Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST — Dipartimento di Scienze e Tecnologie — Università Parthenope

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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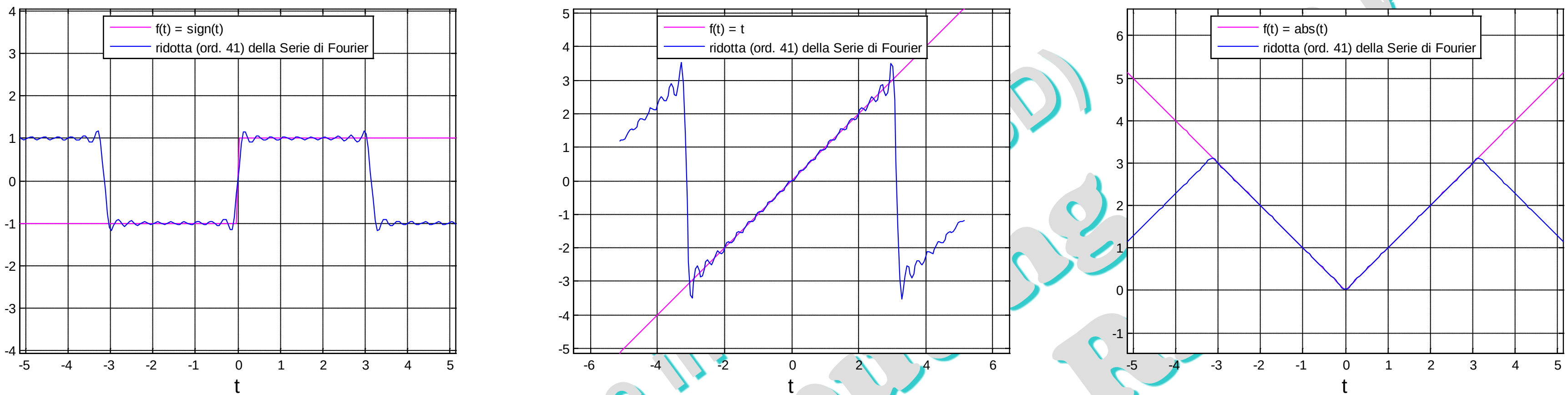
Fourier Transform (FT)



If the ***Fourier Series*** converges to f in an interval $[a,b]$, outside $[a,b]$ the ***Fourier Series*** converges to f only if also f is periodic of period $b-a$. The ***Fourier Transform*** arises from the need to approximate **non-periodic functions** on all $\mathbb{R}$.

DEF The ***Fourier Transform*** $F(\omega)$ of $f(t)$ is

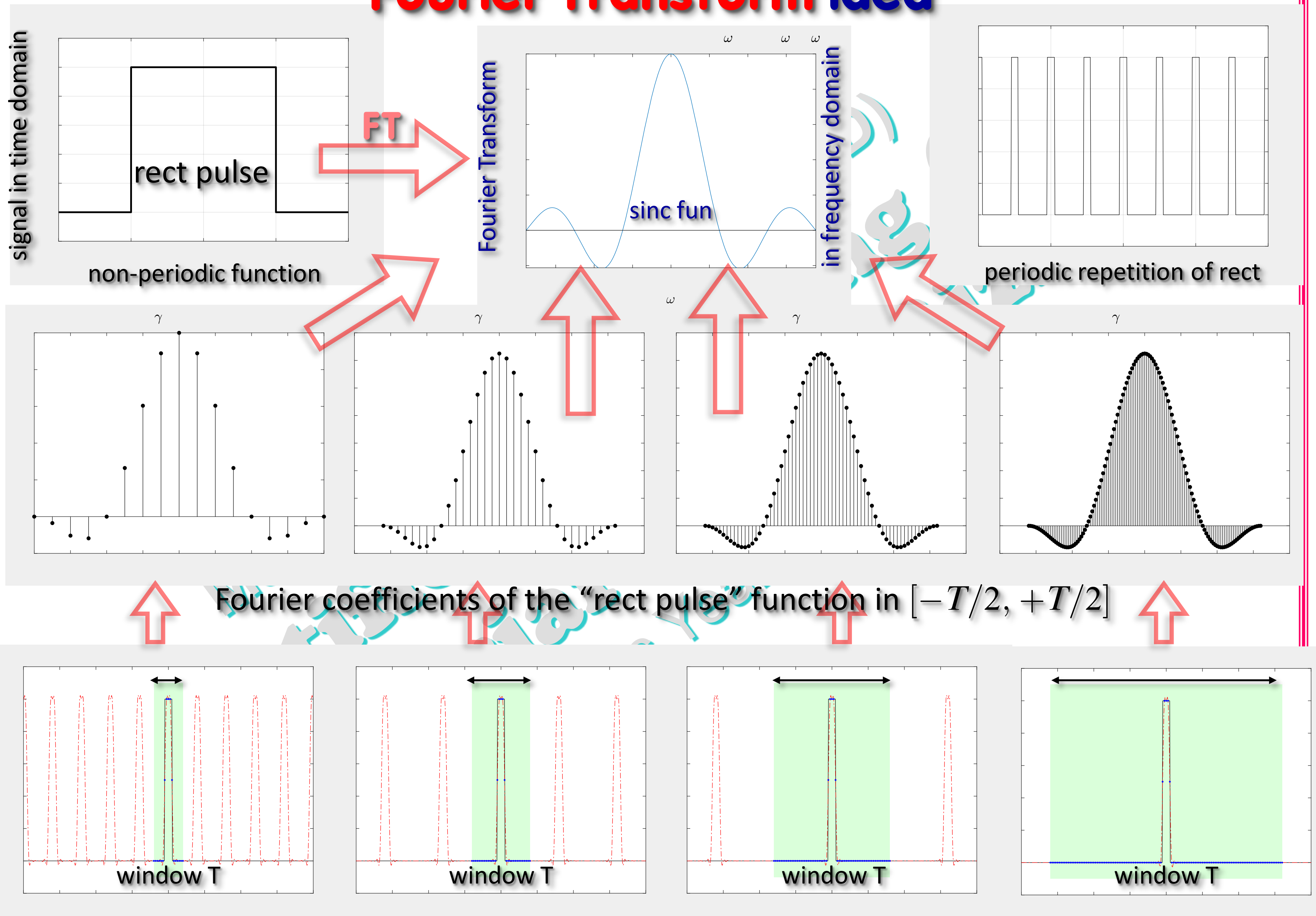
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \, dt$$

$F(\omega)$ is a complex valued function of a real argument ω

when this integral exists (i.e. it is $< \infty$).

The summability of f $[f \in L^1(-\infty,+\infty)]$ represents a sufficient condition, but it is not necessary for the existence of the FT.

Fourier Transform idea



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As the window increases, in the FCs of the "rect pulse" fun the frequencies are getting closer and closer together, and it looks as though the coefficients are tracking some definite curve of the FT function.

Theorem If $f \in L^1(-\infty,+\infty)$ and satisfies the Dirichlet conditions, then the following formulas hold

Fourier Transform (FT)	complex-valued function of a real argument
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} \mathrm{d}t$$	$$F(\nu) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i\nu t} \mathrm{d}t$$
Inverse Fourier Transform (IFT)	
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} \mathrm{d}\omega$$	$$f(t) = \int_{-\infty}^{+\infty} F(\nu) e^{2\pi i\nu t} \mathrm{d}\nu$$

angular frequency ω

circular frequency ν

$$\omega = 2\pi\nu$$

recap

discrete and finite case

Discrete Fourier Transform (DFT)

$$F_k = \sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi}{N}kj}, \qquad k = 0,\dots,N-1$$

Inverse Discrete Fourier Transform (IDFT)

$$f_j = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{+i\frac{2\pi}{N}kj}, \qquad j = 0,\dots,N-1$$

discrete and infinite case

Coefficients of the Fourier Series in $[-\pi,+\pi]$

$$\gamma_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{-ikx}\,\mathrm{d}x, \qquad k = -\infty,\dots,0,\dots,+\infty$$

Fourier Series (FS)

$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{+ikx}$$

continuos case

Fourier Transform (FT)

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\,dt$$

Inverse Fourier Transform (IFT)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) e^{+i\omega t}\,d\omega$$

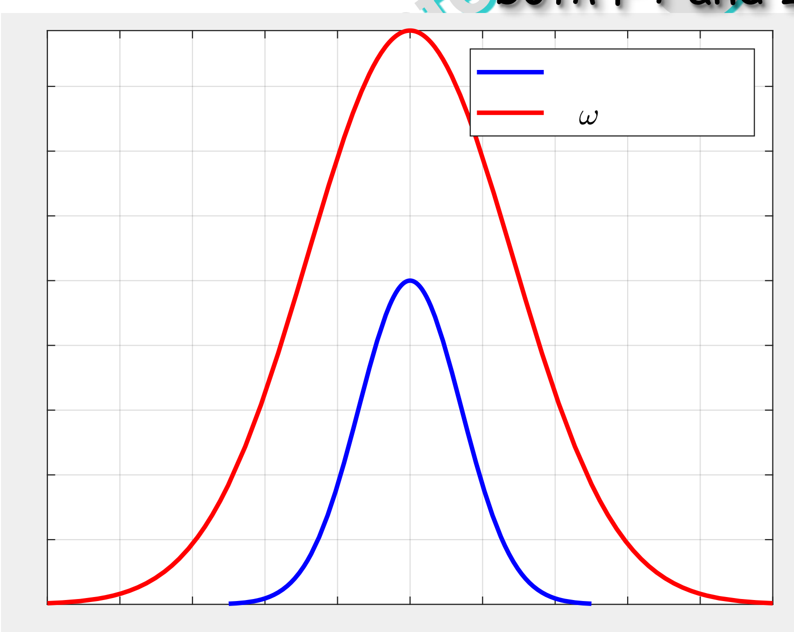
MATLAB **Symbolic Math Toolbox** provides the functions **fourier(…)** and **ifourier()** for the symbolic expression of the FT and of the IFT respectively.

gaussian

```
syms t real
ft=exp(-t^2);  Fw=fourier(ft)
Fw =
pi^(1/2)/exp(w^2/4)
```

```
syms t real;  ft=exp(-t^2);
Fw=fourier(ft);  Ifw=ifourier(Fw)
IFw =
1/exp(x^2)
```

both FT and IFT are gaussian



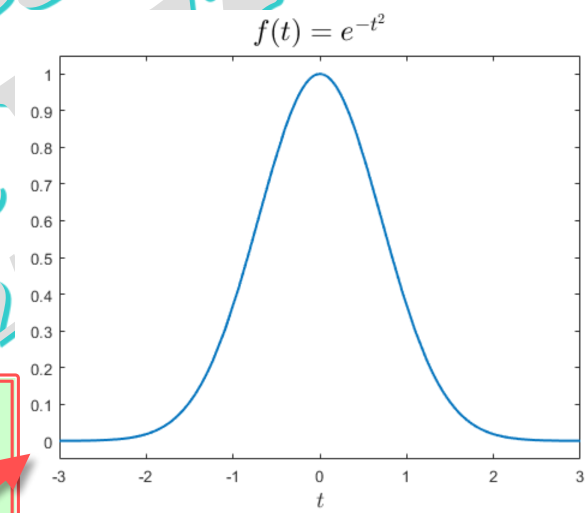
```
fplot(ft,[-5 5],'Color','b','LineWidth',2)
grid on; hold on
fplot(Fw,[-5 5],'Color','r','LineWidth',2)
legend('function f(t)', ...
      'F(\omega)=FT of f(t)','FontSize',14)
```

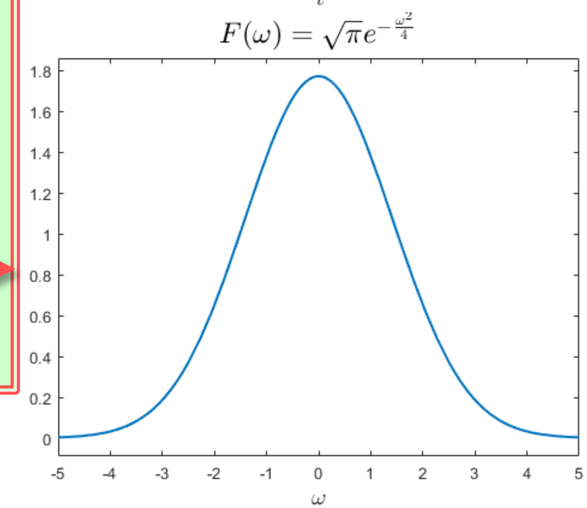
Examples of Fourier Transform

"The Fourier Transform of a Gaussian is still a gaussian"

$$f(t)=e^{-|a|t^2} \quad \circ\!\!-\!\!\!-\!\!\bullet \quad F(\omega)=\sqrt{\frac{\pi}{|a|}}\, e^{-\frac{\omega^2}{4|a|}}$$

```
syms t real; syms a positive
ft=exp(-a*t^2); % gaussian f(t)
fourier(ft)     % FT of f(t)
ans =
(pi^(1/2)*exp(-w^2/(4*a)))/a^(1/2)
ft1=subs(ft,a,1);  fplot(ft1,[-5 5])
title('$f(t)=e^{-t^2}$','FontWeight','normal','FontSize',18,'Interpreter','LaTeX')
xlabel('$t$','FontSize',14,'Interpreter','LaTeX')
fplot(fourier(ft1),[-5 5])
title('$F(\omega)=\sqrt{\pi}e^{-\frac{\omega^2}{4}}$','FontWeight','normal','FontSize',18, ...
'Interpreter','LaTeX')
xlabel('$\omega$','FontSize',14,'Interpreter','LaTeX')
```





Example of Fourier Transform (odd function)

$f(t)$ = **signum**

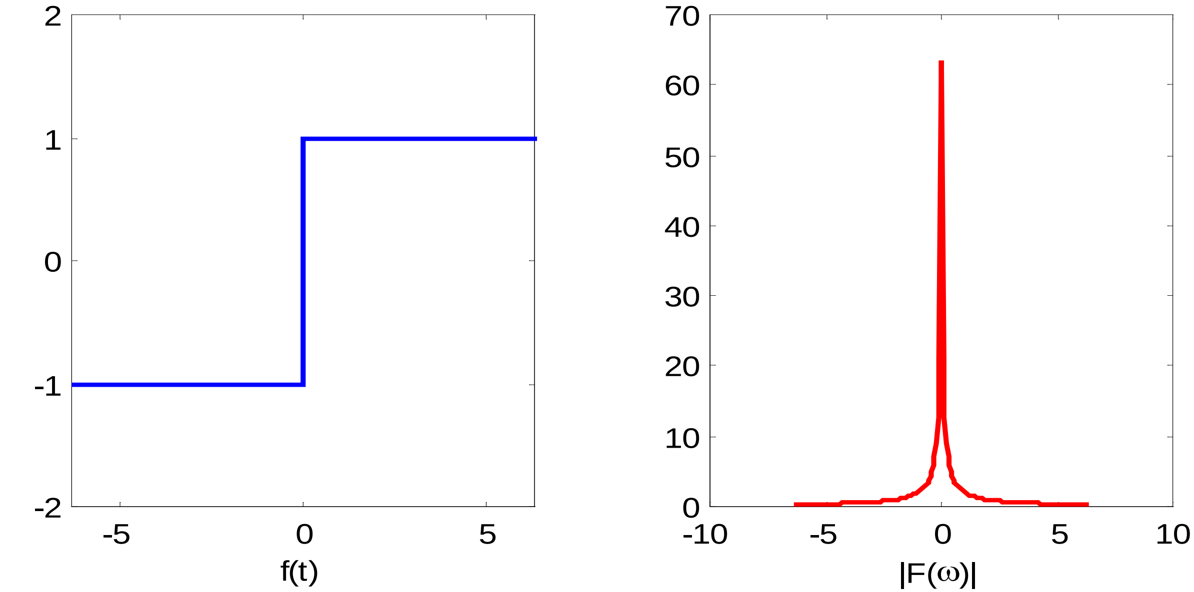
$$f(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}$$

```
syms t real
ft=sign(t);
Fw=fourier(ft)
Fw =
-2i/w
```

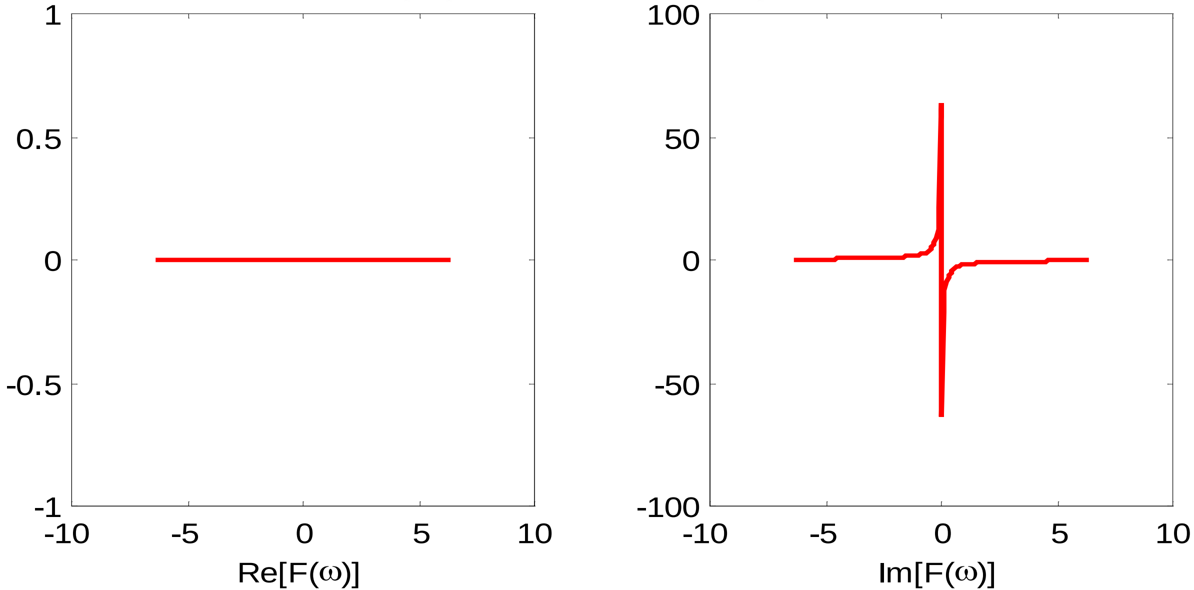
```
sym(2/i)
ans =
-2i
```

$$F(\omega) = \frac{2}{i\omega}$$

$$F(\nu) = \frac{1}{i\pi\nu}$$



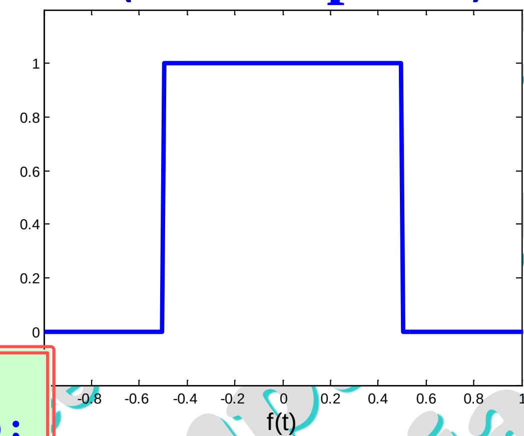
F(ω) is purely imaginary

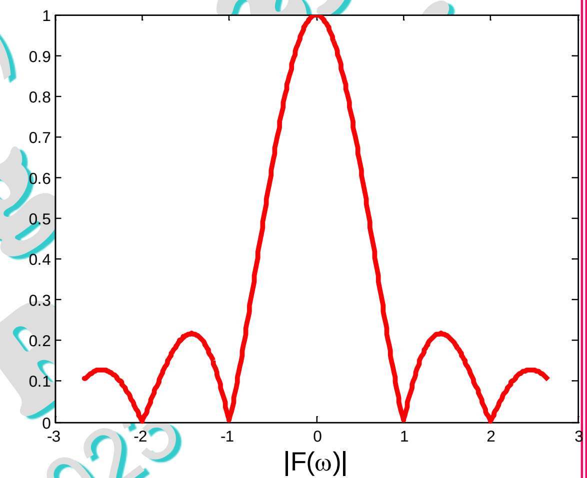


Example of Fourier Transform (even function)

$f(t) =$ **rectangular function** (or **rect pulse**)

$$f(t)=\begin{cases}1 & |t|<\frac{L}{2}\\ 0 & |t|>\frac{L}{2}\end{cases} \qquad L=1$$





MATLAB Symbolic Math Toolbox

```
syms L positive; syms t real
ft=rectangularPulse(-L/2,+L/2,t);
Fw=simplify(fourier(ft))
Fw =
(2*sin((L*w)/2))/w
```

```
syms L positive; syms t real; ft=heaviside(t+L/2)-heaviside(t-L/2);
```

what is heaviside(x)?

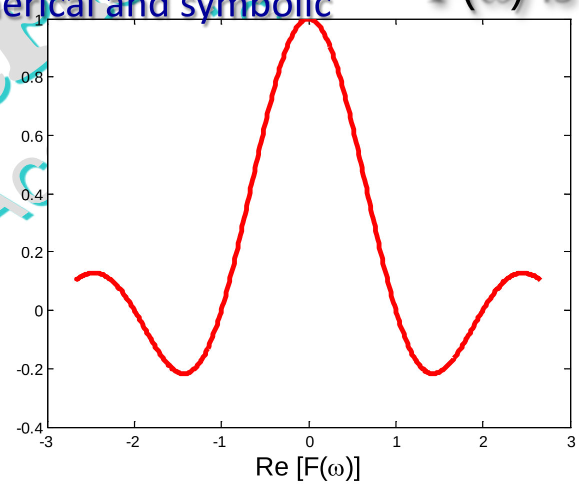
`rectpuls(t)` in MATLAB Signal Toolbox (only numerical)

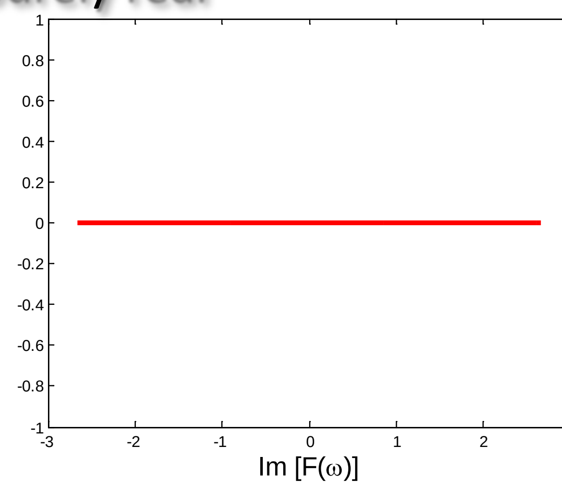
`sinc(v)` sine cardinal or sinc function
`sinc()` is both numerical and symbolic

$$F(\omega)=\frac{2\sin\frac{\omega}{2}}{\omega}$$

$$F(\nu)=\frac{\sin\pi\nu}{\pi\nu}$$

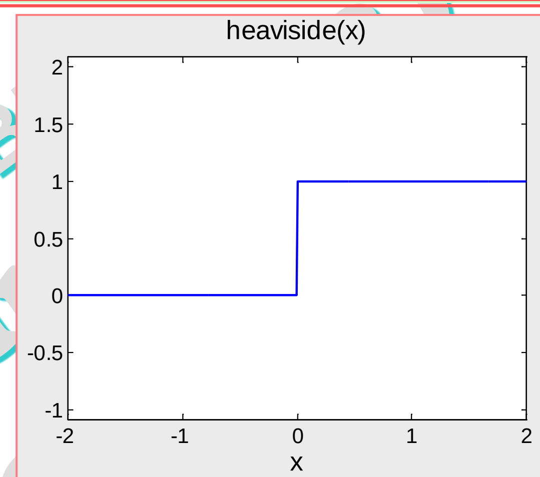
$F(\omega)$ is purely real





what is heaviside(x)?

```
syms x real
ezplot(heaviside(x), [-2 2])
```



Heaviside function
or
Unit step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

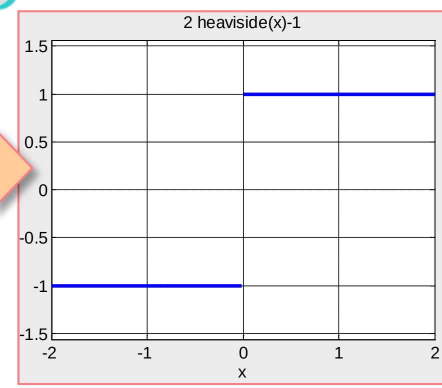
The Dirac delta δ is the derivative of the heaviside function

```
syms x real
diff(heaviside(x))
ans =
dirac(x)
```

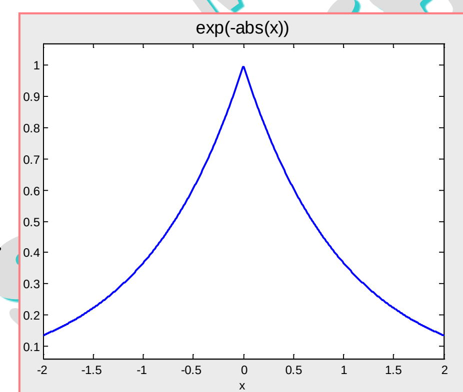
heaviside: what is it for?

before MATLAB R2014a

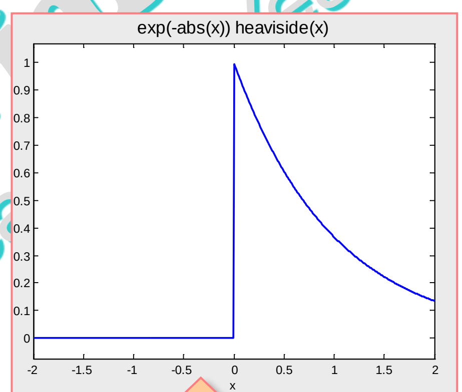
```
syms x real
signum=2*heaviside(x)-1;
ezplot(signum, [-2 2])
```



decay function



decay pulse

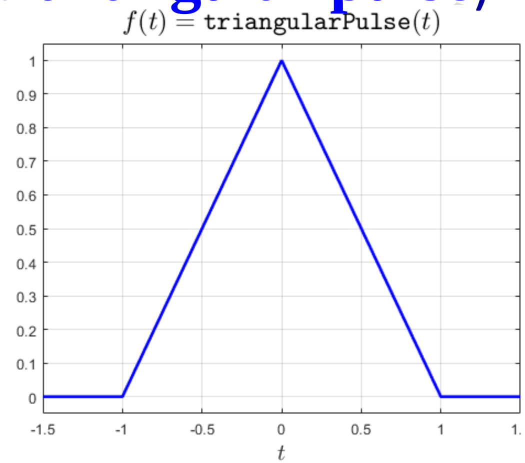


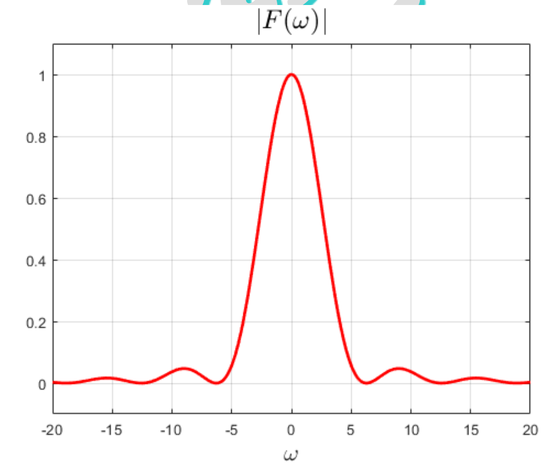

```
syms x real; decayP=exp(-abs(x))*heaviside(x); ezplot(decayP, [-2 2])
```

Example of Fourier Transform (even function)

$f(t)$ = **triangle function** (or **triangular pulse**)

$$f(t)=\begin{cases}1+t & t\in[-L,0]\\ 1-t & t\in[0,L]\\ 0 & |t|>L\end{cases}\qquad L=1$$

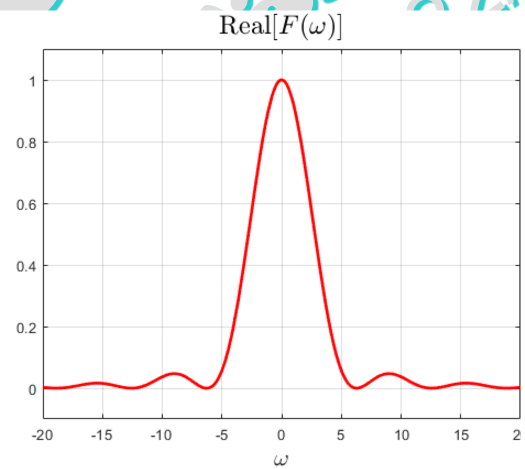


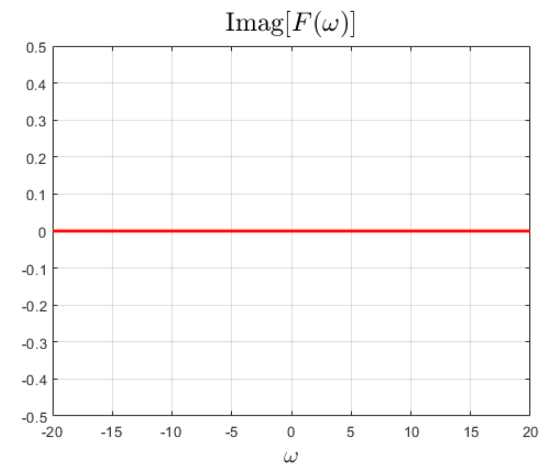


$F(\omega)$ is purely real

MATLAB Symbolic Math Toolbox

```
syms t real
ft=triangularPulse(t);
Fw=simplify(fourier(ft),100)
Fw =
-(2*(cos(w) - 1))/w^2
syms v w
Fv=simplify(subs(Fw,w,2*pi*v),100)
Fv =
sin(pi*v)^2/(v^2*pi^2)
```





$$F(\omega)=2\frac{1-\cos\omega}{\omega^2}$$

$$F(\nu)=\left(\frac{\sin\pi\nu}{\pi\nu}\right)^2 \Leftarrow \texttt{sinc(v)}^2$$

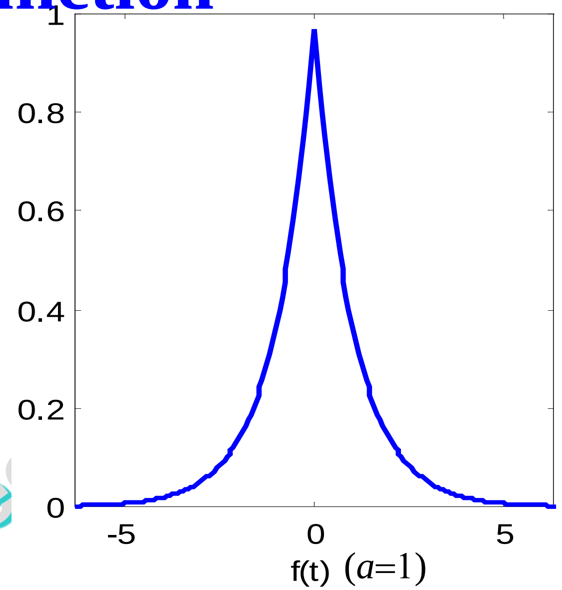
sine cardinal or sinc function

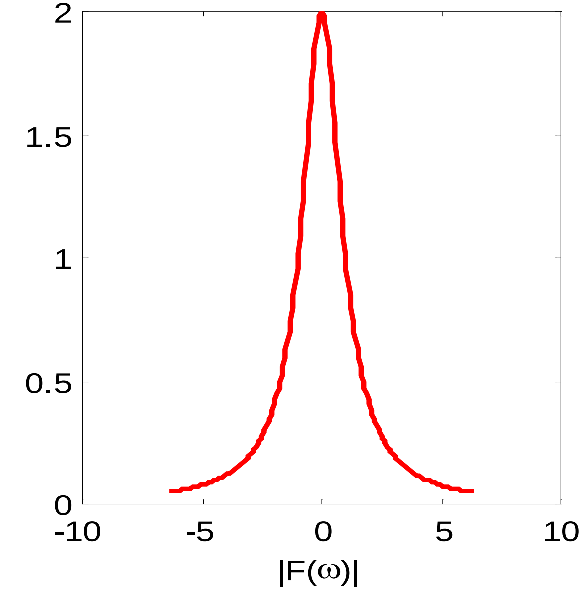
MATLAB `sinc()` is both numerical and symbolic

Example of Fourier Transform (even function)

$f(t)$ = even decay function

$$f(t) = e^{-a|t|}, \quad a > 0$$



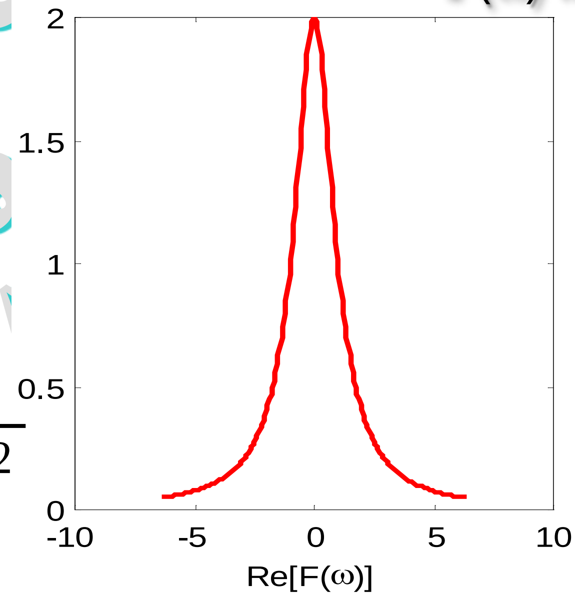


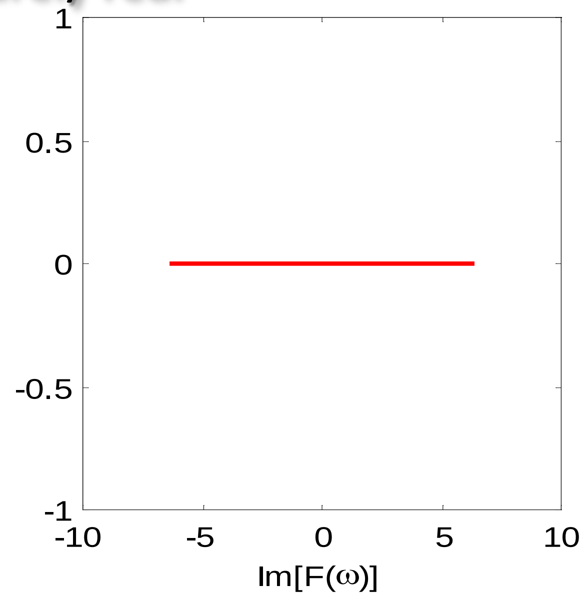

```
syms a t real
syms a positive
ft=exp(-a*abs(t));
Fw=fourier(ft)
Fw =
(2*a)/(a^2 + w^2)
```

$$F(\omega) = \frac{2a}{a^2 + \omega^2}$$

$$F(\nu) = \frac{2a}{a^2 + (2\pi\nu)^2}$$

$F(\omega)$ is purely real





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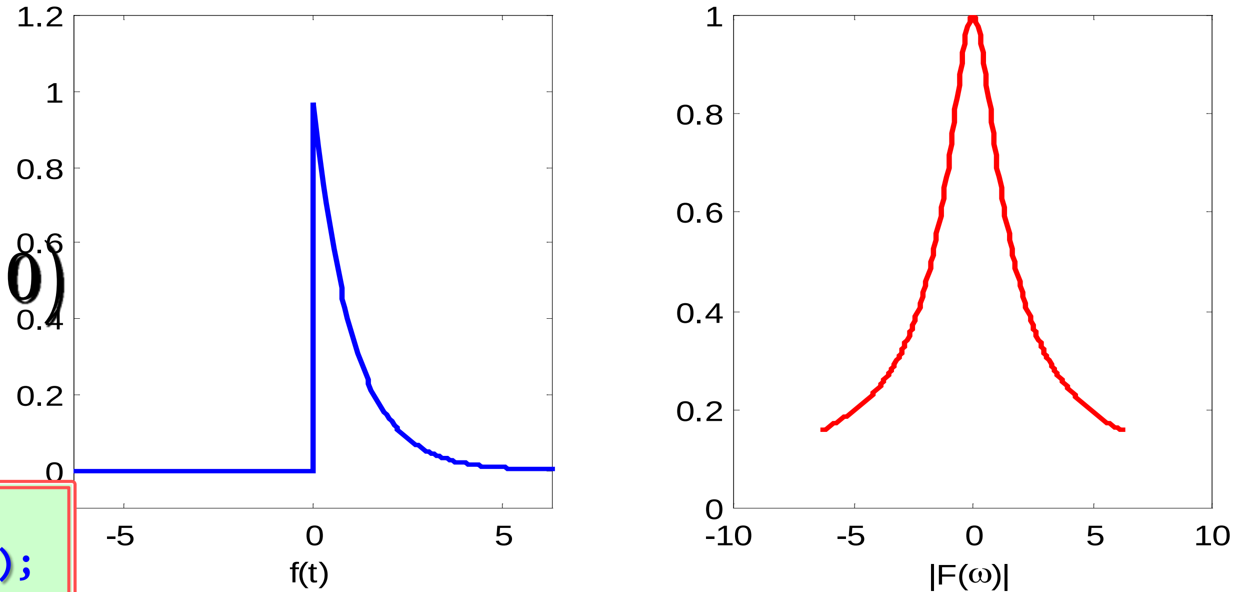
Example of Fourier Transform (neither even nor odd function)

$f(t)$ = **decay pulse**

$$f(t) = \begin{cases} e^{-at} & t>0 \\ 0 & t<0 \end{cases} \quad (a>0)$$

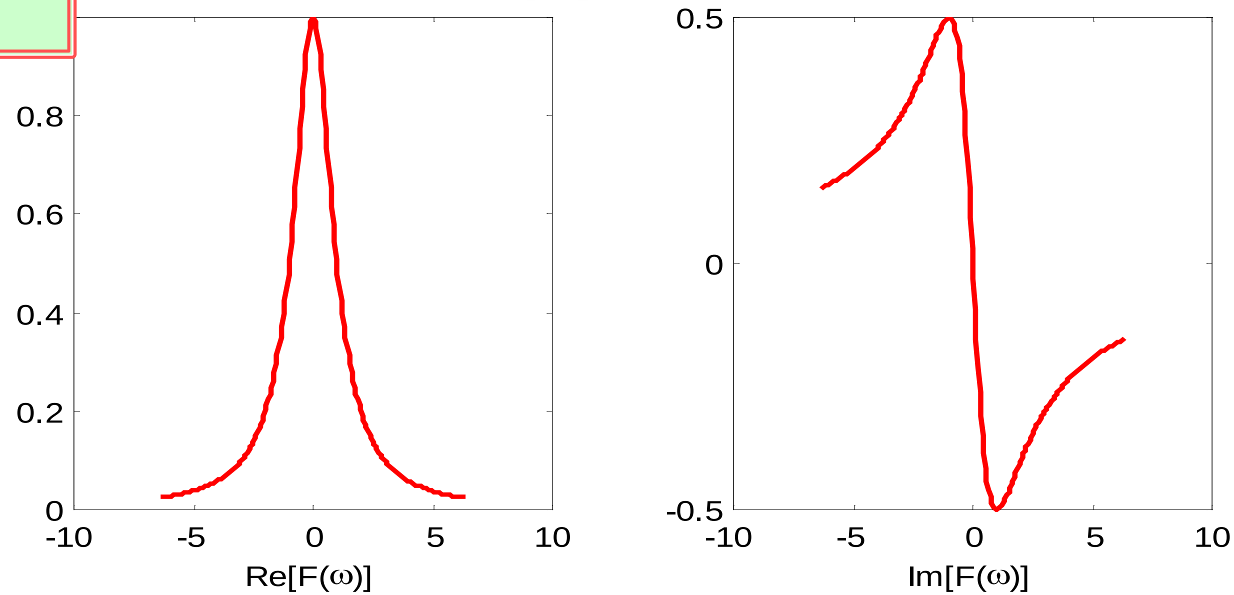



```
syms t real; syms a positive
ft=exp(-a*abs(t))*heaviside(t);
Fw=simplify(fourier(ft))
Fw =
1/(a + w*1i)
```

$$F(\omega) = \frac{1}{a + i\omega}$$

$$F(\nu) = \frac{1}{a + i2\pi\nu}$$

$F(\omega)$ is complex

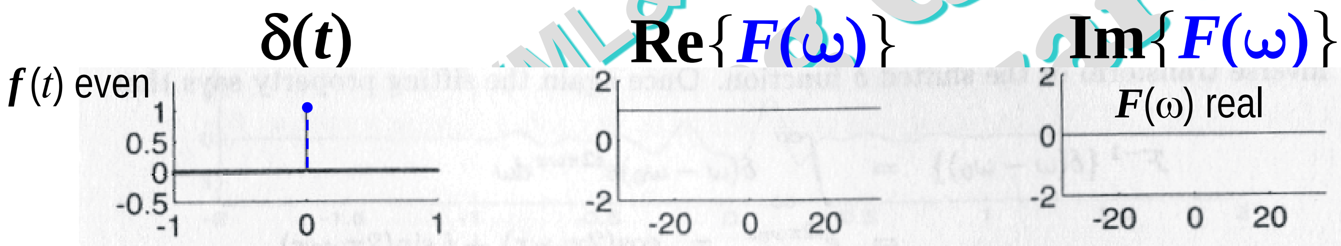


Examples of Fourier Transform

Dirac delta function

$$f(t) = \delta(t) \;\circ\!\!-\!\!\!-\!\!\bullet\; F(\omega) = 1$$

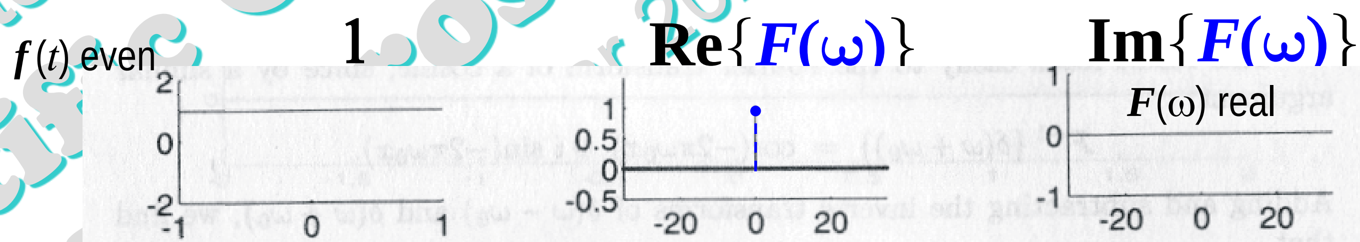
```
syms t; f=dirac(t);
F = fourier(f)
 F =
 1
```



constant function 1

$$f(t) = 1 \;\circ\!\!-\!\!\!-\!\!\bullet\; F(\omega) = 2\pi\delta(\omega)$$

```
fourier(sym(1))
ans =
2*pi*dirac(w)
```



Heaviside function

$$f(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \qquad F(\omega) = \pi\,\delta(\omega) - \frac{i}{\omega}$$

```
syms t real; fourier(heaviside(t))
ans =
pi*dirac(w) - 1i/w
```

Examples of Fourier Transform

trigonometric functions

$f(t)$ even; $F(\omega)$ is real

$$f(t) = \cos \alpha t \quad (\circ\!\!-\!\!\!-\!\!\bullet) \quad F(\omega) = \pi\left[\delta(\omega+\alpha) + \delta(\omega-\alpha)\right]$$

$f(t)$ odd; $F(\omega)$ is imaginary

$$f(t) = \sin \alpha t \quad (\circ\!\!-\!\!\!-\!\!\bullet) \quad F(\omega) = i\pi\left[\delta(\omega+\alpha) - \delta(\omega-\alpha)\right]$$

$$f(t) = e^{i\alpha t} \quad \circ\!\!-\!\!\!-\!\!\bullet \quad F(\omega) = 2\pi\delta(\omega-\alpha)$$

```
syms t a real
disp(fourier(cos(a*t)))
pi*(dirac(w-a)+dirac(w+a))
disp(fourier(sin(a*t)))
pi*(-dirac(w-a)+dirac(w+a))*1i
```

```
syms t a real
disp(fourier(exp(i*a*t)))
2*pi*dirac(w-a)
```