Scuola Interdipartimentale delle Scienze, dell'Ingegneria e della Salute

DiST Dipartimento di Scienze e Tecnologie
Università Parthenope

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- Fourier Transform of a Comb function.
- Superposition of functions.
- Properties of the Fourier Transform.

Examples of Fourier Transform

Comb function (pulse train or sampling function)

$$f(t) = \delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad F(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} e^{i\omega kT}$$

$$F(\nu) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\nu - \frac{k}{T}\right) = \sum_{k=-\infty}^{+\infty} e^{i2\pi\nu kT}$$

pulse train

$$\delta_T(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \tfrac{1}{T}\delta_{\frac{1}{T}}(\nu)$$

period T — period $1/T$

The Fourier Transform of a comb function is still a comb function, but with a period equal to the reciprocal of the period of the original function.

δ_T is a series of T-shifted "δ functions", called as **superposition** or **periodic replication** of $\delta(t)$; δ_T is characterized by

$$\langle \delta_T, g \rangle = \int_{-\infty}^{+\infty} g(x)\delta_T(x)\,dx = \sum_{k=-\infty}^{+\infty} g(kT)$$

(it describes the sampling of g)

by the Sifting Property of the Dirac delta

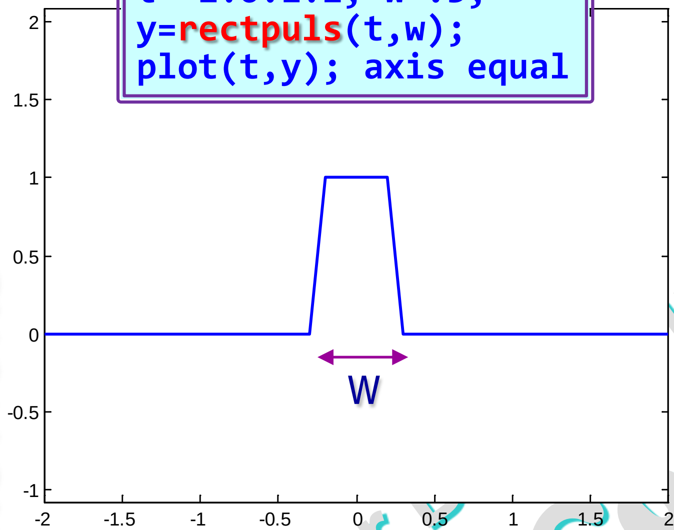
$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,dx = f(x_0)$$

in MATLAB Signal Toolbox `pulstran()`

pulstran(): in MATLAB Signal Toolbox

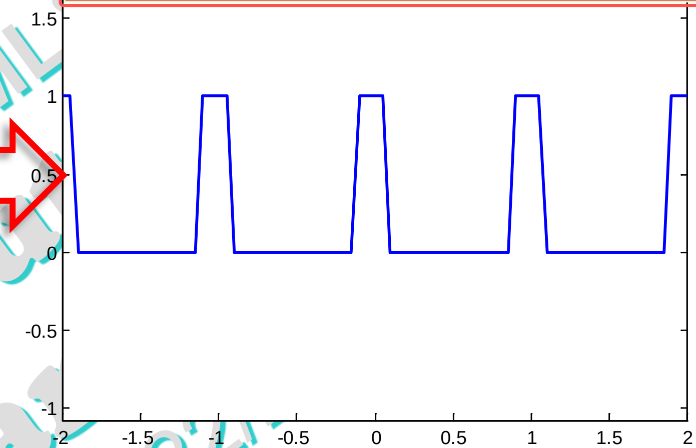
window functions

```
t=-2:0.1:2; w=.5;
y=rectpuls(t,w);
plot(t,y); axis equal
```

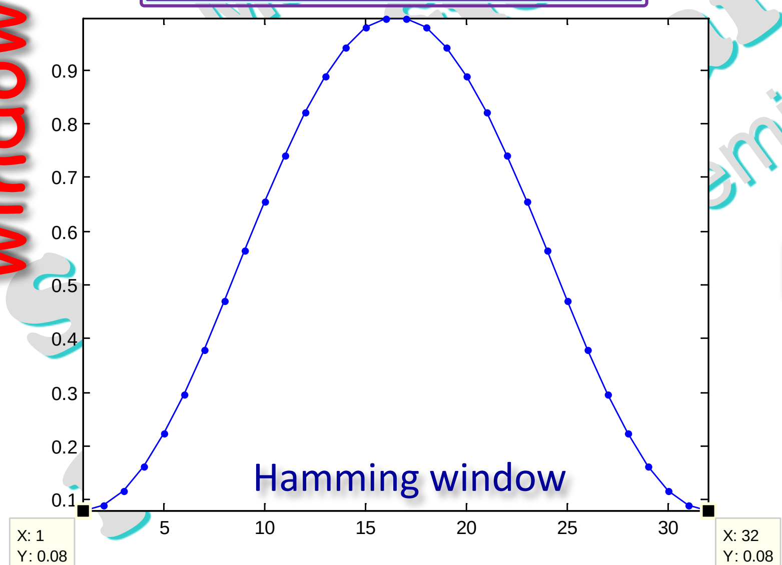


superposition

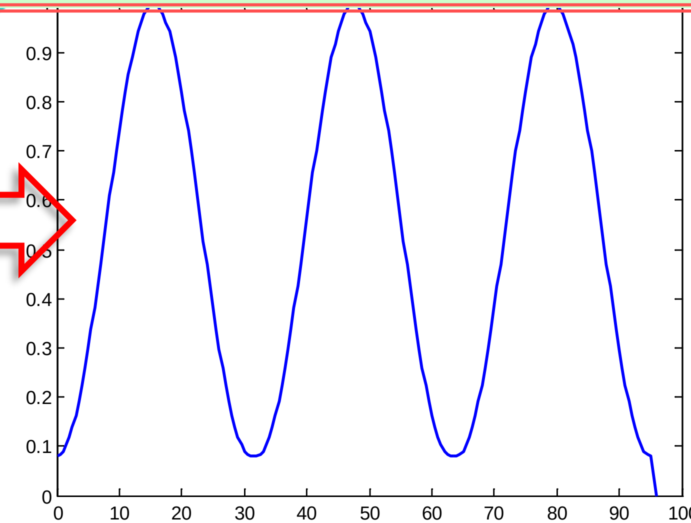
```
t=-2:0.05:2; w=0.2;  d=-2:2;
y=pulstran(t,d,'rectpuls',w);
plot(t,y); axis equal; grid on
```



```
y=hamming(32); plot(y)
```




```
W=32; N=3; d=(0:N-1)'*W; t=0:0.5:N*W;
p=hamming(H); y=pulstran(t,d,p);
plot(t,y)
```



(prof. M. Rizzardi)

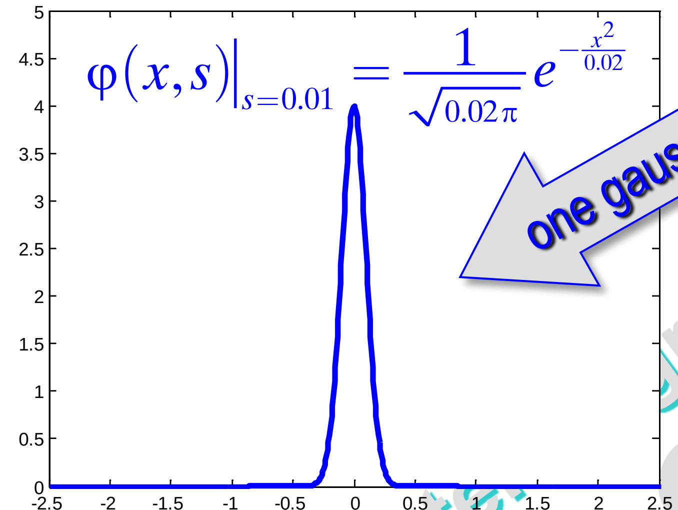
What is a superposition $\varphi_T(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - kT)$?

periodic replication

$$p(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

for $\mu=0$ and $s=\sigma^2$ it becomes $\varphi(x,s) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$

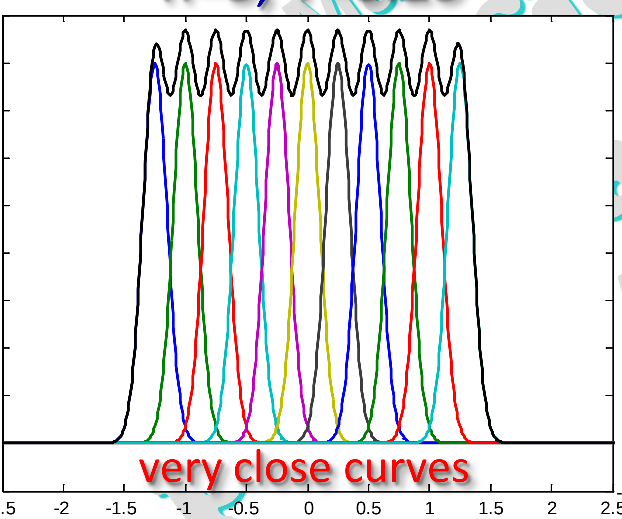
one gaussian



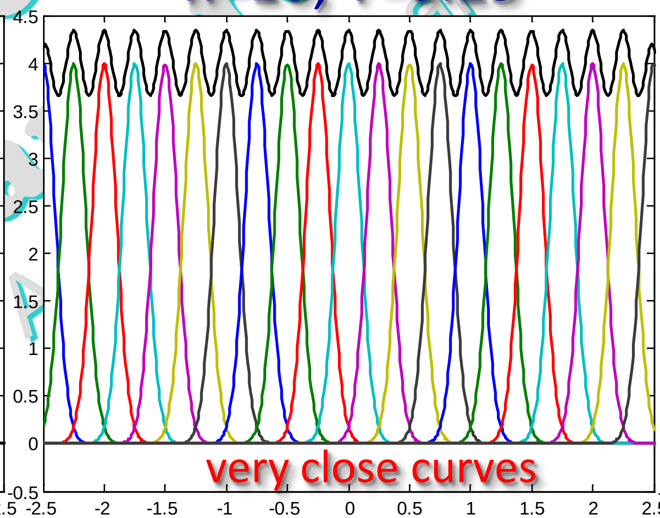
We give the idea of a superposition, but in a finite case

```
f=@(x,k,t)exp(-(x-k*t).^2/0.02)/sqrt(pi*0.02);
n=5;   k=-n:n;  x=linspace(-4,4,1001);
[K,X]=meshgrid(k,x); T=0.25;
F=f(X,K,T); plot(x',F, x',(sum(F,2)),'k')
axis([-2.5 2.5 -.5 4.5])
```

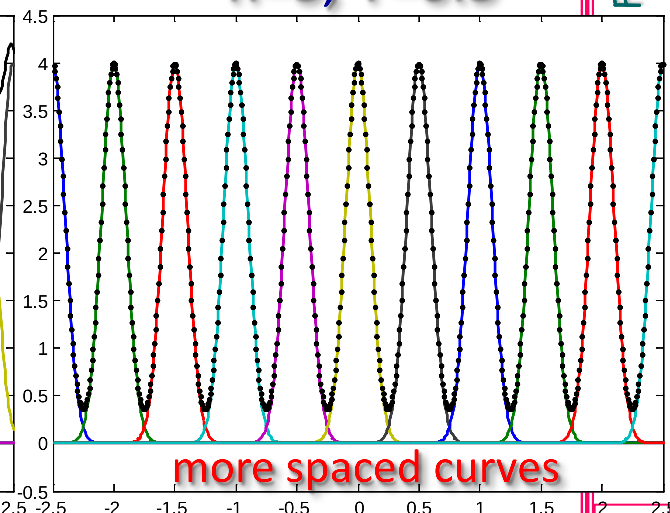
n=5; T=0.25



n=10; T=0.25



n=5; T=0.5

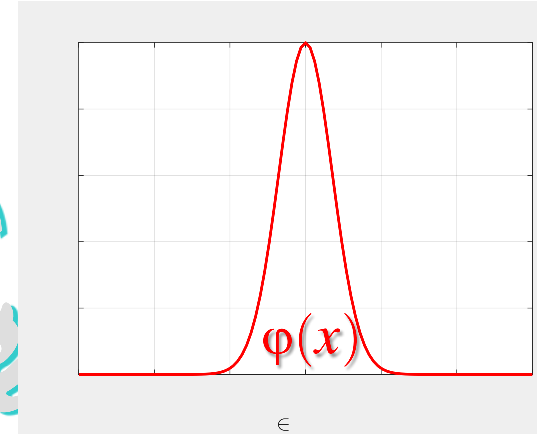


The **black curve** represents the (finite) superposition of n gaussians with a T period

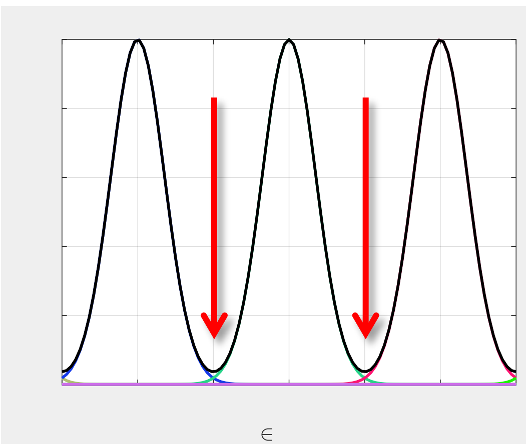
Download: `finite_superposition.p`

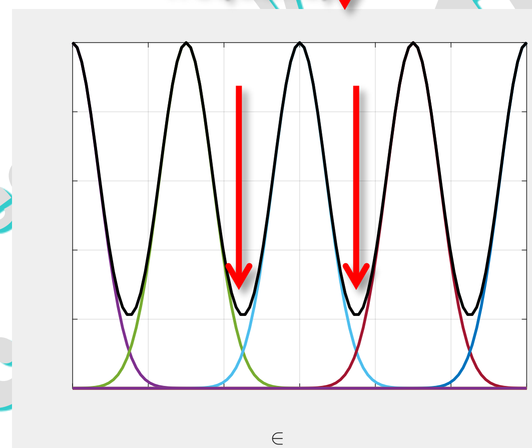
$$\varphi_T(x) = \sum_{k=-\infty}^{+\infty} \varphi(x - kT)$$

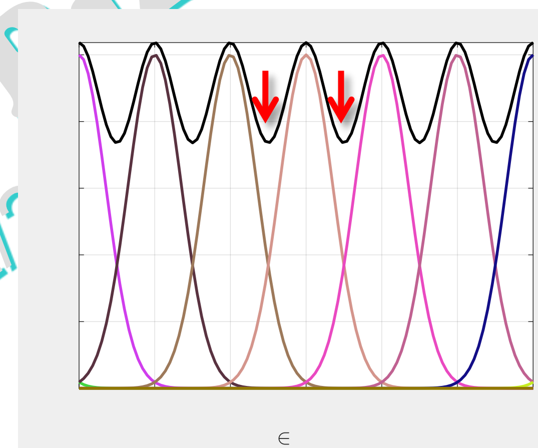
(infinite) superposition



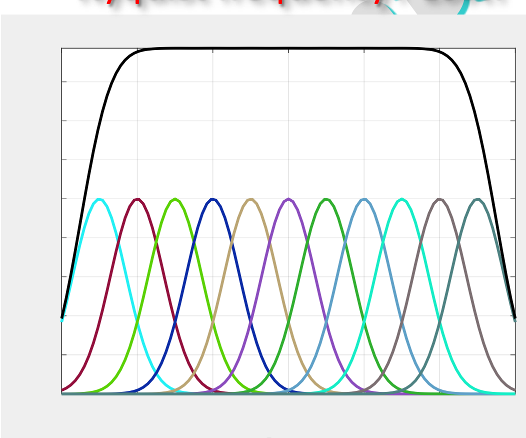
Nyquist frequency



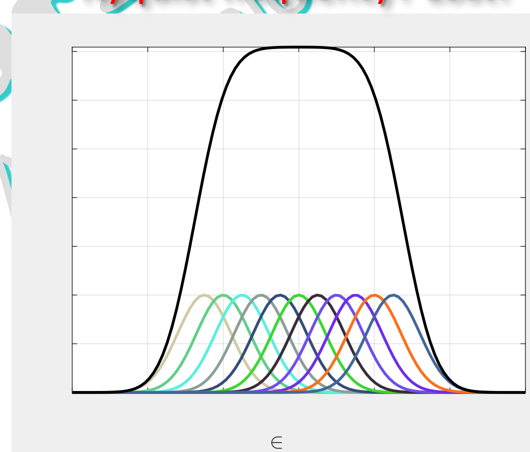




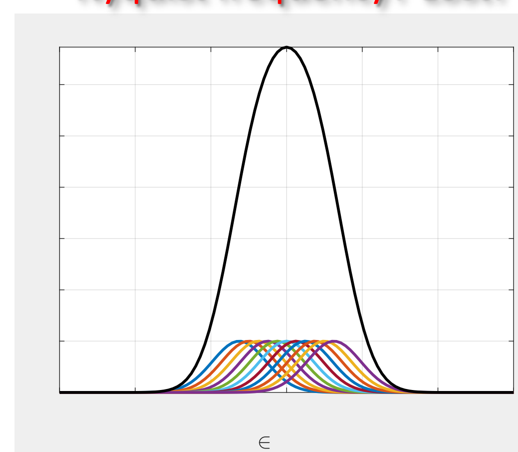
Nyquist frequency? Lost!



Nyquist frequency? Lost!



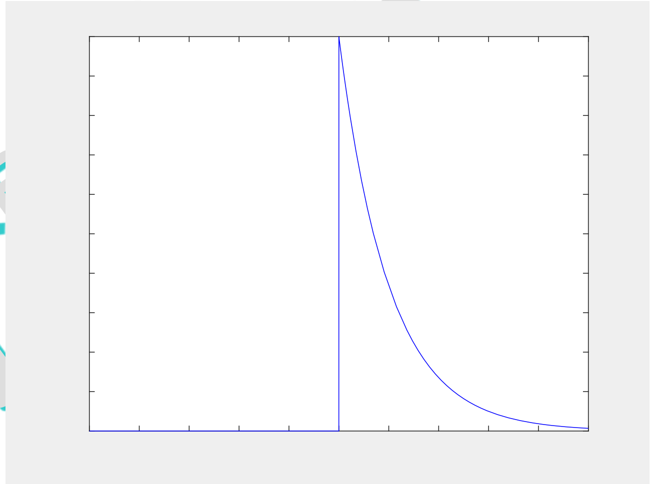
Nyquist frequency? Lost!



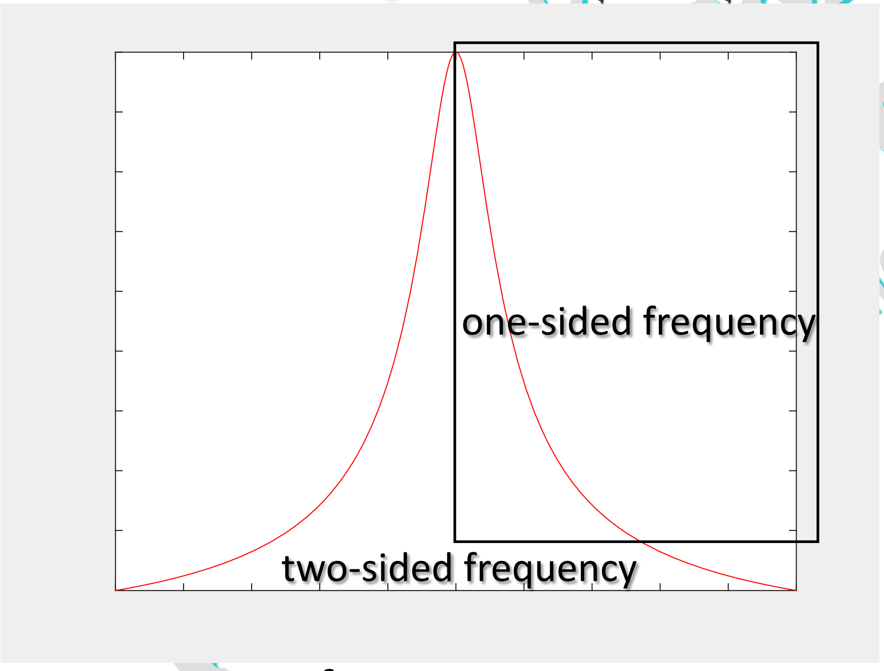
a bit of terminology …

```
syms t real
ft=exp(-abs(t))*heaviside(t);
figure; fplot(ft,[-5 5]); axis tight
Fw=simplify(fourier(ft));
figure; fplot(abs(Fw), [-10 10])
figure; fplot(abs(Fw)^2, [-10 10])
```

signal: decay pulse $e^{-|t|}$

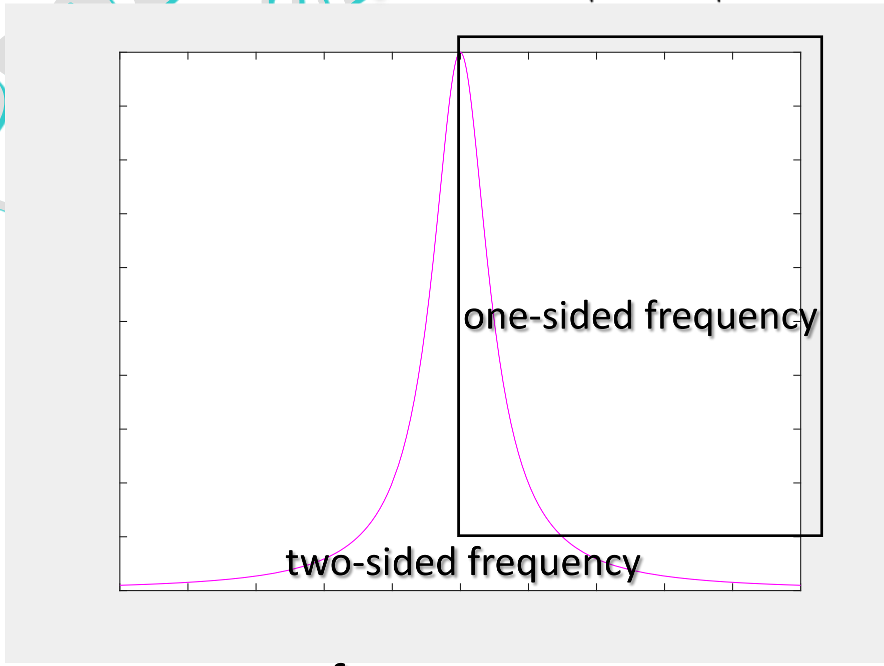


Fourier Spectrum: $\left|F(\omega)\right|$



frequency ω

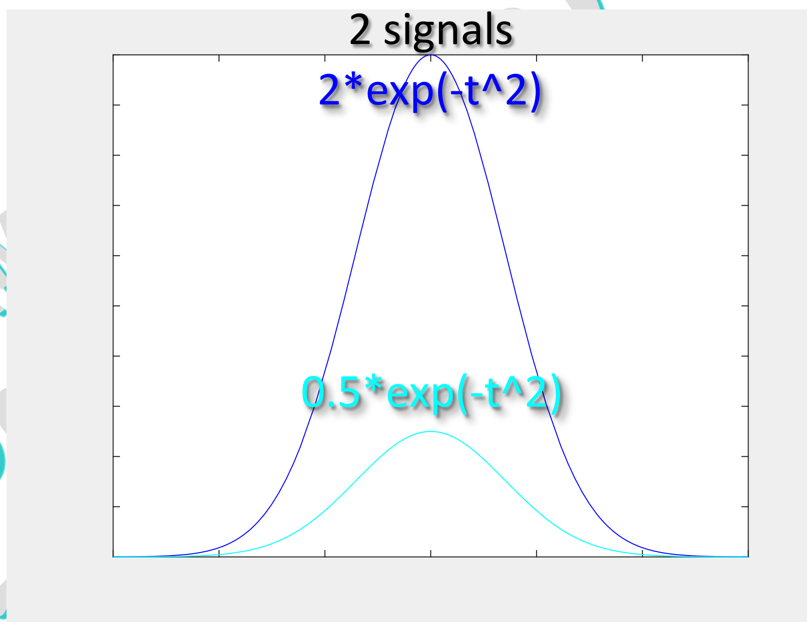
Power Spectrum: $\left|F(\omega)\right|^2$



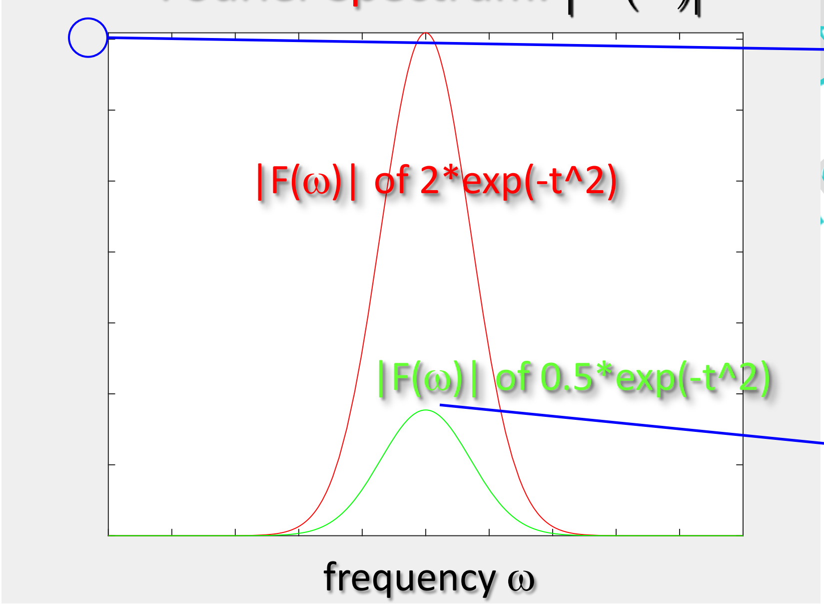
frequency ω

What is the Power Spectrum for?

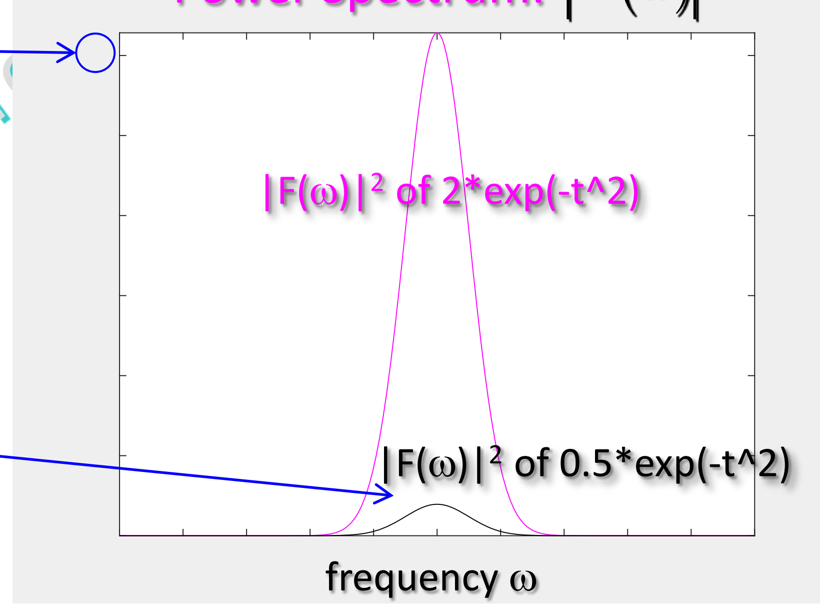
The **Power Spectrum** amplifies the large values of the Fourier Spectrum ($|F(\omega)|>1$) and shrinks its small values ($|F(\omega)|<1$).



Fourier Spectrum: $|F(\omega)|$



Power Spectrum: $|F(\omega)|^2$



Main properties of the Fourier Transform [1]

Let $\mathscr{F}$ be the map which transforms a function $f(x) \in L^1(-\infty, +\infty)$ into its Fourier Transform $F(\omega)$

$$\mathscr{F} : f \longrightarrow \mathscr{F}[f, \omega] = F(\omega) \qquad f(t) \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ F(\omega)$$

- $\mathscr{F}[f, \omega]$ is a ***linear operator***: $\mathscr{F}[\alpha f + \beta g, \omega] = \alpha F(\omega) + \beta G(\omega)$.
- If f is an ***even function***, then $F(\omega)$ is ***real*** and it is in turn an ***even function***.
- If f is an ***odd function***, then $F(\omega)$ is ***purely imaginary*** and it is in turn an ***odd function***.
- If f is a ***real-valued function***, then $F(\omega)$ is a ***complex valued function***, and

$$F(-\omega) = \overline{F(\omega)}$$

- ***Shift Properties***: *shifting* (or *translating*) a function *in the time domain* $t \pm h$ corresponds to a *rotation* by an angle $\pm h\omega$ *in the frequency domain*, i.e.

$$f(t \pm h) \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ e^{\pm ih\omega} F(\omega) \qquad \text{Time shift property}$$

  and *shifting* (or *translating*) *in the frequency domain* $\omega \mp \lambda$ corresponds to a *rotation* by an angle $\pm \lambda t$ *in the time domain*, i.e.

$$e^{\pm i\lambda t} f(t) \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ F(\omega \mp \lambda) \qquad \text{Frequency shift property}$$

Main properties of the Fourier Transform [2]

- ***Time scaling*** (or ***Similarity Property***): if $f(t) \bullet\!\!-\!\!\!-\!\!\circ\ \mathscr{F}[f,\omega] = F(\omega)$ and $c \in \mathbb{R} - \{0\}$, then

$$f(ct) \bullet\!\!-\!\!\!-\!\!\circ\ F(\omega/c)/|c|$$

- ***Convolution Property***: if $f, g \in L^1(-\infty, +\infty)$ also $f * g \in L^1$, then *the convolution of signals* in the time domain will be transformed into *the multiplication of their Fourier Transforms* in the frequency domain, and, conversely, *the multiplication of signals* in the time domain will be transformed into *the convolution of their Fourier Transforms* in the frequency domain:

$$\mathscr{F}[f * g, \omega] = \frac{1}{2\pi} F(\omega) \cdot G(\omega) \quad \text{and} \quad \mathscr{F}[f \cdot g, \omega] = F(\omega) * G(\omega)$$

$$\mathscr{F}[f * g, \nu] = F(\nu) \cdot G(\nu) \quad \text{and} \quad \mathscr{F}[f \cdot g, \nu] = F(\nu) * G(\nu)$$

where the ***convolution*** between $f, g \in L^1(-\infty, +\infty)$ is defined as

$$[f * g](\tau) = \int_{-\infty}^{+\infty} f(t) g(\tau - t)\, \mathrm{d}t$$

- ***Parseval's Identity*** (or ***Rayleigh's Energy Theorem***)

$$\int_{-\infty}^{+\infty} |f(t)|^2\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2\, \mathrm{d}\omega$$

Main properties of the Fourier Transform [3]

- ***Differentiation of $f(t)$***: If f is absolutely continuous and summable, and f' is summable, then

$$\mathscr{F}[f',\omega] = i\omega\,\mathscr{F}[f,\omega]$$

More generally, if f is absolutely continuous and summable with its first $k-1$ derivatives, and $f^{(k)}$ is summable, then

$$\mathscr{F}[f^{(k)},\omega] = (i\omega)^k\,\mathscr{F}[f,\omega]$$

- ***Differentiation of $F(\omega)$***: If $f(t)$, $tf(t) \in L^1(-\infty,+\infty)$, then F has a continuous derivative, and

$$F'(\omega) = \mathscr{F}[-it\,f(t),\omega]$$

More generally, if $f(t)$, $tf(t)$, …, $t^k f(t) \in L^1(-\infty,+\infty)$, then F has continuous derivatives up to order k, and

$$F^{(k)}(\omega) = \mathscr{F}[-(it)^k f(t),\omega]$$

- ***Symmetry*** (or ***Duality***) ***Property***:

$$\begin{cases} \mathscr{F}[F(\omega),y] = \displaystyle\int_{-\infty}^{+\infty} F(\omega)e^{-i\omega y}\,d\omega = 2\pi f(-y) \\ \mathscr{F}[F(\nu),y] = \displaystyle\int_{-\infty}^{+\infty} F(\nu)e^{-2\pi i\nu y}\,d\nu = f(-y) \end{cases}$$

reversed signal

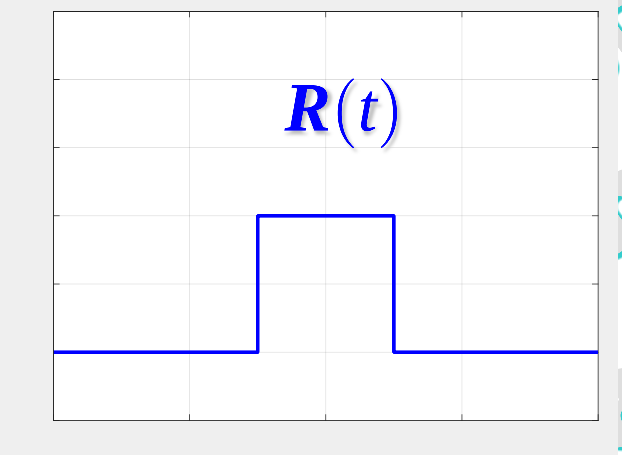
- ***Riemann-Lebesgue Lemma***: If $f(t) \in L^1(-\infty,+\infty)$, then $F(\omega)$ is a continuous function and it is infinitesimal at ∞.

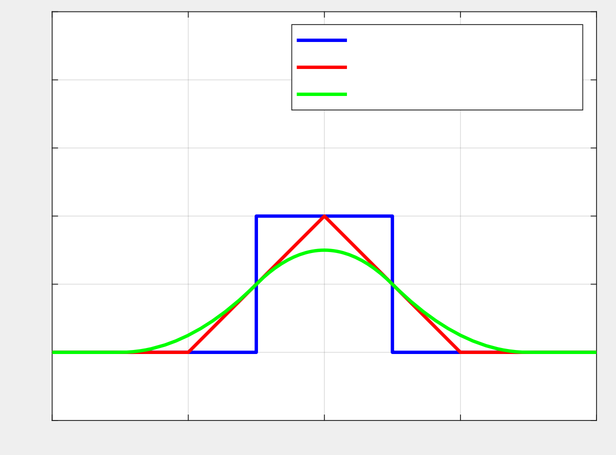
Application of Convolution Property (w.r.t. the circular freq)

$$[f * g](\tau) = \int_{-\infty}^{+\infty} f(t)g(\tau - t)\mathrm{d}t \qquad f * g = \mathscr{F}^{-1}\{\mathscr{F}[f]\cdot\mathscr{F}[g]\}$$

rect pulse

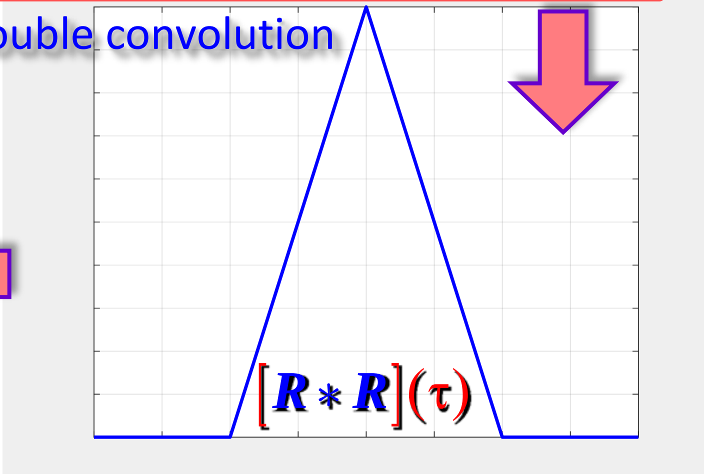
$$R(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$



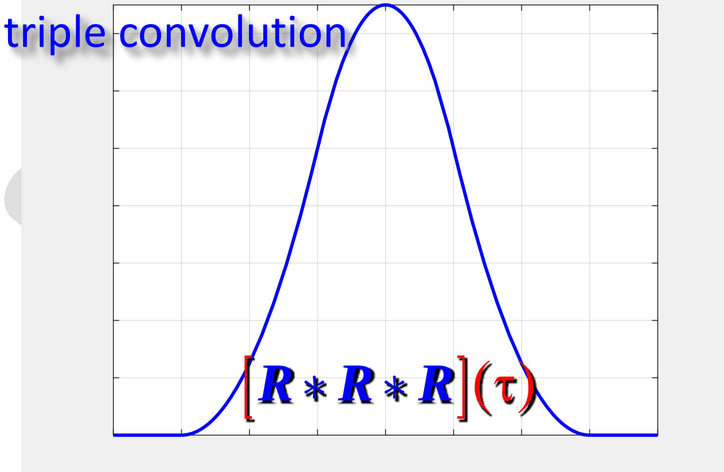


```
syms t w real;  Rt=rectangularPulse(t);  Fw=simplify(fourier(Rt))
fplot(Rt,[-2 2],'Color','b','LineWidth',2); hold on
F2=Fw*Fw; R2=simplify(ifourier(F2),100); % double convolution R(t)*R(t)
fplot(R2,[-2 2],'Color','r','LineWidth',2)
F3=F2*Fw; R3=simplify(ifourier(F3),100); % triple convolution R(t)*R(t)*R(t)
fplot(R3,[-2 2],'Color','g','LineWidth',2)
```

double convolution



triple convolution

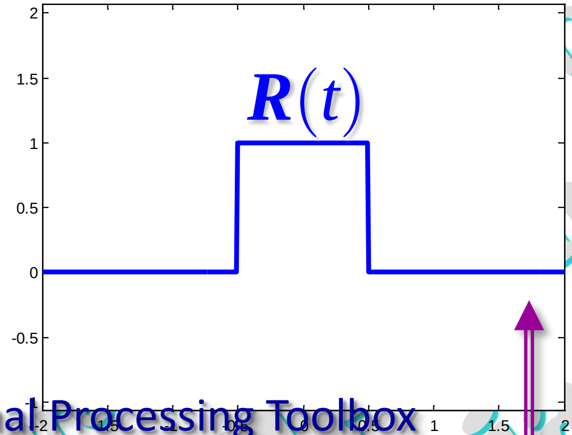


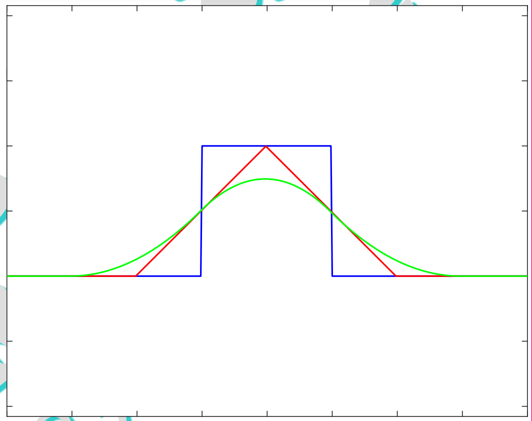
Numerical application of Convolution Property (w.r.t. the circular freq)

Equivalently by conv() ...
```
RR=conv(R,R,'same')*T/N;
```

rect pulse

$$R(t)=\begin{cases}1 & |t|<\frac{1}{2}\\ 0 & |t|>\frac{1}{2}\end{cases}$$

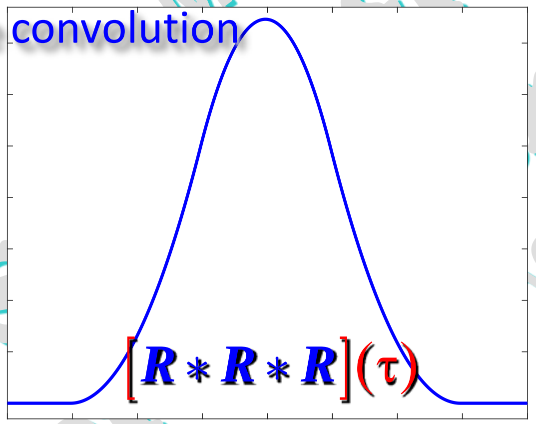




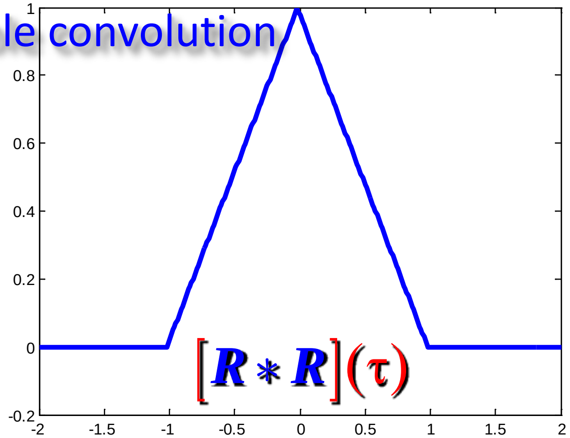
rectpuls(): in MATLAB Signal Processing Toolbox

```
a=-2; b=2; T=b-a; t=a:.01:b; N=numel(t)-1; R=rectpuls(t); plot(t,R,'b')
axis equal; hold on
RR=conv(R,R,'same')*T/N; plot(t,RR,'r')      % double convolution
RRR=conv(RR,R,'same')*T/N; plot(t,RRR,'g')  % triple convolution
```

triple convolution



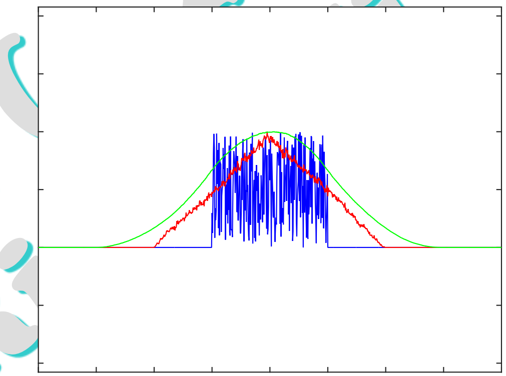
double convolution



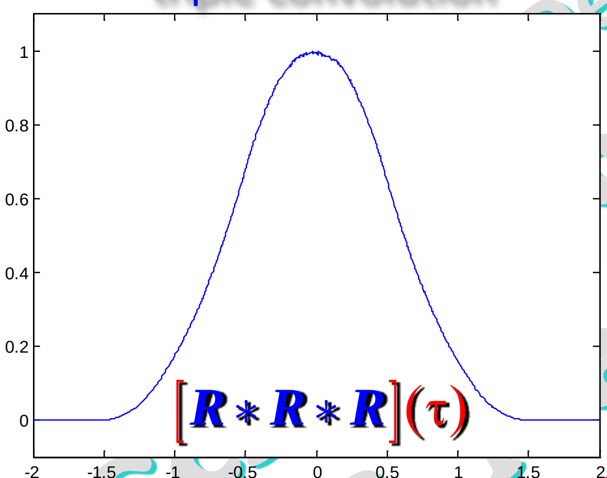
Exercise: repeat with **DFT** (**fft**) and with **cconv(R,R,N)**

Numerical application of Convolution Property (w.r.t. the circular freq)

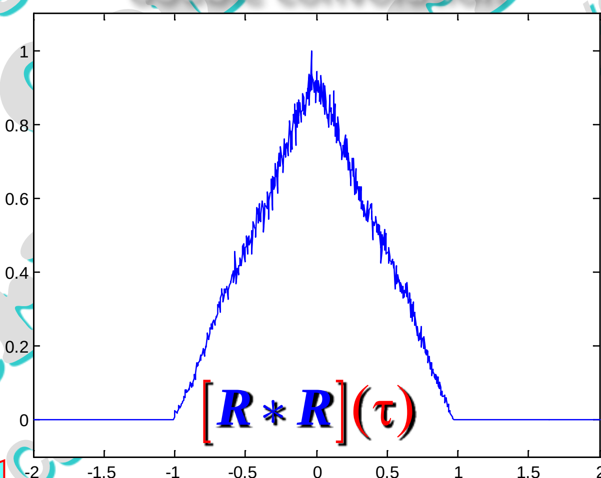
```
a=-2; b=2; T=b-a;
t=a:.01:b; N=numel(t)-1;
rng('default');
R=rectpuls(t).*rand(size(t)); % with uniform random noise
RR=conv(R,R,'same')*T/N;     % RR double convolution R(t)*R(t)
RRR=conv(RR,R,'same')*T/N;  % RRR triple convolution R(t)*R(t)*R(t)
plot(t,R,'b'); axis equal; hold on
plot(t,RR/max(RR),'r')
plot(t,RRR/max(RRR),'g')
```

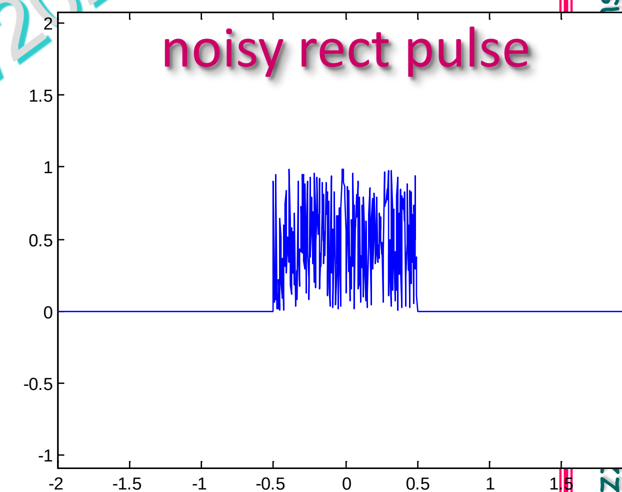


triple convolution



double convolution





the noise has been abated! (convolution as data filtering)

Exercise: repeat with DFT (`fft`) and with `cconv(R,R,N)`

From the previous example, we get the following:

From sinc(ω) being the Fourier Transform of the rect pulse:

$$\mathscr{F}[\text{rect pulse},\omega] = \mathrm{sinc}(\omega)$$

it follows that the Fourier Transform of the triangular function is $\mathrm{sinc}^2(\omega)$:

$$\mathscr{F}[\text{triangular pulse},\omega] = \mathrm{sinc}^2(\omega)$$

Quiz: ... Why?
Explain your answer