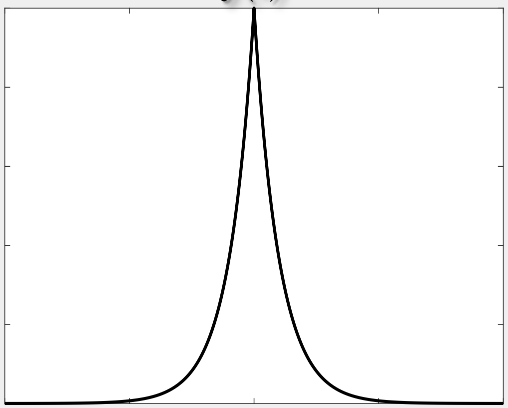
$H(\omega) = F(\omega) * \delta_{1/T}(\omega)$

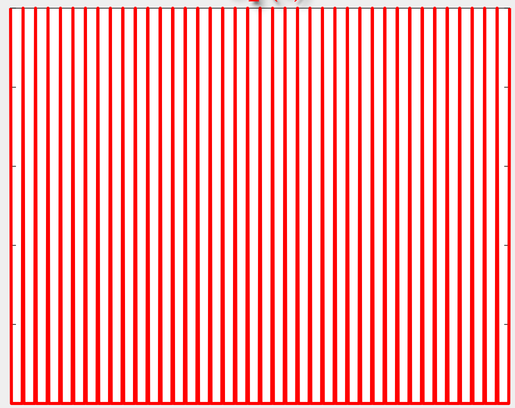
Superposition of $F(\omega)$

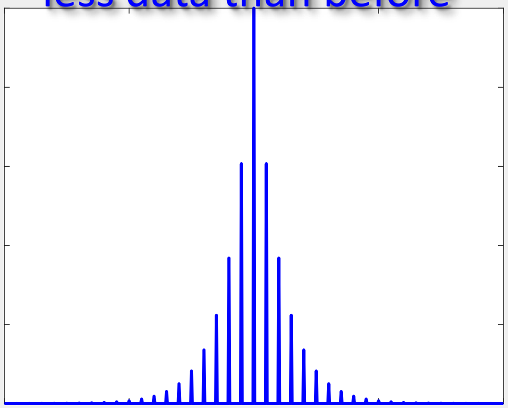
Aliasing effect: example (cont.)

Now, let's double the period T of the Comb Function δ_T

$f(t)$ | $\delta_T(t)$ | less data than before

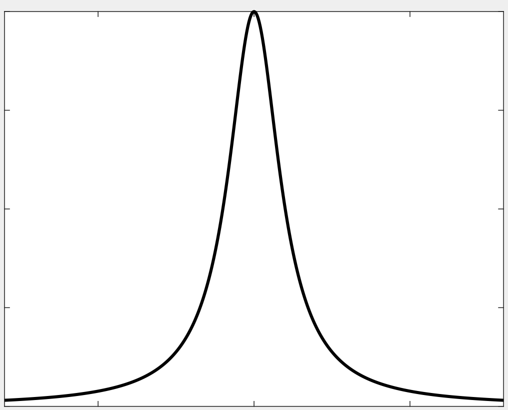


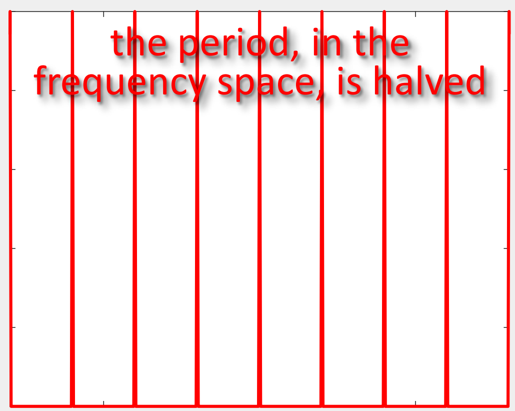




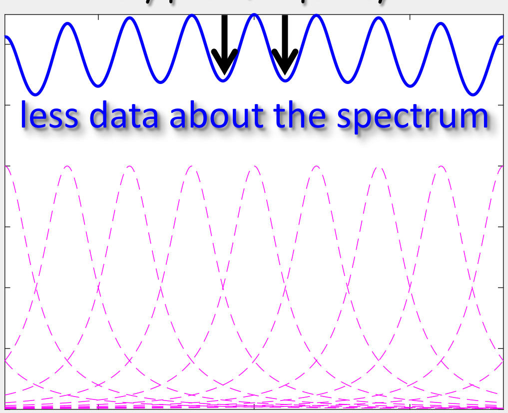
original signal: f(t)=even decay $e^{-|t|}$ | pulse train $\delta_T(t)$, T = 0.5 | observed signal: $h(t) = f(t) \cdot \delta_T(t)$

$\mathscr{F}[f(t),\omega]$ | $\mathscr{F}[\delta_T(t),\omega]$ | $\mathscr{F}[f(t),\omega] * \mathscr{F}[\delta_T(t),\omega]$





Nyquist frequency

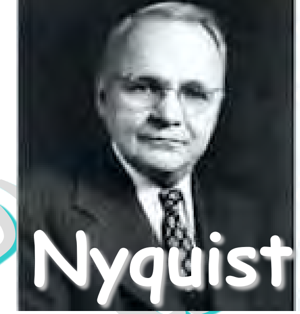


$F(\omega) = 2/(1+\omega^2)$ = FT of $f(t) = e^{-|t|}$ | $\delta_P(\omega)$ = FT of $\delta_T(t)$, P = 1/T = 2 | $H(\omega) = F(\omega) * \delta_{1/T}(\omega)$

Superposition of $F(\omega)$

The Nyquist-Shannon Sampling Theorem

stated by Harry Nyquist and proved by Claude Shannon in 1948



w.r.t. the circular frequency ν

If the ***FT*** of $f(t)$, $F(\nu)$ $[\omega=2\pi\nu]$, is such that

$$F(\nu)=0,\ |\nu|>H/2$$

(f is a **band limited function** of bandwidth H)

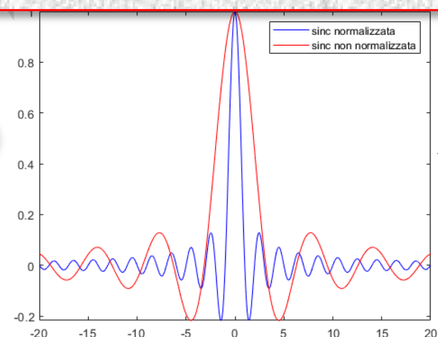
then the values of f at the sampling points $t_k=k\Delta t$, taken with frequency $f_s=1/\Delta t$, if $f_s>H$ (greater than twice the maximum frequency) allow f to be fully restored, i.e.:

$$f(t)=\sum_{k=-\infty}^{+\infty} f(k\Delta t)\ \mathrm{sinc}\left(\frac{t}{\Delta t}-k\right)$$

$$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

normalized sinc: $\sin(\pi x)/(\pi x)$

unnormalized sinc $\sin(x)/(x)$



sinc:
"sine cardinal" function

new expansion basis

The **Aliasing Error** is governed by the Sampling Theorem.

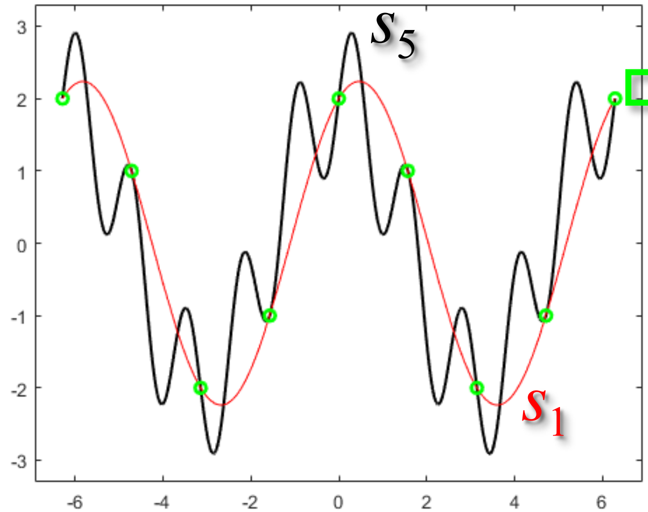
The **Sampling Theorem** can be thought of as the conversion of an analog signal into a discrete form by taking the sampling frequency at least as twice the maximum frequency of the input analog signal.

This **Theorem** is the basis of digital recordings able to reach a degree of fidelity much higher than analog ones.

Example: application of Sampling Theorem

Let us consider **two signals**: $s_5 = 2\cos(t) + \sin(5t)$ and $s_1 = 2\cos(t) + \sin(t)$.

We want to reconstruct s_5 starting from a sequence of its equispaced samples.
We sample s_5 with a step $h = \pi/2$ in the interval $[-2\pi, +2\pi]$ getting 9 samples.



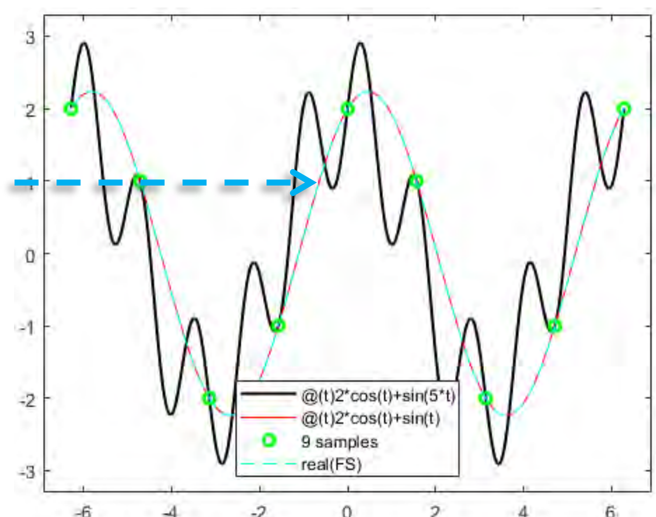
But s_5 and s_1 share these samples and we are not able to distinguish the two signals starting from these 9 samples, since the sampling rate **does not satisfy** the Sampling Theorem.

max freq in s_5: $\omega_{max} = 5$ ⇒ $\nu = \omega/(2\pi)$

min useful freq for s_5: $f_s > 2\nu_{max} = 5/\pi$ ⇒

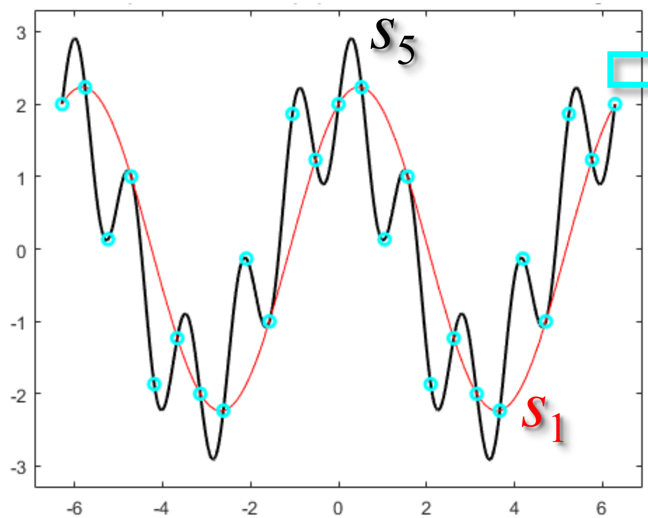
According to the Sampling Theor. we should use a step: $\Delta t < \pi/5$ but used step = $\pi/2 > \pi/5$

In facts, if we reconstruct a signal by means of the Fourier Series starting from the 9 samples (whose frequency is $f_s = 2/\pi < 5/\pi = 2\nu_{max}$), we get s_1 and not s_5 (aliasing).



Example: application of Sampling Theorem (cont.)

Now we sample s_5 with a step $h=\pi/6<\pi/5$, that satisfies the Sampling Theor.: we get 25 samples.



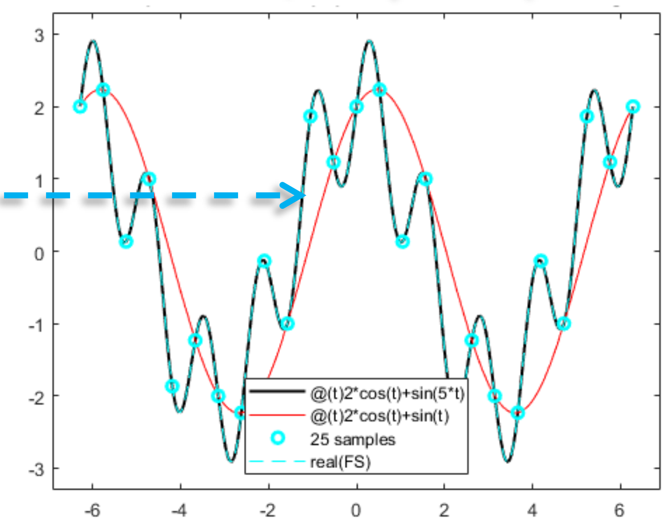
The figure shows that some of the 25 samples of s_5 no longer fall on s_1, so that we are able to distinguish the two signals starting from these 25 samples.

max freq in s_5: $\omega_{max}=5$ ⇨ $\nu=\omega/(2\pi)$

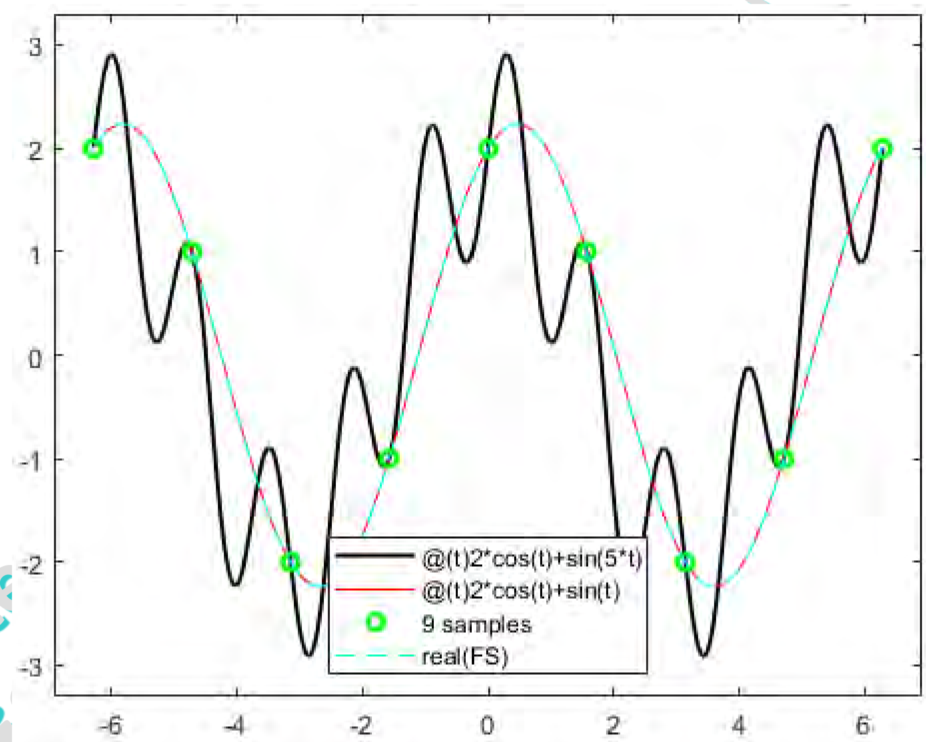
min useful freq for s_5: $f_s>2\nu_{max}=5/\pi$ ⇨

According to the Sampling Theor. we should use a step: $\Delta t<\pi/5$ and used step = $\pi/6<\pi/5$

In facts, if we reconstruct a signal by means of the Fourier Series starting from the 25 samples (whose frequency is $f_s=6/\pi>5/\pi=2\nu_{max}$), now we get s_5 (no aliasing).



Quiz



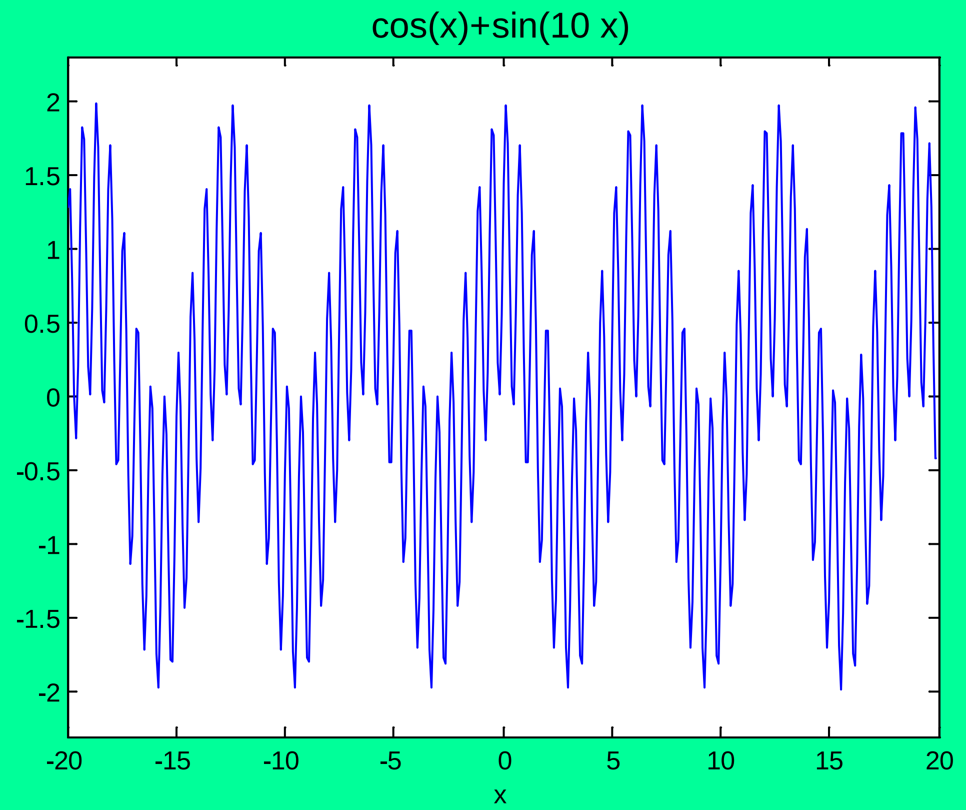
Why, starting from the samples of $s_5 = 2\cos(t) + \sin(5t)$ taken with frequency $f_s = 2/\pi < 5/\pi = 2\nu_{max}$, are we able to reconstruct the signal $s_1 = 2\cos(t) + \sin(t)$ with angular frequency $= 1$, and not:

$$s_k = 2\cos(t) + \sin(kt),\ k = 2,3,4$$

with maximum frequency $\omega = k < 5$?

Exercise

What is the Nyquist frequency of the following signal?



Check it by means the Sampling Theor. and display what happens with a lower frequency.