SIS Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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- Fourier Series.
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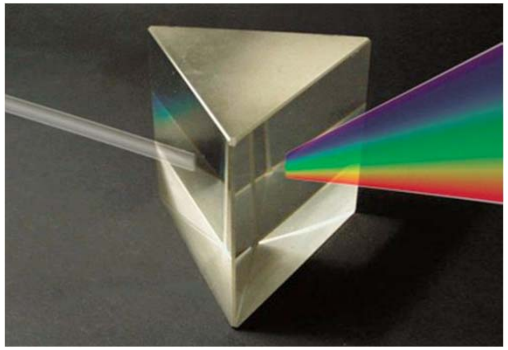
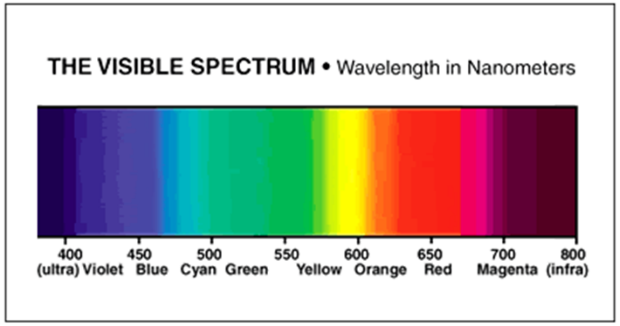
A light ray (as an electromagnetic wave) is formed by "lights" of different colors or frequencies, which can be separated by an optical prism. Each component is a monochromatic light with a sinusoidal pattern of a given frequency.





Fourier's idea

Jean-Baptiste Joseph Fourier
(1768 - 1830)

Fourier claimed that any function of a real variable, whether continuous or discontinuous, can be expanded in a series of sine functions of multiples of the variable. Though this result is not correct without additional conditions, Fourier's observation that some discontinuous functions are the sum of infinite series was a breakthrough.

continuous $f(x)=\frac{1}{2}x=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots$

discontinuous $f(x)=\operatorname{sgn}(x)=\frac{4}{\pi}\left[\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\cdots\right]$

?... what can we say

about convergence in $\|\cdot\|_2$

for Spaces having an...

infinite dimension

???

already seen in SC2_11f

It can be proved that the ***trigonometric functions***

$$\left\{\frac{1}{\sqrt{2\pi}},\left\{\frac{\cos(kx)}{\sqrt{\pi}},\frac{\sin(kx)}{\sqrt{\pi}}\right\}_k\right\}$$

or, equivalently, the ***exponential functions***

$$\left\{\frac{e^{ikx}}{\sqrt{2\pi}}\right\}_k$$

Euler's formula
$e^{i\theta}=\cos\theta+i\sin\theta$

form a ***complete orthonormal system*** w.r.t. $\|\cdot\|_2$ in the **Hilbert space** $L^2([-\pi,+\pi])$ of square integrable (or square summable) functions over $[-\pi,+\pi]$.

This implies that the **Fourier Series** of $f(x)\in L^2([-\pi,+\pi])$ **converges in mean square** or **in quadratic mean** (i.e., w.r.t. $\|\cdot\|_2$) to $f(x)$.

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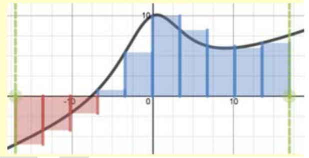
Theory

The **summability** of a function $\varphi(x)$ in the interval $[a, b]$ is a more general property than the Riemann integrability* because, even if the function is not continuous, it guarantees the existence and finiteness of the integral, i.e.:

$$\int_{\alpha}^{\beta} |\varphi(x)|\,\mathrm{d}x = \lim_{n} \int_{\alpha_n}^{\beta_n} |\varphi(x)|\,\mathrm{d}x < \infty$$

* The Riemann integral is the limit of the Riemann sums



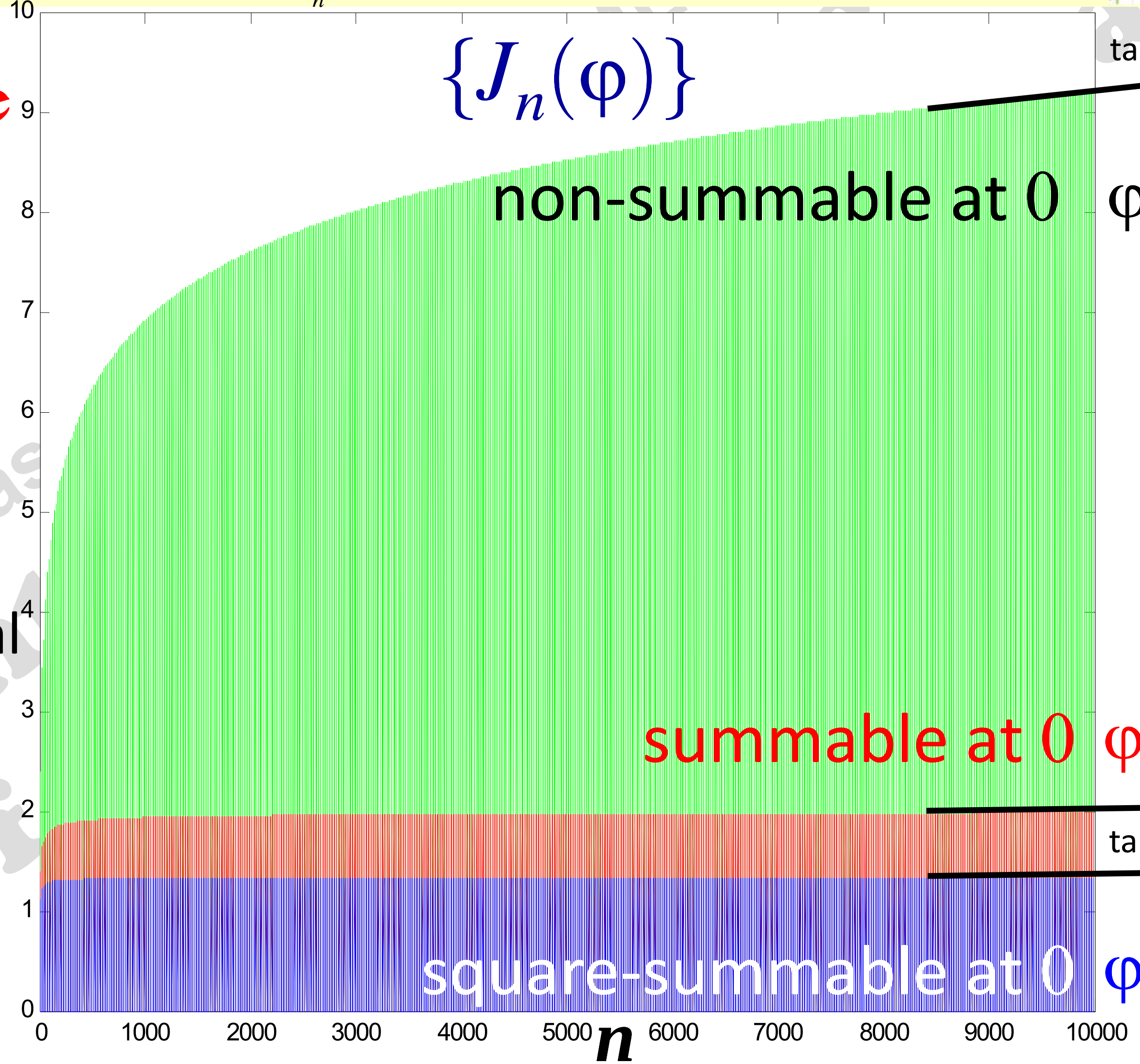
Example

plots of the sequence $\{J(n)\}$

$$J(n) = \int_{1/n}^{1} \varphi(x)\,\mathrm{d}x$$

for the integral

$$\int_{0}^{1} \varphi(x)\,\mathrm{d}x$$



$$\varphi(x) = \frac{1}{x}$$

$$\varphi(x) = \frac{1}{\sqrt{x}}$$

$$\varphi(x) = \frac{1}{\sqrt[4]{x}}$$

(prof. M. Rizzardi)

Examples: summable functions and not $\int_0^1 \varphi(x)\,dx$

```
syms x n positive; f0=1/x;
J0=int(f0,1/n,1)
J0 =
log(n)
[int(f0,0,1) limit(J0,n,inf)]
ans =
[  Inf,  Inf ]
fplot(J0,[1,1000],'Color','g')
```

$\{J_n(f)\}$

non-summable at 0

```
syms x n positive; f1=1/sqrt(x);
J1=int(f1,1/n,1)
J1 =
2 - 2/n^(1/2)
[int(f1,0,1) limit(J1,n,inf)]
ans =
[  2,  2]
fplot(J1,[1,1000],'Color','r')
```

$\{J_n(f)\}$

summable at 0

```
syms x n;  f2=1/sqrt(sqrt(x));
J2=int(f2,1/n,1)
J2 =
4/3 - 4/(3*n^(3/4))
[int(f2,0,1) limit(J2,n,inf)]
ans =
[  4/3,  4/3]
fplot(J2,[1,1000],'Color','b')
```

$\{J_n(f)\}$

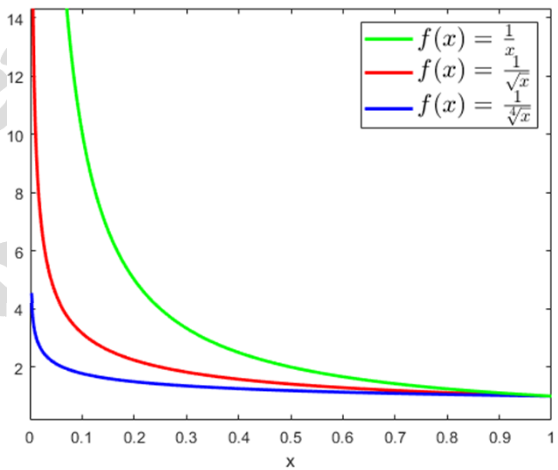
square summable at 0

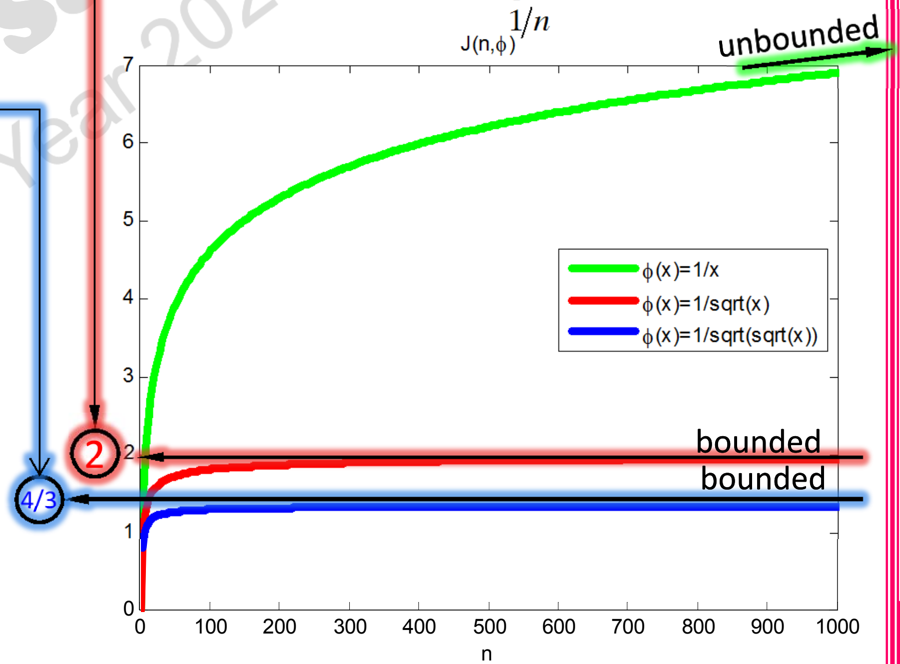
$$J(n) = \int_{1/n}^{1} \varphi(x)\,dx$$

∞

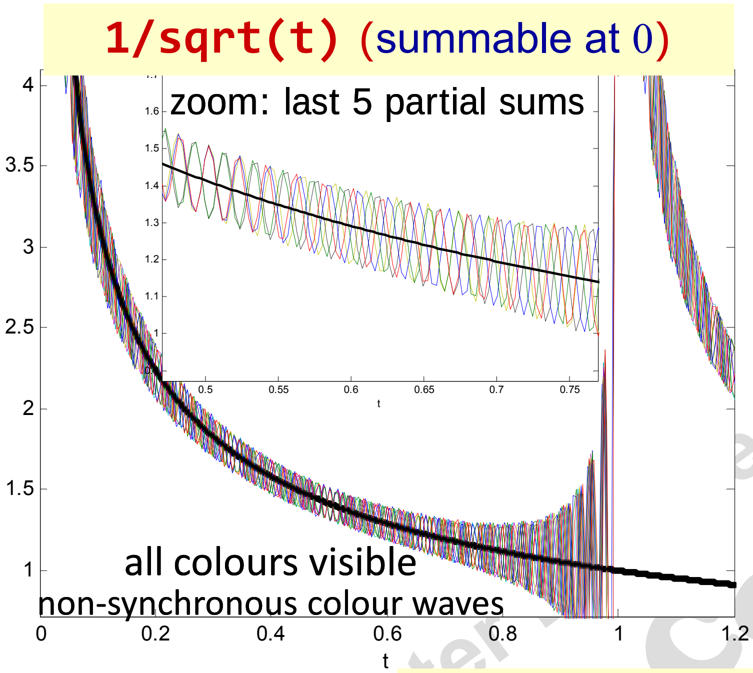
$\varphi(x)$

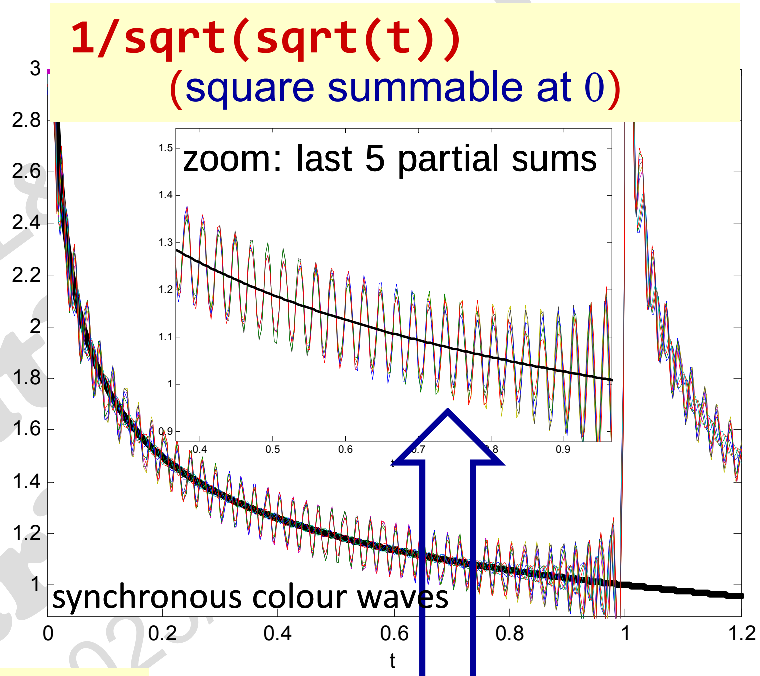
a singularity at $x=0$



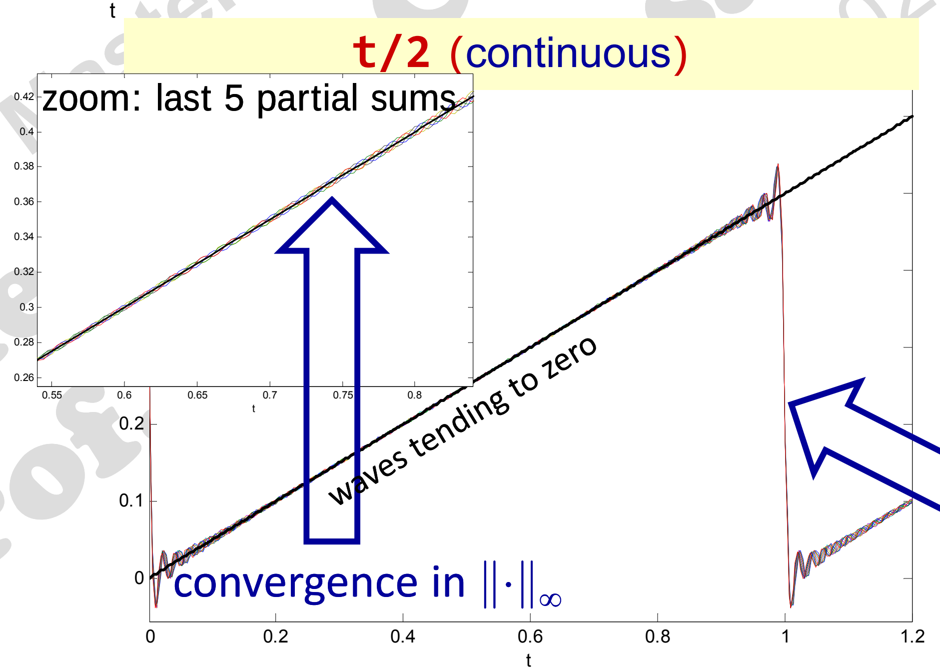


Examples: partial sums of Fourier Series in [0,1] of …





convergence in $\|\cdot\|_2$



convergence in $\|\cdot\|_\infty$

pointwise convergence

all the partial sums pass through the mean of the left and right limits

Fourier Series (FS) of f in $[-\pi,+\pi]$

A generic trigonometric series

$$\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\left[\alpha_k\cos(kx)+\beta_k\sin(kx)\right] \qquad \text{(real form)}$$

$$\sum_{k=-\infty}^{+\infty}\gamma_k e^{ikx} \qquad \text{(complex form)}$$

Partial Sum of order (N+1)
(N^th degree trigonometric polynomial)

$$\frac{\alpha_0}{2}+\sum_{k=1}^{N/2}\left[\alpha_k\cos(kx)+\beta_k\sin(kx)\right]$$

$$\sum_{k=-N/2}^{+N/2}\gamma_k e^{ikx}$$

N must be even

is said a **Fourier Series** of $f(x)$ in $[-\pi,+\pi]$, **by definition**, if its coefficients are defined as:

$\langle\cdot,\cdot\rangle$: scalar product

$$\alpha_k=\frac{\langle f,\cos kx\rangle}{\langle\cos kx,\cos kx\rangle}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,dx$$

$$\beta_k=\frac{\langle f,\sin kx\rangle}{\langle\sin kx,\sin kx\rangle}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx$$

$$\gamma_k=\frac{\langle f,e^{-ikx}\rangle}{\langle e^{-ikx},e^{-ikx}\rangle}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,e^{-ikx}\,dx$$

connection between real and complex coefficients

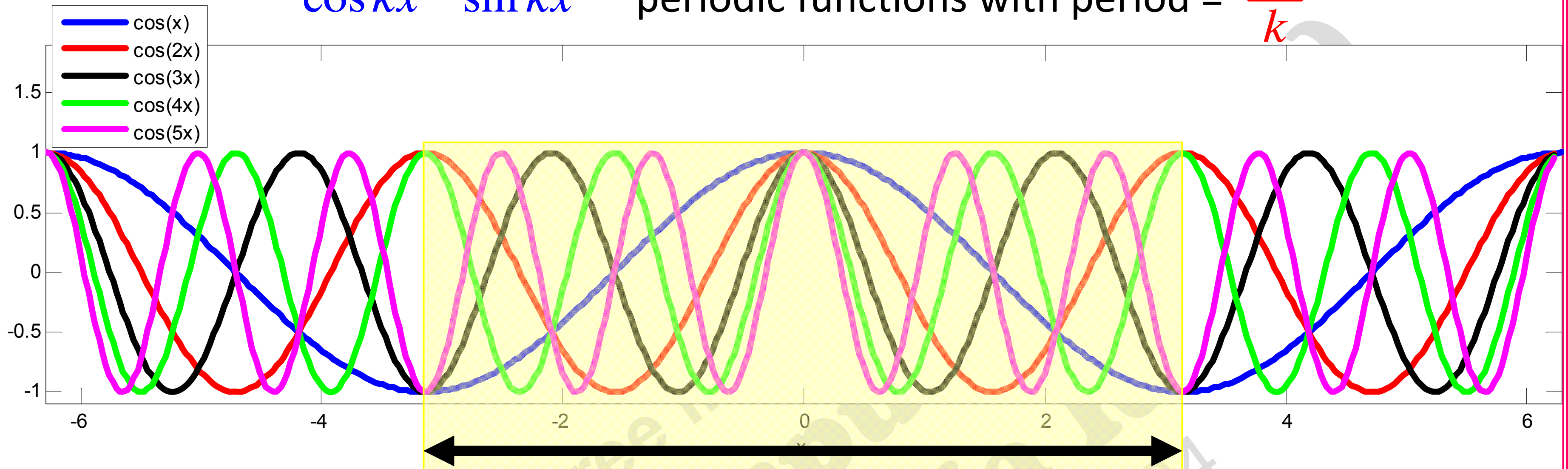
$$\gamma_0=\frac{\alpha_0}{2}$$

$$\left.\begin{aligned}\gamma_k&=\frac{\alpha_k-i\beta_k}{2}\\ \gamma_{-k}&=\frac{\alpha_k+i\beta_k}{2}\end{aligned}\right\}\quad k=1,2,\dots,+\infty$$

$$\begin{aligned}\alpha_0&=2\gamma_0\\ \alpha_k&=\gamma_k+\gamma_{-k}\\ \beta_k&=-i(\gamma_{-k}-\gamma_k)\end{aligned}$$

Exercise: Using the Symbolic Math Toolbox, verify relationships between real and complex Fourier coefficients

$\cos kx \quad \sin kx$ periodic functions with period = $\dfrac{2\pi}{k}$



common period: 2π

$$\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\left[\boxed{\alpha_k \cos kx}+\boxed{\beta_k \sin kx}\right]$$

elementary waves

A trigonometric series, if convergent, always has a continuous and periodic sum with period 2π.

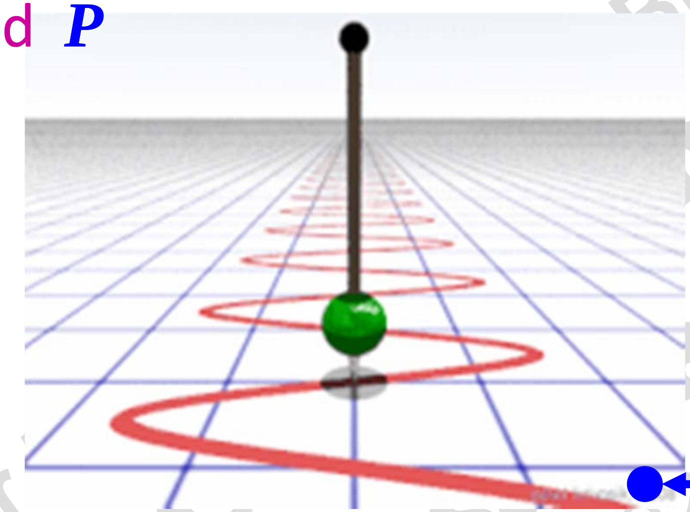
Partial sums of a trigonometric series are trigonometric polynomials

... physical interpretation: recalls

elementary waves

periodic with period P

$$P = \frac{2\pi}{\omega}$$



simple harmonic motion

$$x(t) = A \sin[\alpha + \omega t]$$

or equivalently

$$x(t) = A \cos[\underbrace{\alpha + \omega t}_{\text{phase}}]$$

amplitude (A)

phase

initial phase (α)

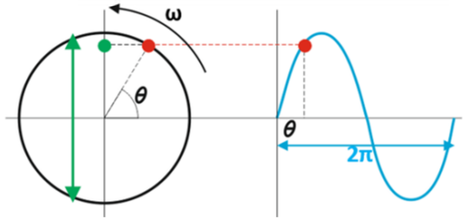
frequency $\nu = \dfrac{1}{P}$

If the period P is measured in **seconds**, then the frequency ν is measured in Hertz = cycles sec^{-1}

circular frequency ν — Hz = cycles sec^{-1}

angular frequency ω — radians sec^{-1}

$$\boxed{\omega = 2\pi\nu}$$



frequency ν:
number of complete cycles or oscillations per time unit.

frequency ω:
angular speed, i.e. the change speed of the angle

Trigonometric polynomials

N th degree algebraic polynomial ($\Leftrightarrow$ N+1 coefficients)

$$P_N(x) = c_0 + c_1\,x + c_2\,x^2 + \cdots + c_N\,x^N$$

N th degree trigonometric polynomials (N+1 coefficients)

periodic functions of period 2π

$$S_N(x) = \alpha_0 + \alpha_1 \cos x + \beta_1 \sin x + \alpha_2 \cos 2x + \beta_2 \sin 2x + \cdots$$

$$\cdots + \alpha_{\frac{N}{2}} \cos \frac{N}{2} x + \beta_{\frac{N}{2}} \sin \frac{N}{2} x$$

in **real form** $(S_N(x) \in \mathbb{R})$

by replacing: $\cos kx = \dfrac{e^{ikx} + e^{-ikx}}{2}; \quad \sin kx = \dfrac{e^{ikx} - e^{-ikx}}{2i}$

... it becomes:

N even

$$T_N(x) = \gamma_{-\frac{N}{2}} e^{-i\frac{N}{2}x} + \cdots + \gamma_{-2} e^{-i2x} + \gamma_{-1} e^{-ix} + \gamma_0 +$$

$$+ \gamma_1 e^{ix} + \gamma_2 e^{i2x} + \cdots + \gamma_{+\frac{N}{2}} e^{+i\frac{N}{2}x}$$

in **complex form** $(T_N(x) \in \mathbb{C})$

Trigonometric polynomials

A particular **trigonometric polynomial** $Q(x)$ with $N+1$ coefficients is the following:

$$Q(x) = c_0 + c_1 e^{ix} + c_2 e^{i2x} + c_3 e^{i3x} + \cdots + c_N e^{iNx}$$

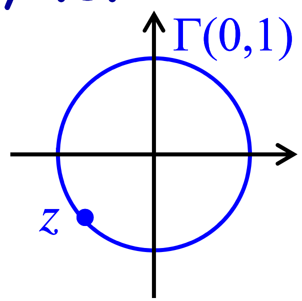
in **complex form** $(Q(x) \in \mathbb{C})$

$Q(x)$ is a particular $2N^{\text{th}}$ trigonometric polynomial with N null coefficients (those with negative indices)

it is a periodic function of period 2π

by setting $z = e^{ix}$, for $x \in \mathbb{R}$, $Q(x)$ becomes $Q(z)$, i.e.

$$Q(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_N z^N$$



$Q(z)$ looks like an N^{th} degree algebraic polynomial, computed for $z = e^{ix} \in \Gamma(0,1)$ (i.e. on the unitary circle centered at $\mathbf{O}$): z is periodic

Connection between trigonometric and algebraic polynomials

N even

In general, if we write

$$T_N(x) = \gamma_{-\frac{N}{2}} e^{-i\frac{N}{2}x} + \cdots + \gamma_{-2} e^{-i2x} + \gamma_{-1} e^{-ix} + \gamma_0 + \gamma_1 e^{ix} + \gamma_2 e^{i2x} + \cdots + \gamma_{+\frac{N}{2}} e^{+i\frac{N}{2}x}$$

as

$$T_N(x) = \boxed{e^{-i\frac{N}{2}x}} \Big(\gamma_{-\frac{N}{2}} + \gamma_{-\frac{N}{2}+1} e^{ix} + \gamma_{-\frac{N}{2}+2} e^{i2x} + \cdots + \gamma_0 e^{i\frac{N}{2}x} + \\ + \gamma_1 e^{i(\frac{N}{2}+1)x} + \gamma_2 e^{i(\frac{N}{2}+2)x} + \cdots + \gamma_{+\frac{N}{2}} e^{iNx} \Big)$$

the common factor was extracted

by setting $z = e^{ix}$, for $x \in \mathbb{R}$, $T_N(x)$ changes into $T_N(z)$, i.e.

$$T_N(z) = z^{-\frac{N}{2}} \Big(\gamma_{-\frac{N}{2}} + \gamma_{-\frac{N}{2}+1} z + \gamma_{-\frac{N}{2}+2} z^2 + \cdots + \gamma_0 z^{\frac{N}{2}} + \\ + \gamma_1 z^{\frac{N}{2}+1} + \gamma_2 z^{\frac{N}{2}+2} + \cdots + \gamma_{+\frac{N}{2}} z^N \Big)$$

$\boxed{e^{-i\frac{N}{2}x}}$

$$T_N(z) = z^{-\frac{N}{2}} Q(z) \Leftarrow Q(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_N z^N$$

How can we evaluate a trigonometric polynomial in MATLAB?

any N

$$Q(x) = c_0 + c_1 e^{ix} + c_2 e^{i2x} + \cdots + c_N e^{iNx}$$

$$Q(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_N z^N \quad \Leftarrow z = e^{ix}$$

```
z=exp(i*x);
Q=polyval(fliplr(c), z);
```

fliplr : if c=[c_0, c_1, c_2,..., c_N] is a row-wise vector,
flipud: if c is a column-wise vector

`fliplr(A)` flip left right: flip array left to right

`flipud(A)` flip up down: flip array up to down

even N

$$T_N(z) = z^{-\frac{N}{2}}\left(\gamma_{-\frac{N}{2}} + \gamma_{-\frac{N}{2}+1} z + \gamma_{-\frac{N}{2}+2} z^2 + \cdots + \gamma_0 z^{\frac{N}{2}} + \right.$$
$$\left. + \gamma_1 z^{\frac{N}{2}+1} + \gamma_2 z^{\frac{N}{2}+2} + \cdots + \gamma_{+\frac{N}{2}} z^N \right)$$

$$T_N(z) = \boxed{z^{-\frac{N}{2}}}\underbrace{\left(c_0 + c_1 z + c_2 z^2 + \cdots + \cdots + c_N z^N\right)}_{Q(z)} \quad \Leftarrow z = e^{ix}$$

```
z=exp(i*x);   T_N = z.^(-N/2) .* polyval(fliplr(c), z);
```

Example of Fourier Series in $[-\pi,+\pi]$

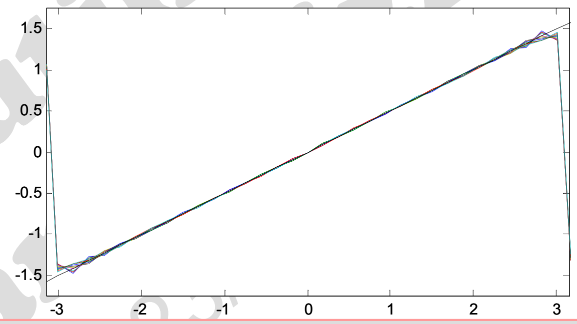
$$\phi(x) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}[\alpha_k \cos kx + \beta_k \sin kx], \qquad \begin{cases} \alpha_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \phi(t)\cos(kt)\, dt \\ \beta_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \phi(t)\sin(kt)\, dt \end{cases}$$

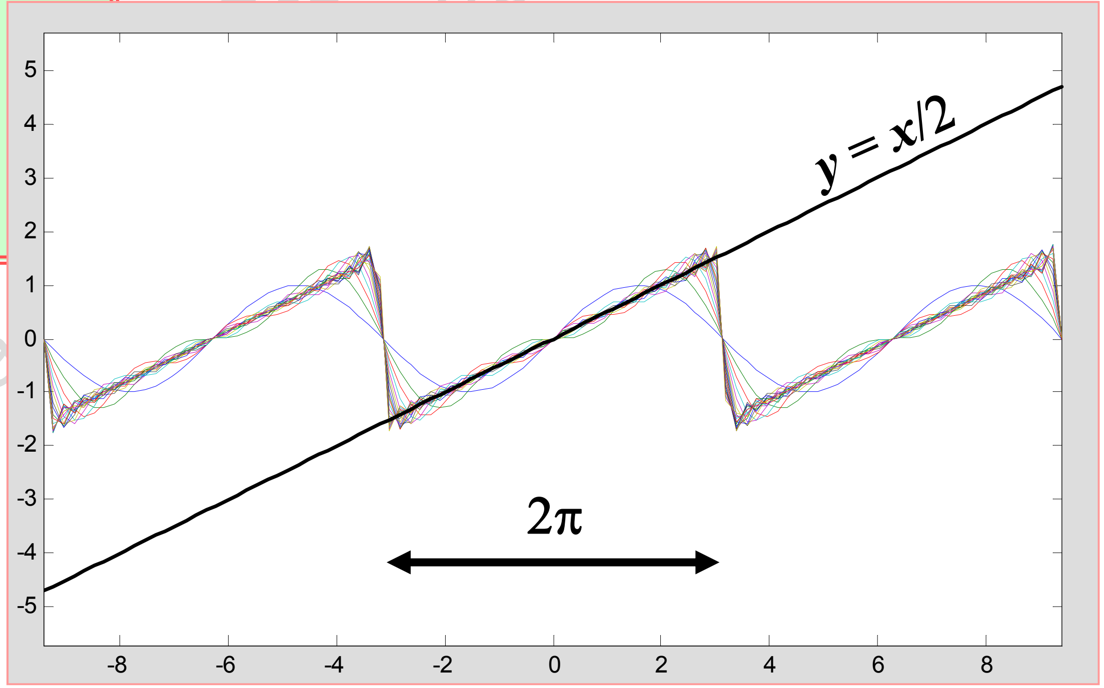
$$\phi(x) = \frac{1}{2}x \sim \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

order of partial sums

```
syms t;  f = t/2;  k=(0:60)';
a(1+k)=1/pi*int(f.*cos(k*t),t,-pi,pi);
b(1+k)=1/pi*int(f.*sin(k*t),t,-pi,pi);
double([a' b'])
ans =
    0         0
    0         1              1
    0      -0.5           -1/2
    0    0.3333            1/3
    0     -0.25           -1/4
    0       0.2            1/5
…
```

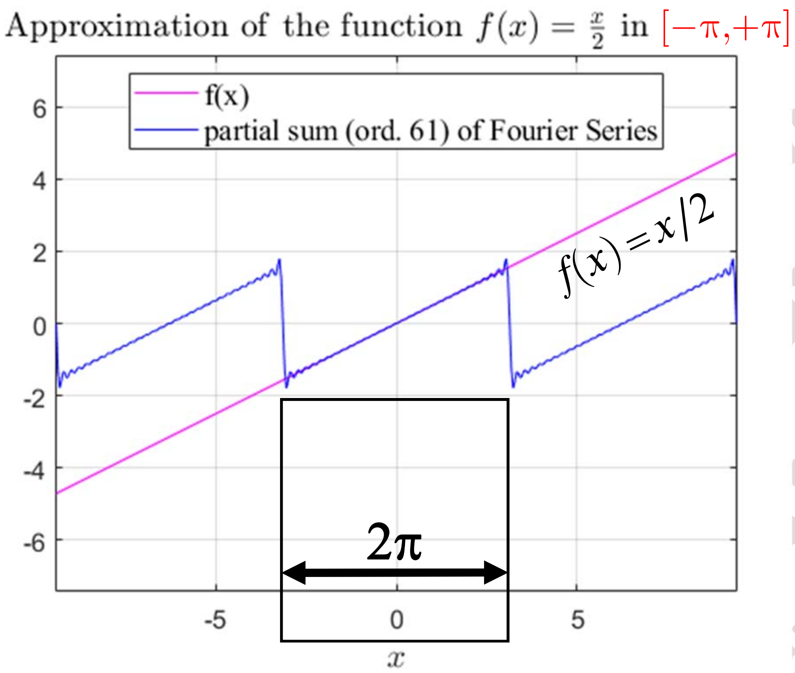
zoom of last 10 partial sums

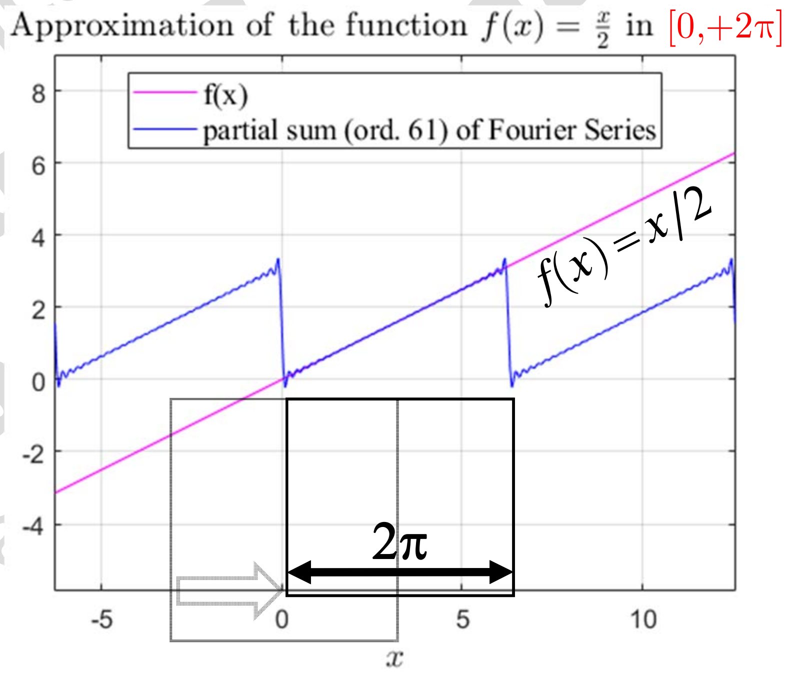




Example of Fourier Series

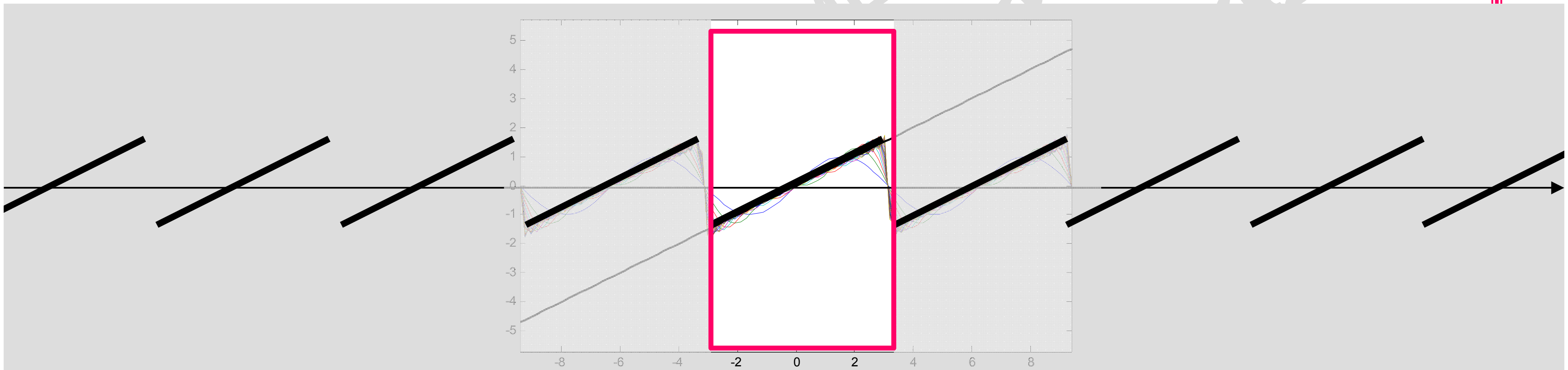
$$\phi(x)=\frac{1}{2}x \sim \sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots$$





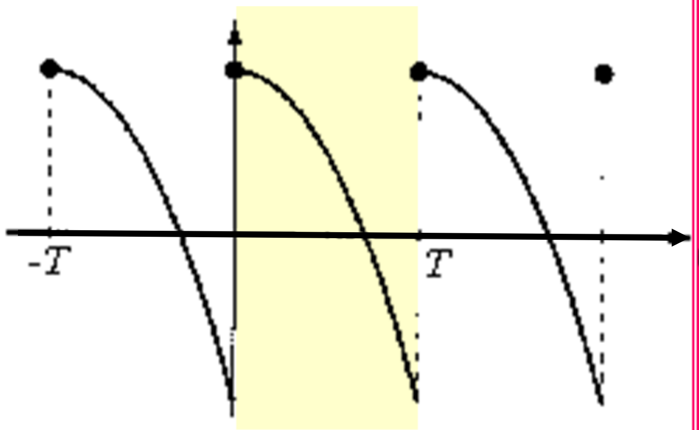
The Fourier Series fits the restriction of the function $f(x)$ to the interval where it has been constructed, and then repeats itself periodically.

The Fourier Series di $\phi(t)$ in an interval of width T describes the periodic repetition of $\phi_T(x)$, where $\phi_T(x)$ is the restriction of $\phi(x)$ to that interval.



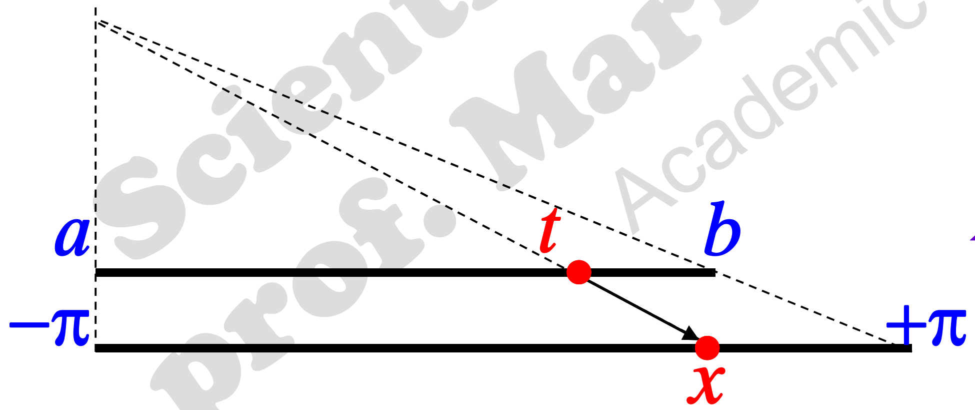
In general the periodic repetition of $\phi_T(t)$ with period T is defined, in the interval $[0,T]$, as

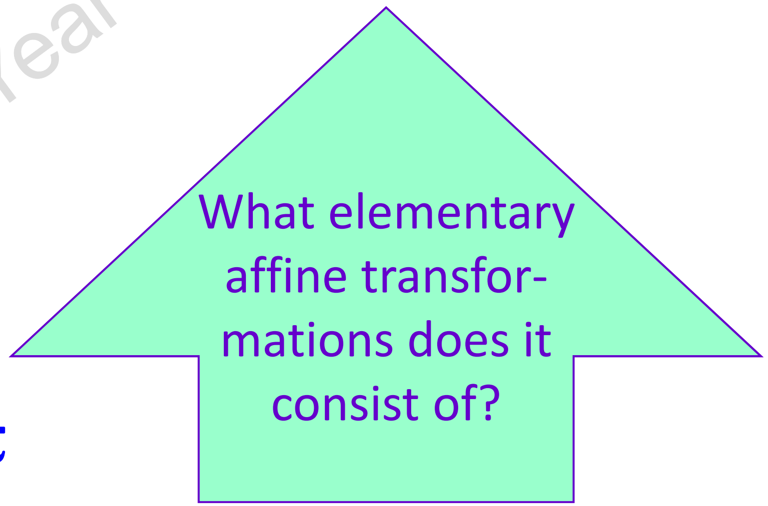
$$\tilde{\phi}_T(t) = \begin{cases} \phi(t) & t \in [0,T[\\ \tilde{\phi}_T(t-T) & t \in [T,+\infty[\\ \tilde{\phi}_T(t+T) & t \in]-\infty,0[\end{cases}$$



Of course, in the previous definition of Fourier Series, both the interval $[-\pi,+\pi]$ and the period 2π are not a limitation. In facts, by means of the following **affinity** a FS can be written for the generic interval $[a, b]$ and with $b-a$ as its period:

$$\boxed{t \in [a,b] \longrightarrow x = \frac{2\pi}{b-a}(t-a) - \pi \in [-\pi,+\pi]}$$





Fourier Series (FS) of f in $[a,b]$ with period $b-a$

$$\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}[\alpha_k\cos(kx)+\beta_k\sin(kx)]$$

$$t\in[a,b]\longrightarrow x=\frac{2\pi}{b-a}(t-a)-\pi\in[-\pi,+\pi]$$

$$\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\left\{\alpha_k\cos\left\{k\left[\frac{2\pi}{b-a}(t-a)-\pi\right]\right\}+\beta_k\sin\left\{k\left[\frac{2\pi}{b-a}(t-a)-\pi\right]\right\}\right\}$$

real form

$$\begin{cases}\alpha_k=\frac{2}{b-a}\int_a^b f(\tau)\cos\left\{k\left[\frac{2\pi}{b-a}(\tau-a)-\pi\right]\right\}d\tau\\ \beta_k=\frac{2}{b-a}\int_a^b f(\tau)\sin\left\{k\left[\frac{2\pi}{b-a}(\tau-a)-\pi\right]\right\}d\tau\end{cases}$$

real coefficients

$$\sum_{k=-\infty}^{+\infty}\gamma_k e^{ik\left[\frac{2\pi}{b-a}(t-a)-\pi\right]}$$

complex form

$$\gamma_k=\frac{1}{b-a}\int_a^b f(\tau)\,e^{-ik\left[\frac{2\pi}{b-a}(\tau-a)-\pi\right]}d\tau$$

complex coefficients

particular case

Fourier Series (FS) of f with period T in $\left[-\frac{T}{2},+\frac{T}{2}\right]$

$$t\in\left[-\frac{T}{2},\frac{T}{2}\right]\longrightarrow x=\frac{2\pi}{T}t\in[-\pi,+\pi]$$

$$\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\left[\alpha_k\cos\left(\frac{2k\pi}{T}t\right)+\beta_k\sin\left(\frac{2k\pi}{T}t\right)\right] \longleftarrow \text{(real form)}$$

$$\begin{cases}\alpha_k=\frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos\left(\frac{2k\pi}{T}\tau\right)d\tau\\ \beta_k=\frac{2}{T}\int_{-T/2}^{+T/2} f(x)\sin\left(\frac{2k\pi}{T}\tau\right)d\tau\end{cases}$$

$$\sum_{k=-\infty}^{+\infty}\gamma_k e^{ik\frac{2\pi}{T}t} \longleftarrow \text{(complex form)}$$

$$\gamma_k=\frac{1}{T}\int_{-T/2}^{T/2} f(x)\,e^{-ik\frac{2\pi}{T}\tau}d\tau$$

(prof. M. Rizzardi)

Example of Fourier Series in $[a,b]$

$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{ik\left[\frac{2\pi}{b-a}(t-a)-\pi\right]}$$

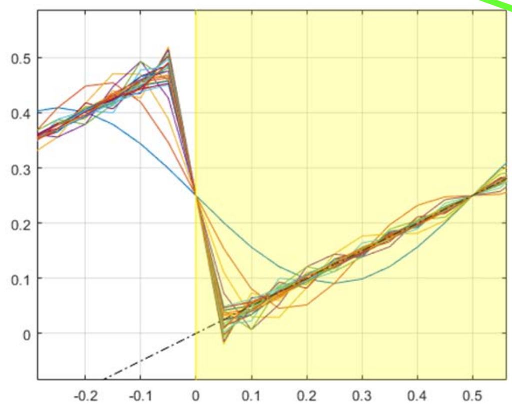
$$\boxed{\gamma_k = \frac{1}{b-a}\int_a^b f(\tau)\, e^{-ik\left[\frac{2\pi}{b-a}(\tau-a)-\pi\right]} d\tau}$$ in complex form

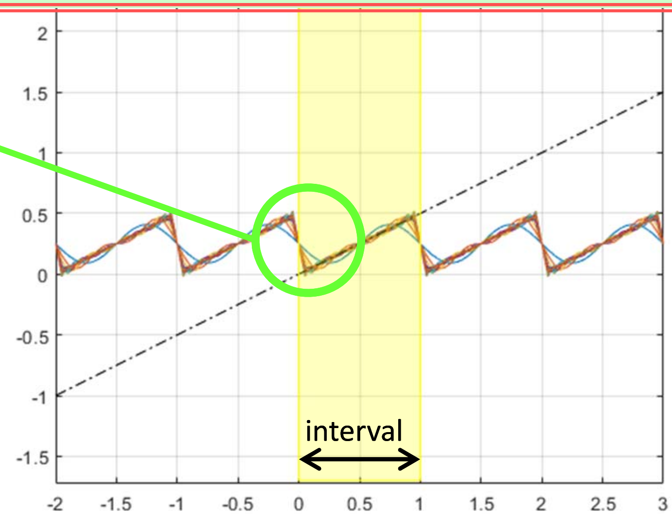
draw all the partial sums

```
f=@(tau)tau/2;  a=0;  b=1;  T=b-a;
syms t real
Nmez=30;  k=(-Nmez:Nmez)';  m=Nmez+1; % middle index of coefficients
c(m+k)=double( 1/T*int(f(t).*exp(-1i*k*(2*pi/T*(t-a)-pi)), t,a,b));
c=fliplr(c); % to use polyval
t=linspace(a-2,b+2,101);  x=2*pi/T*(t-a)-pi; % project [a,b] onto [-pi,+pi]
z=exp(1i*x); % change of variable
S=zeros(numel(t),Nmez); % matrix of partial sums
for k=1:Nmez
    S(:,k)=z.^(-k).*polyval(c(m-k:m+k),z); % partial sum of order 2*k+1
end
figure; plot(t,real(S)',t,f(t),'-.k'); axis equal; grid
```

evaluate a trigonmetric polynomial

zoom





Existence and convergence of Fourier Series
(sufficient conditions)

$f(x) \in \boldsymbol{L}^1[a,b]$
(f summable) ⟹ **Fourier Series** of f exists

$f(x) \in \boldsymbol{L}^2[a,b]$
(f square summable) ⟹ **Fourier Series** of f converges to f in quadratic mean ($\|\cdot\|_2$)

$f(x) \in \boldsymbol{L}^2[a,b]$
and f satisfies the Dirichlet conditions ⟹ **Fourier Series** of f converges pointwise to f and uniformly ($\|\cdot\|_\infty$) where f is continuous

Dirichlet conditions

- f is a bounded function in $[a,b]$;
- a finite partition of $[a,b]$ exists:
  $$a = x_1 < x_2 < \cdots < x_m = b$$
  such that f is monotonic in every $[x_i, x_{i+1}]$.

i.e.: f is bounded and with only discontinuities of the first kind

(prof. M. Rizzardi)

Convergence of Fourier Series
(sufficient conditions)

Teor.: $f(x) \in \boldsymbol{L}^2[a,b]$ and f satisfies the Dirichlet conditions ⟹ **Fourier Series** of f converges pointwise to f and uniformly ($\|\cdot\|_\infty$) where f is continuous

⇓

more precisely:
the **Fourier Series of** f converges pointwise in $[a,b]$:
its **sum** equals

- $f(x)$ if f is continuous at x;
- $\frac{1}{2}[f(x_-) + f(x_+)]$ if f is discontinuous at x;

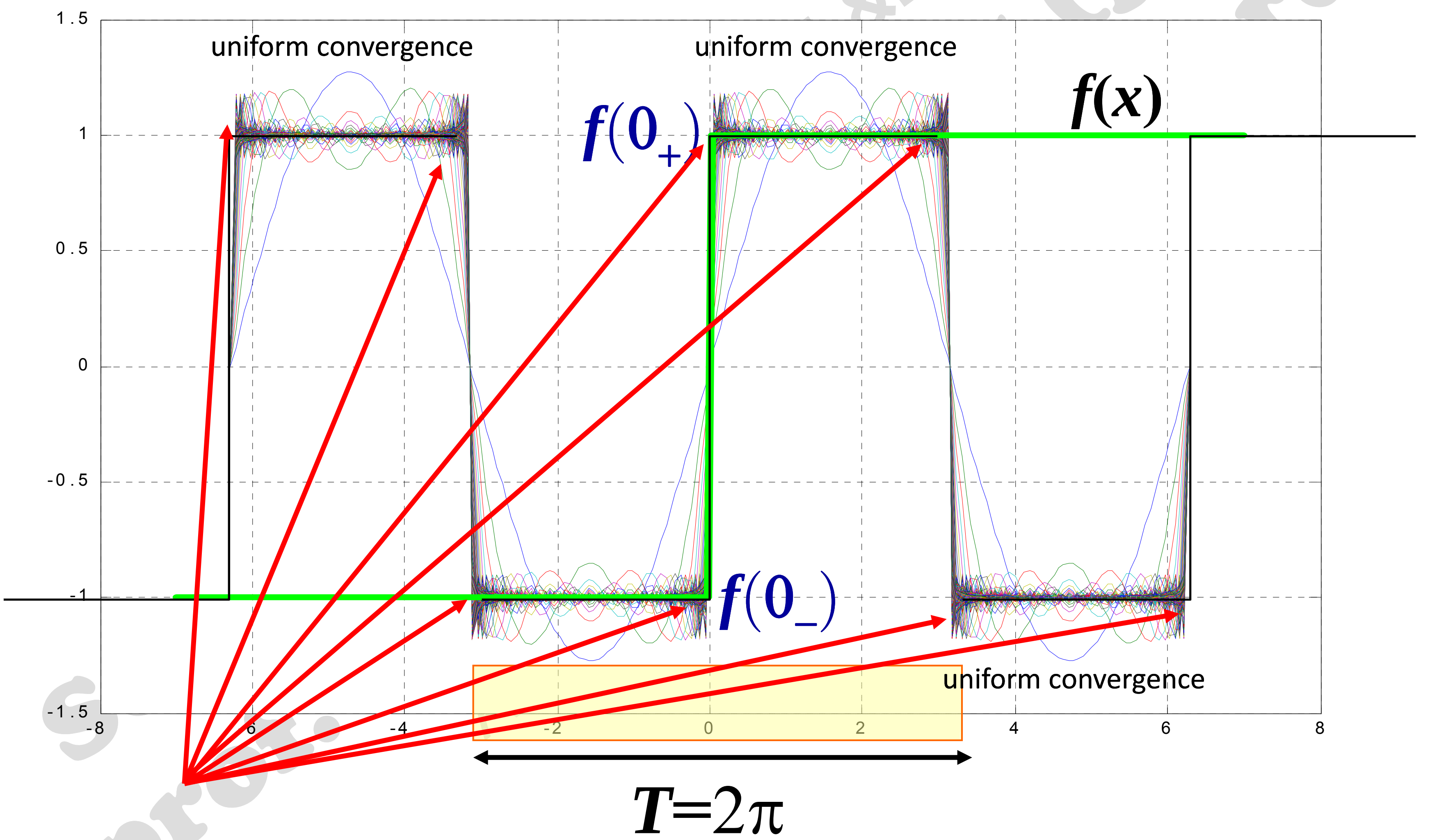
around a discontinuity jump, it shows the Gibbs' phenomenon.

Moreover, the convergence is uniform in every subinterval of $[a,b]$ where f is continuous.

DEF ⟹ f can be expanded as a Fourier Series

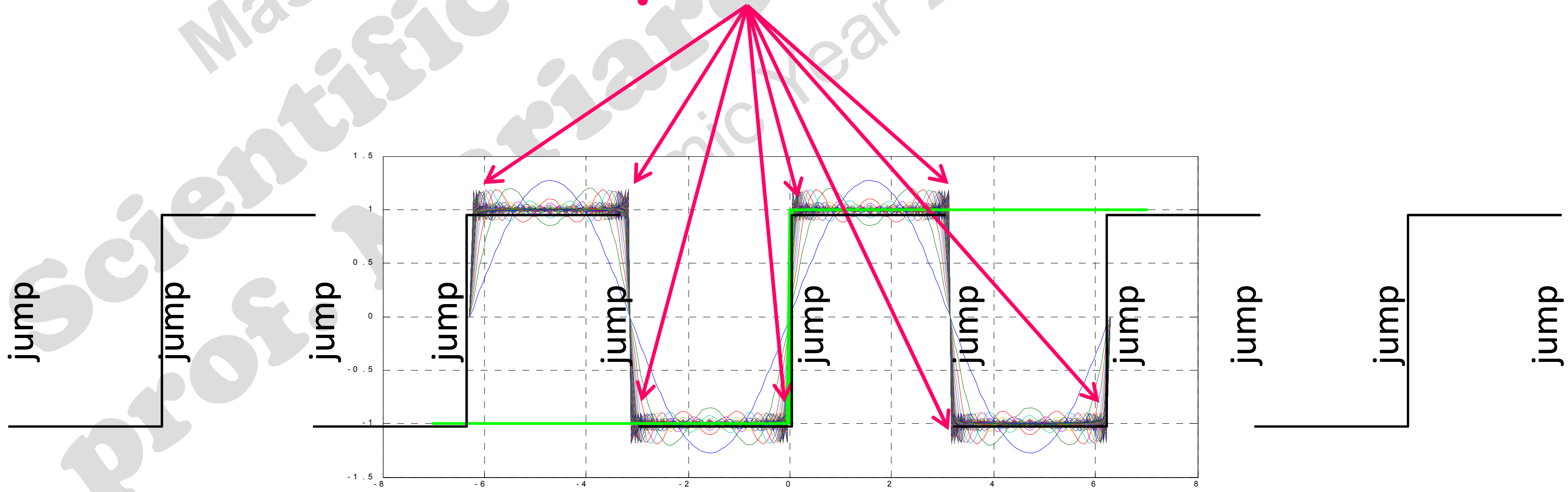
Example: function that can be expanded as a FS

$$x \in [-\pi, \pi] \quad f(x) = \mathrm{signum}(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$



Gibbs' phenomenon at each jump discontinuity

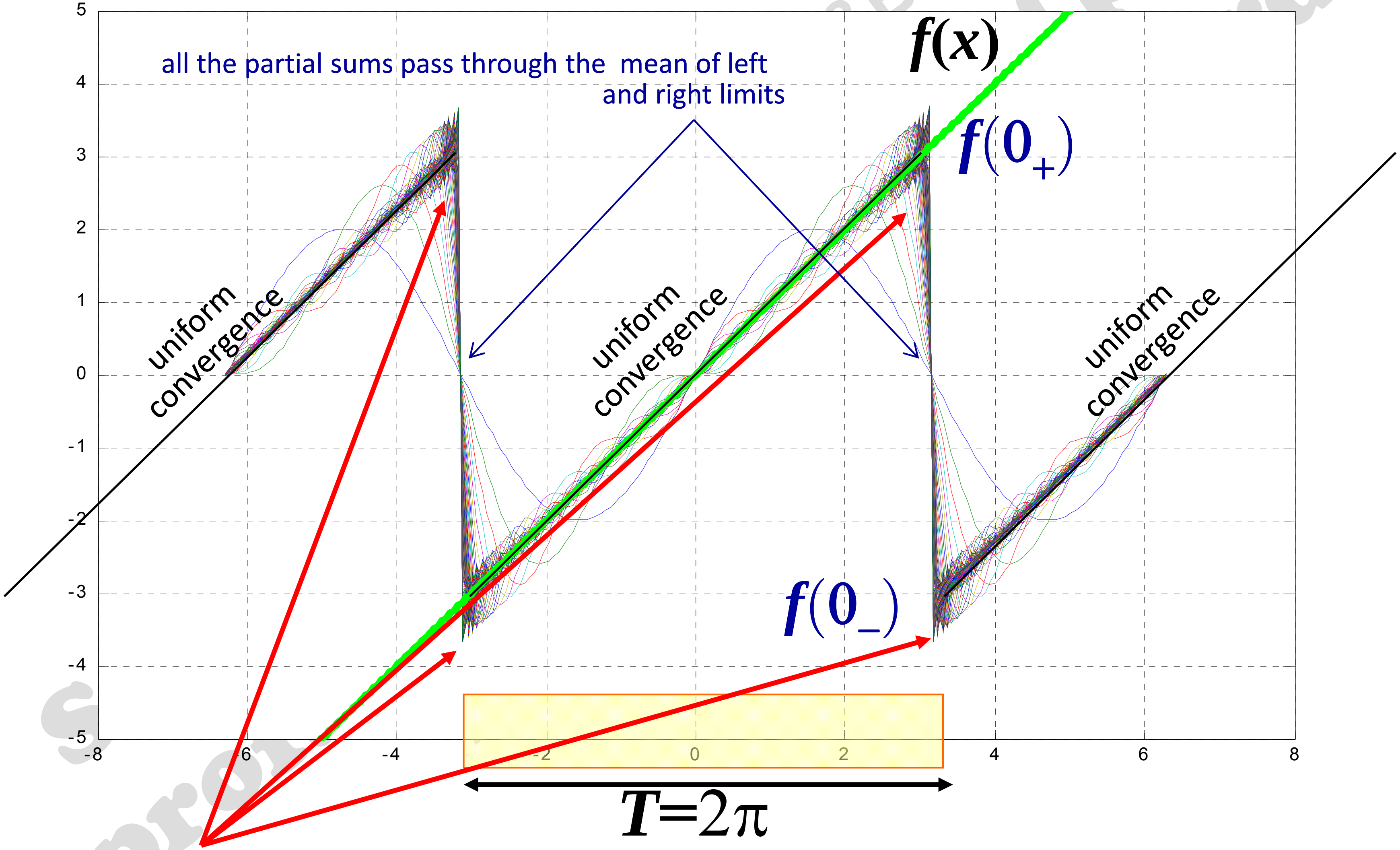
Around each jump discontinuity of the periodic repetition of $f(x)$, **partial sums of the Fourier series of $f(x)$** show spurious oscillations which, as the order N increases, do not decrease in amplitude even if they always occur in increasingly narrow intervals. This phenomenon is called the **Gibbs phenomenon**.



periodic repetition of the signum() function in $[-\pi,+\pi]$

Example: function that can be expanded as a FS

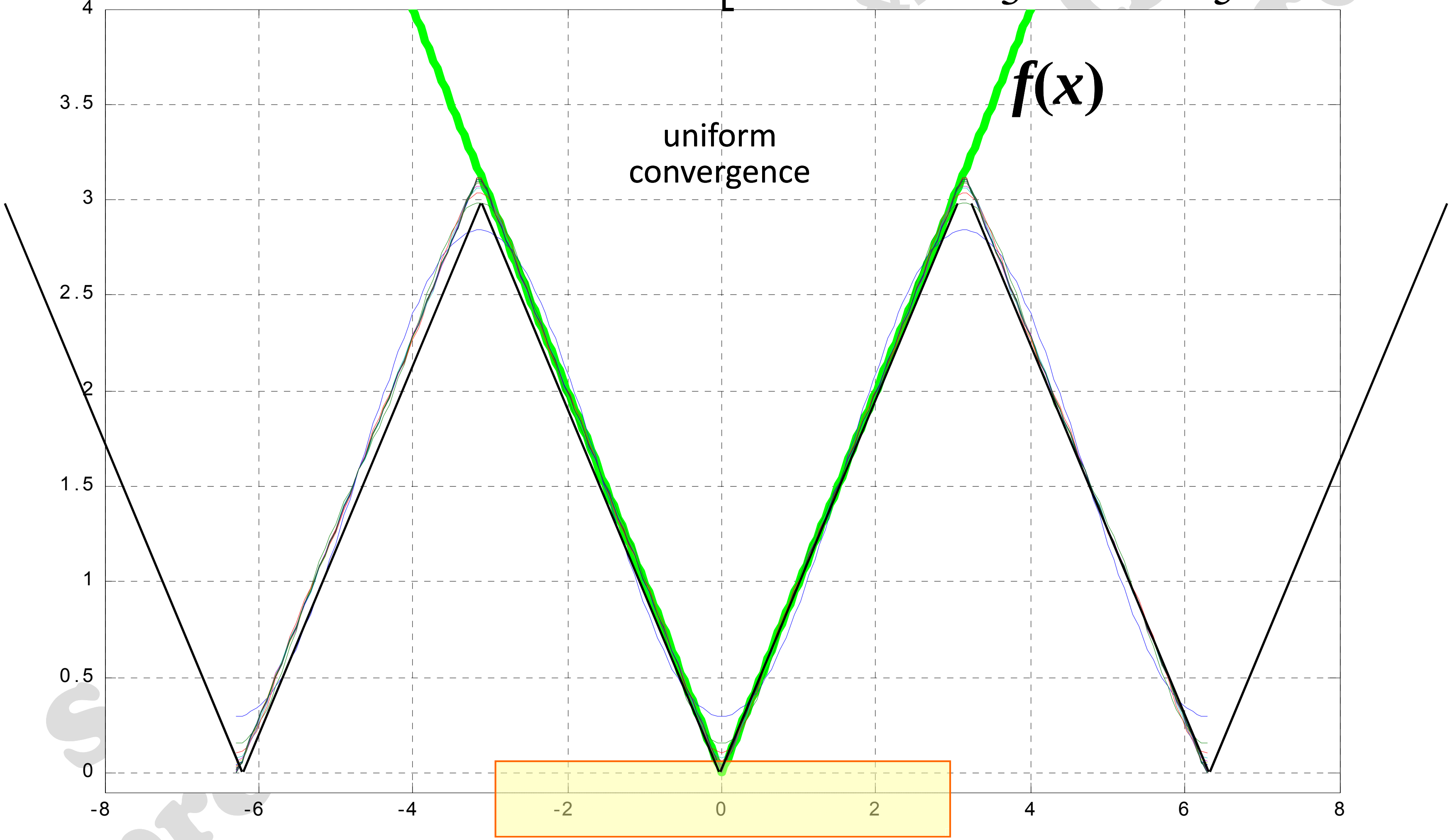
$$x \in [-\pi, \pi] \quad f(x) = x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$



Gibbs' phenomenon at each jump discontinuity

Example: function that can be expanded as a FS

$$x\in[-\pi,\pi] \quad f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$



$T=2\pi$

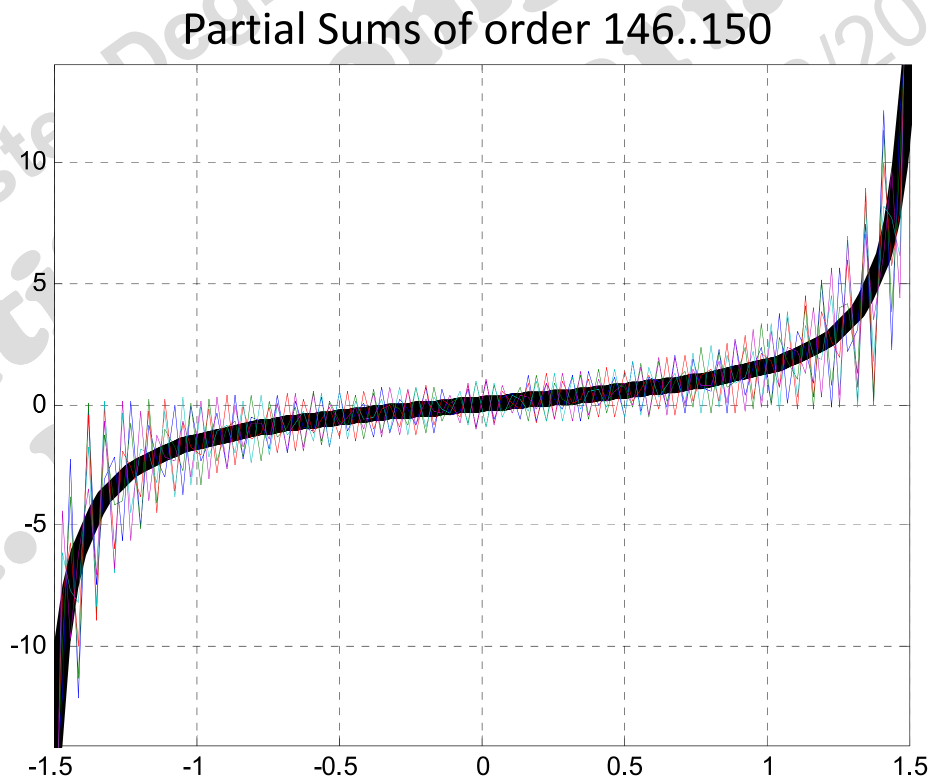
No Gibbs' phenomenon

Example: function that cannot be expanded as a FS

$$x \in \left]-\frac{\pi}{2}, +\frac{\pi}{2}\right[\qquad \tan(x) \sim 2\left[\sin(2x) - \sin(4x) + \sin(6x) - \sin(8x) + \cdots\right]$$

Fourier Series of $\tan(x)$ exists,

but it does not converge at any point, except 0



Example: function that cannot be expanded as a FS

distributions

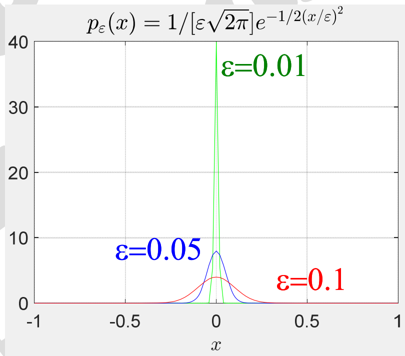
the Dirac* delta function δ

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\boxed{\int_{-\infty}^{+\infty} \delta(x)\,dx = 1} \Rightarrow \int_{-\infty}^{+\infty} f(x)\,\delta(x-c)\,dx = f(c)$$

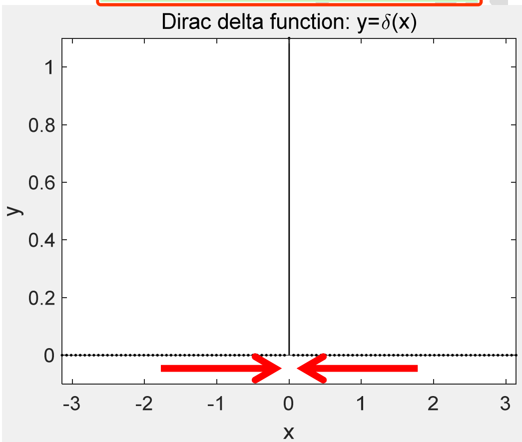
(sifting property)

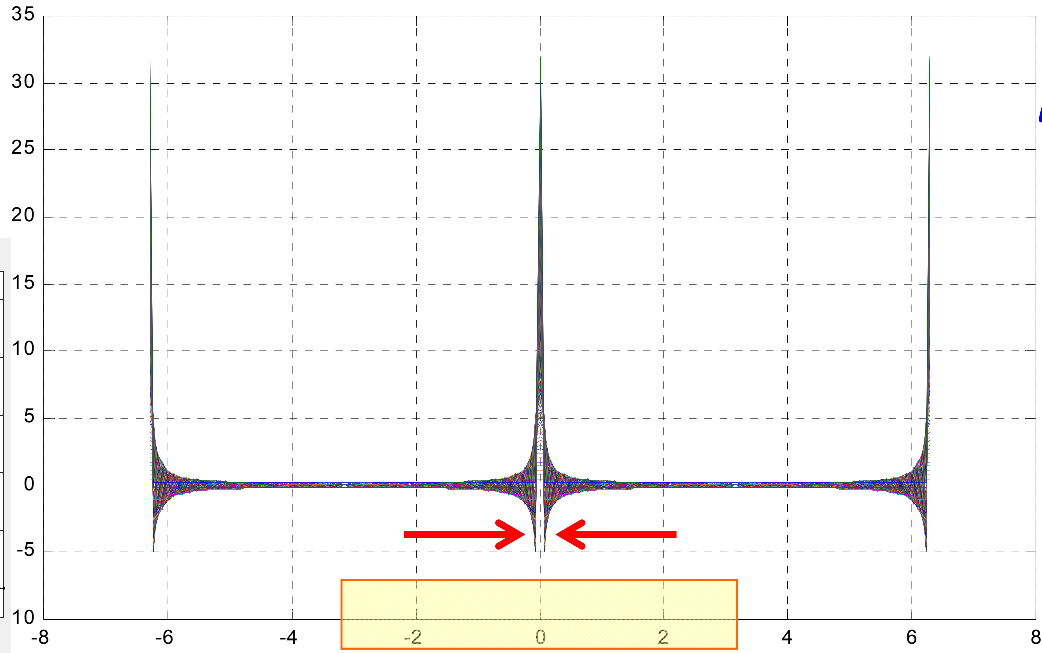
$$\delta(x) = \lim_{\varepsilon \to 0} \frac{e^{-\frac{1}{2}\left(\frac{x}{\varepsilon}\right)^2}}{\varepsilon\sqrt{2\pi}} \Leftarrow \text{Normal distribution } \mathcal{N}(0,\varepsilon) \Leftarrow$$



$$x \in \left]-\pi, +\pi\right[\qquad \delta(x) \sim \frac{1}{2\pi}\left[1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right]$$

```
a=-pi; b=pi; N=101;
maxY=1.1;
x=linspace(a,b,N)';
y=dirac(x);
idx=y == Inf;
y(idx)=maxY;
stem(x,y,'.k')
```





* Paul Adrien Maurice Dirac

Nobel Prize in Physics (1933) with Erwin Schroedinger

Fourier Serie exists, but it does not converge around $x=0$ to $\delta(x)$

Dirac comb function (or pulse train or sampling function)

The Fourier Series of δ(t), w.r.t. the interval $[-T/2, T/2]$, is:

$$\delta(t) \sim \frac{1}{T}\left[1 + 2\sum_{k=1}^{\infty} \cos\frac{2k\pi}{T}t\right] \quad \text{real form}$$

$$\sim \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{i\frac{2k\pi}{T}t} \quad \text{complex form}$$

and it is assumed to define a periodic function of period T, $\delta_T(t)$, called ***periodic impulse function*** (or ***pulse train*** or ***Dirac comb funct***):

convergence in distribution

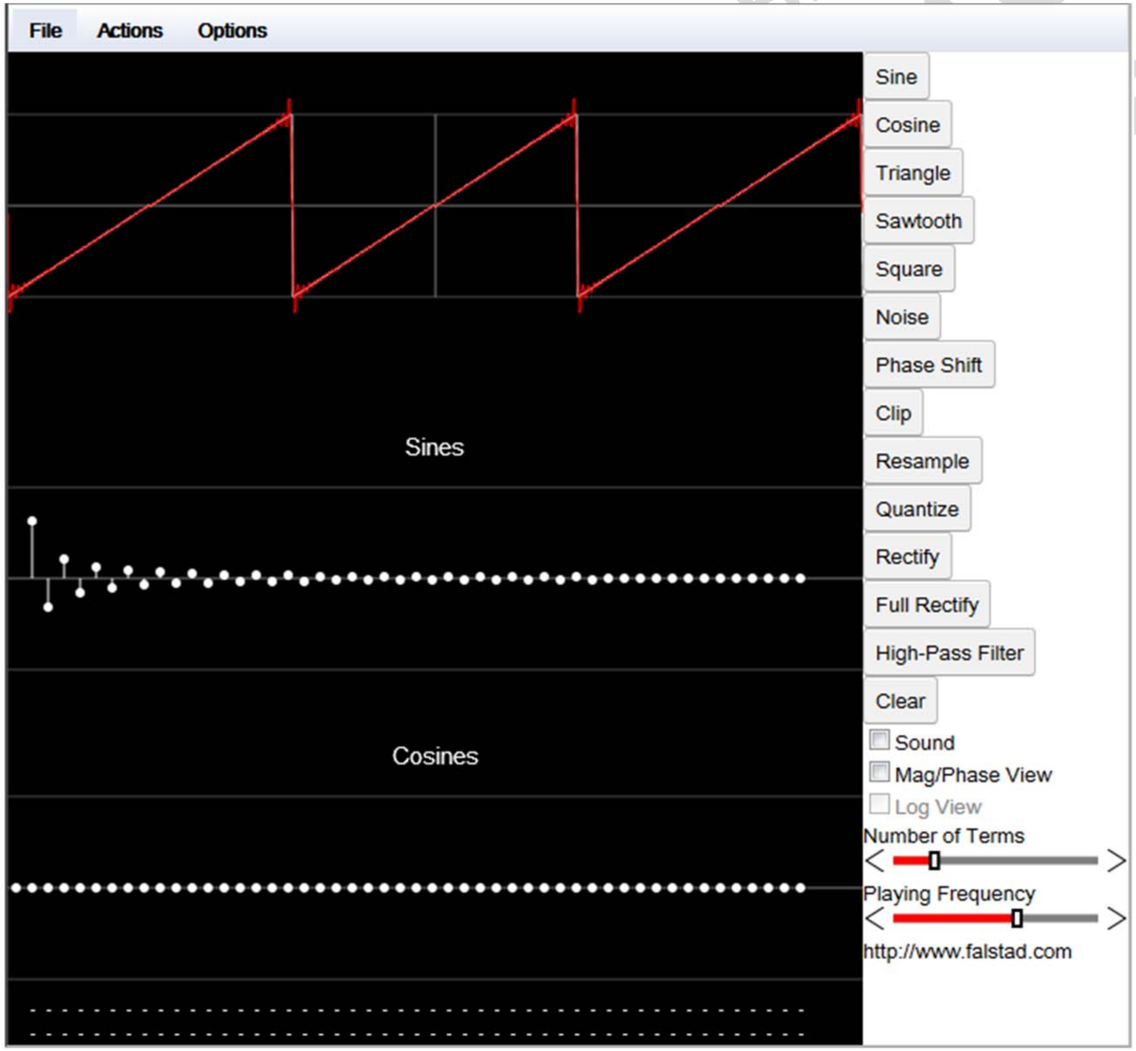
$$\delta_T(t) \overset{\text{def}}{=} \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{i\frac{2k\pi}{T}t}$$

usually it is used in applications to describe sampling

To emphasize its characteristic of being by convention a periodic repetition of ***Dirac*** δ , this function is also denoted as

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - k\mathrm{T})$$

Fourier Series Applet



https://www.falstad.com/fourier/

Contents

➢ **Properties of Fourier Coefficients.**

Main properties of Fourier coefficients [1]

Let $\mathscr{F}$ be the mapping that connects a function $f(x)$ to its Fourier coefficients $\{\mathrm{FC}_k[f]\}_k$ (real coefficients $\{(\alpha_k,\beta_k)\}_k$ or complex coefficients $\{\gamma_k\}_k$)

$$\mathscr{F}: f \longrightarrow \mathscr{F}[f] = \{\mathrm{FC}_k[f]\}_k$$

FC: Fourier Coefficients

- $\mathscr{F}[f]$ is ***linear***.
- If f is an ***even function***, then $\beta_k = 0 \ \forall k$, i.e. its FS only contains cosines.
- If f is an ***odd function***, then $\alpha_k = 0 \ \forall k$, i.e. its FS only contains sines.
- If f is ***real valued***, then $\gamma_{-k} = \overline{\gamma}_{+k}, \quad \forall k$.
- ***Time Shifting Property***: if $f(x)$ is shifted by a constant $f(x-h)$ then its Fourier coefficient $\#k$ is rotated by an angle $-kh$, i.e.

$$\mathrm{FC}_k[f(x-h)] = e^{-ikh}\,\mathrm{FC}_k[f(x)] \quad \forall k$$

- ***Differentiation of $f(x)$***: if also $f'(x)$ can be expanded in FS, then its FS is obtai-ned by differentiating the FS of $f(x)$ term-by-term, i.e.

$$\boxed{\mathrm{FC}_k[f'(x)] = ik\,\mathrm{FC}_k[f(x)] \quad \forall k}$$

This property can be used to approximate $f'(x)$ starting from samples of $f(x)$ (numerical differentiation)

(prof. M. Rizzardi)

Main properties of Fourier coefficients [2]

- ***Convolution Property***: given the Fourier coefficients of two functions, $\mathrm{FC}[f]$ and $\mathrm{FC}[g]$, then the Fourier coefficients of their product $\boldsymbol{f \cdot g}$ are given by the convolution product $\mathrm{FC}[f]*\mathrm{FC}[g]$ of their coefficients and vice versa, i.e.:

$$\mathrm{FC}[f \cdot g] = \mathrm{FC}[f] * \mathrm{FC}[g] \quad \text{and} \quad \mathrm{FC}[f * g] = \mathrm{FC}[f] \cdot \mathrm{FC}[g]$$

where

$$\begin{matrix} \mathrm{FC}[f] = \{\gamma_k\} \\ \mathrm{FC}[g] = \{\mu_h\} \end{matrix} \Rightarrow \begin{matrix} \mathrm{FC}[f \cdot g] = \{\varepsilon_n\}: \; \varepsilon_n = \sum_{k=-\infty}^{+\infty} \gamma_k \mu_{n-k} \\ \mathrm{FC}[f * g] = \{a_n\}: \; a_n = \gamma_n \mu_n \end{matrix}$$

and the ***convolution*** $*$ between $f, g \in L^1([-\pi, +\pi])$ is defined as:

$$[f * g](\tau) = \int_{-\pi}^{+\pi} f(t)\, g(\tau - t)\, \mathrm{d}t$$

- ***Parseval's identity***

$$\int_{-\pi}^{+\pi} |f(x)|^2 \,\mathrm{d}x = 2\pi \sum_{k=-\infty}^{\infty} |\gamma_k|^2$$

It gives the signal energy in terms of the Fourier coefficients of the signal

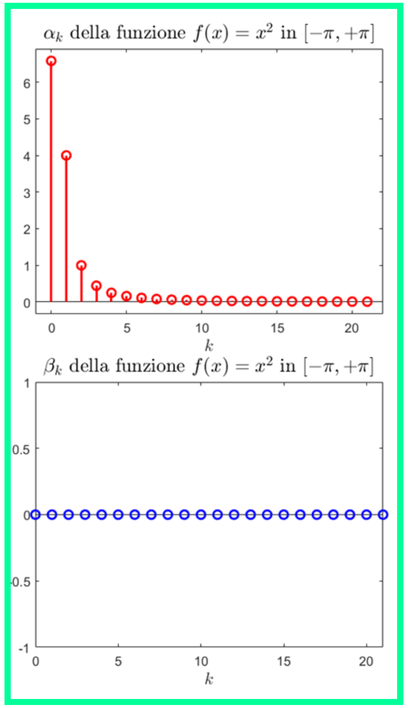
$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{ikx}$$

$$\frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\left[\alpha_k \cos(kx) + \beta_k \sin(kx)\right]$$

Properties of FCs: examples

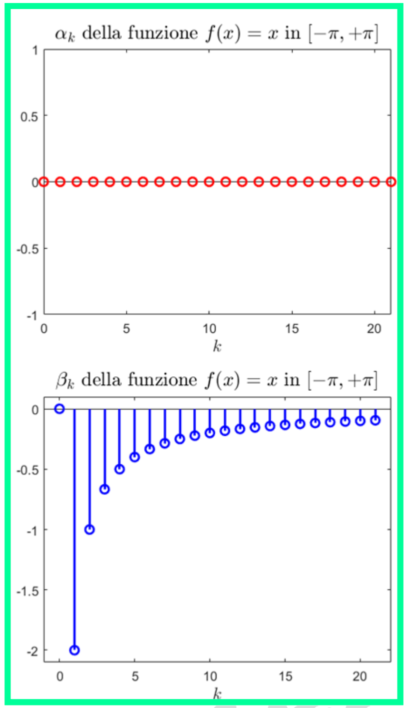
If f is an ***even function***, then $\beta_k = 0\ \forall k$, i.e. its FS only contains cosines.

Example: $f(x) = x^2$



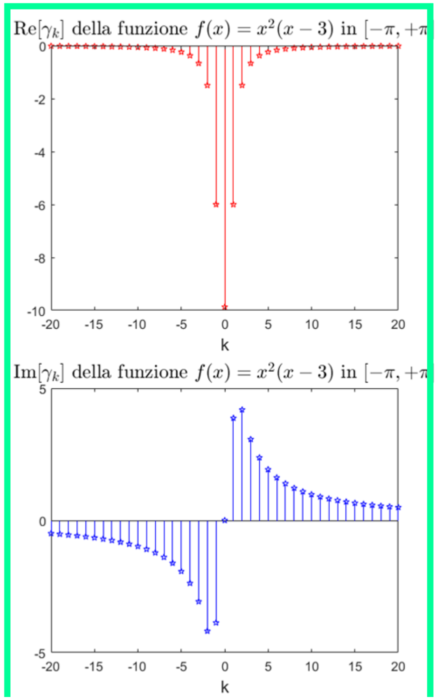
If f is an ***odd function***, then $\alpha_k = 0\ \forall k$, i.e. its FS only contains sines.

Example: $f(x) = x$



Se f is a ***real valued function***, then its FCs are complex and such that $\gamma_{-k} = \overline{\gamma}_k$.

Example: $f(x) = x^2(x-3)$



Application of Time Shifting Property

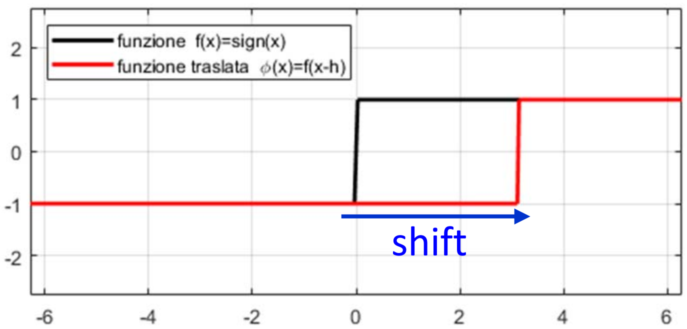
Starting from the FCs of $f(x)$ in $[-\pi,+\pi]$, we can write the Fourier Series expansion of the shifted function $f(x-h)$, $h=\pi$, in the interval $[0,2\pi]$ or in any other interval of width 2π:

$$\mathrm{FC}_k[f(x-h)] = e^{-ihk}\,\mathrm{FC}_k[f(x)]$$

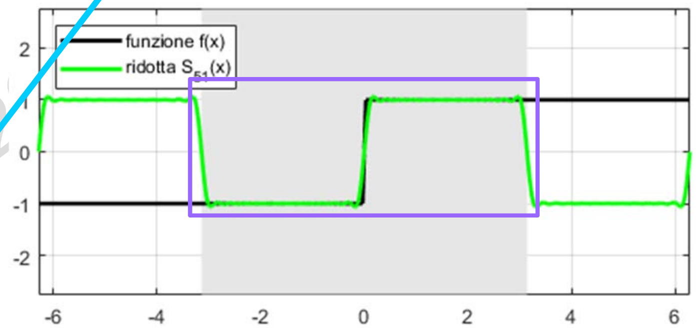
$\mathrm{FC}_k[f(x-h)]$ in $[-\pi+h,\pi+h]$; $\mathrm{FC}_k[f(x)]$ in $[-\pi,+\pi]$

$$\underbrace{\mathrm{FC}_k[f(x-\pi)]}_{\text{in } [0,2\pi]} = \boxed{e^{-i\pi k}}\,\underbrace{\mathrm{FC}_k[f(x)]}_{\text{in } [-\pi,+\pi]}$$

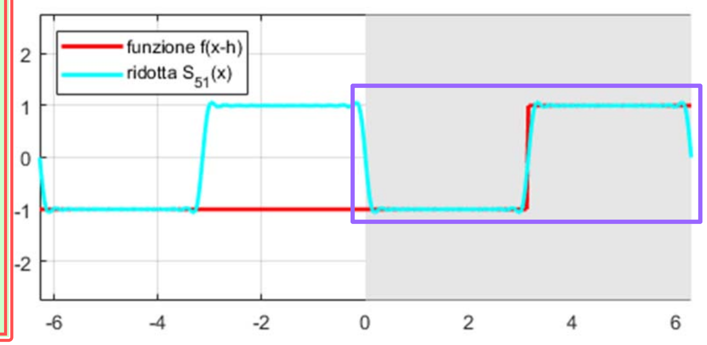
Function $\phi(x) = f(x-\pi)$



Fourier Series in $[-\pi,+\pi]$



Fourier Series in $[0,2\pi]$




```
pf=@sign; T=2*pi; N=60;
… c: (column-wise vector) FCs in [-π,+π]
 cT: FCs of the shifted function
 ST: partial sum of FS of the shifted function
h=pi; k=(-N/2:N/2)'; cT=exp(-1i*h*k).*c;
ST=exp(-i*N*pi/T*x).*polyval(flipud(cT),exp(2i*pi/T*x));
```

$$\mathrm{FC}_k[f(x-h)] = e^{-ihk}\,\mathrm{FC}_k[f(x)]$$

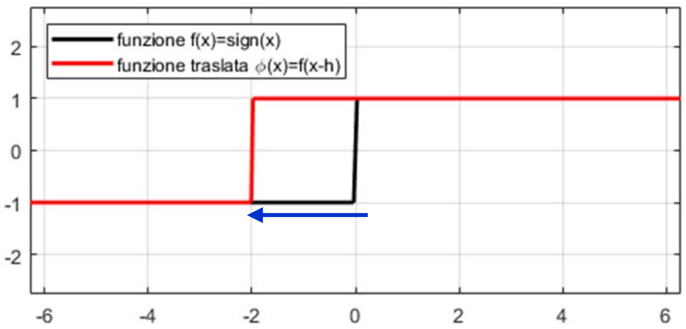
$\uparrow$ in $[-\pi+h,\ \pi+h]$ $\qquad$ $\uparrow$ in $[-\pi,+\pi]$

in $[-\pi-2,\pi-2]$ $\qquad$ in $[-\pi,+\pi]$

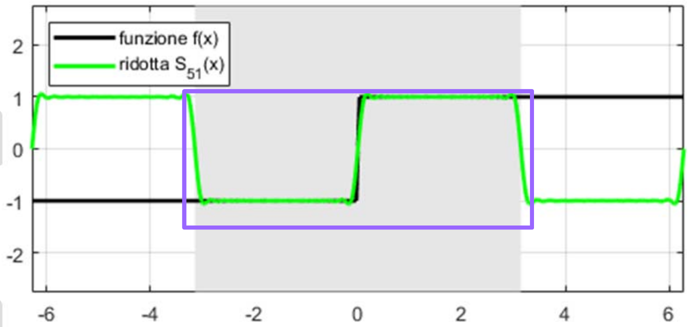
$$\mathrm{FC}_k[f(x+2)] = e^{+2ik}\,\mathrm{FC}_k[f(x)]$$

$h=-2$

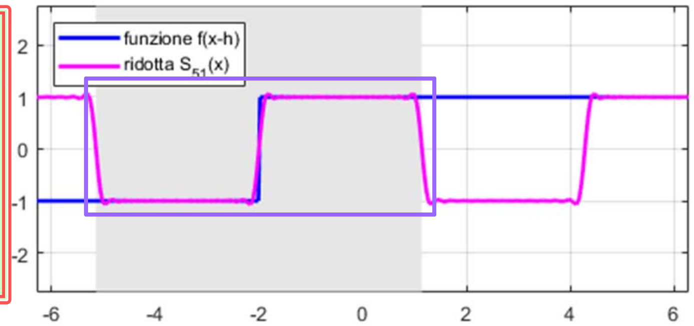
Function $\phi(x)=f(x+2)$



Fourier Series in $[-\pi,+\pi]$



Fourier Series in $[-\pi-2,+\pi-2]$




```
pf=@sign; T=2*pi; N=60;
… c: (column-wise vector) FCs in [−π,+π]
 cT: FCs of the shifted function
 ST: partial sum of FS of the shifted function
h=-2; k=(-N/2:N/2)'; cT=exp(-1i*h*k).*c;
ST=exp(-i*N*pi/T*x).*polyval(flipud(cT),exp(2i*pi/T*x));
```

Exercise

Given the Fourier Coefficients of $f(x)$ for the interval $[-\pi,+\pi]$, obtain the formulas to get the Fourier Coefficients in the interval $[0,2\pi]$ by applying the *Shift Property* to the Fourier Coefficients in the interval $[-\pi,+\pi]$. Similarly for the interval $[0,T]$ w.r.t. $[-T/2,+T/2]$. What changes between the two algorithms?

Properties of FCs: examples

Differentiation of f

If also f' can be expanded as a FS, then its Fourier Series is obtained by differentiating the Fourier Series of $f(x)$ term-by-term, i.e.

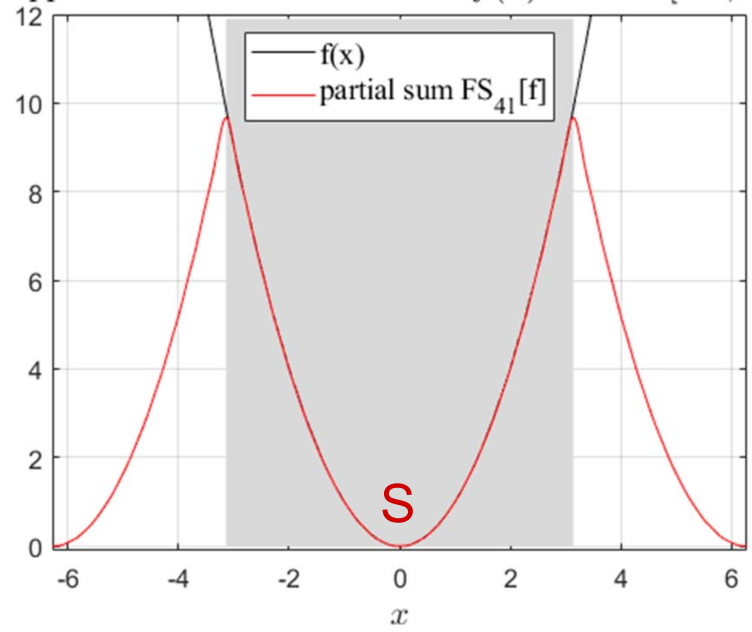
$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{ikx} \quad \Rightarrow \quad \mathrm{FC}_k[f'(x)] = ik\,\mathrm{FC}_k[f(x)]$$

Example

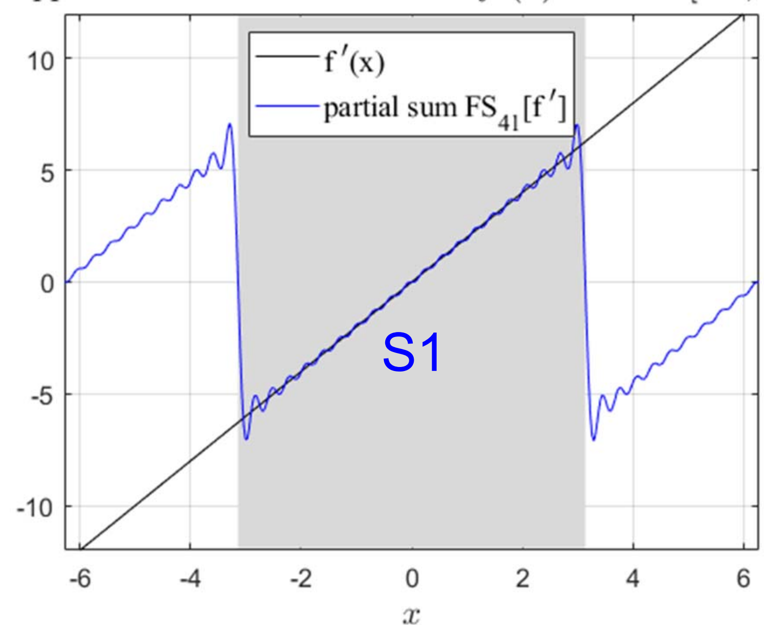
$$f(x) = x^2$$
$$f'(x) = 2x$$

```
fun=@(x) x.^2; fun1=@(x) 2*x;  T=2*pi; N=50;
% … c, S: coefficients and partial sum of the FS of fun in [−π,+π]
k=(-N/2:N/2)'; c1=i*k.*c; % coefficients of the differentiated series
S1=exp(-i*N*pi/T*x).*polyval(flipud(c1),exp(2i*pi/T*x));
```

Approximation of the function $f(x) = x^2$ in $[-\pi, +\pi]$



Approximation of the function $f'(x) = 2x$ in $[-\pi, +\pi]$



Properties of FCs: examples

Differentiation of f

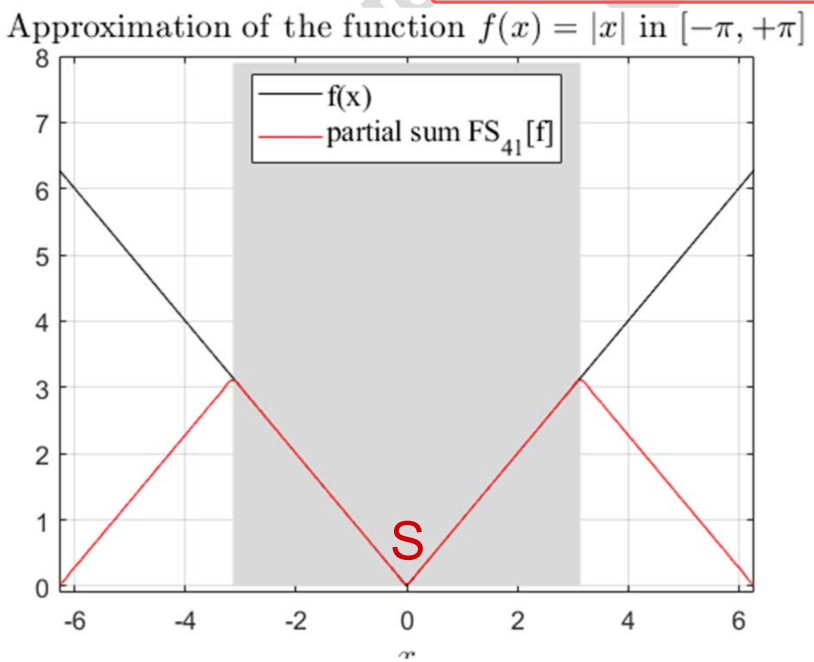
If also f' can be expanded as a FS, then its Fourier Series is obtained by differentiating the Fourier Series of $f(x)$ term-by-term, i.e.

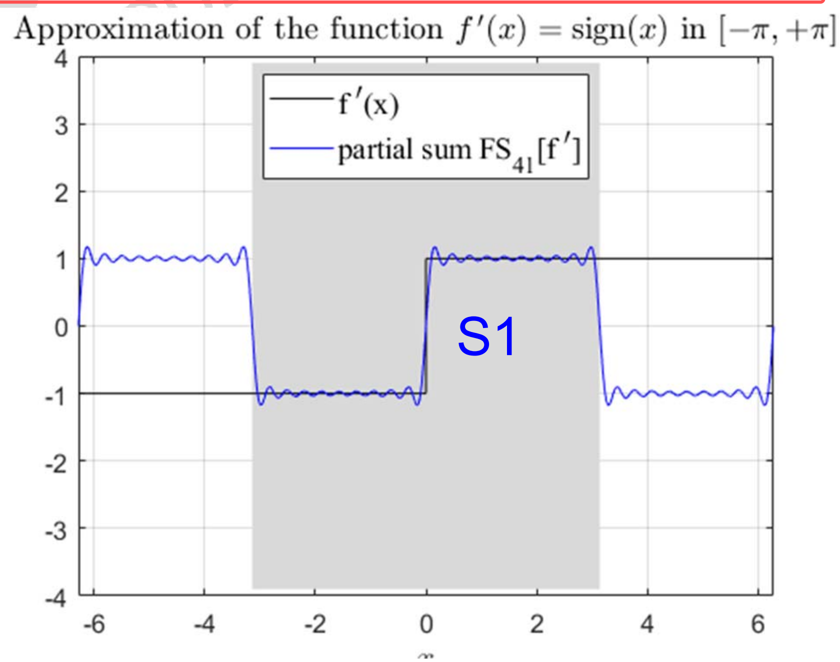
$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{ikx} \quad \Longrightarrow \quad \mathrm{FC}_k[f'(x)] = ik\,\mathrm{FC}_k[f(x)]$$

Example

$$f(x) = |x|$$
$$f'(x) = \mathrm{signum}(x)$$

```
fun=@abs; fun1=@sign;  T=2*pi; N=50;
% ... c, S: coefficients and partial sum of the FS of fun in [−π,+π]
k=(-N/2:N/2)'; c1=i*k.*c; % coefficients of the differentiated series
S1=exp(-i*N*pi/T*x).*polyval(flipud(c1),exp(2i*pi/T*x));
```

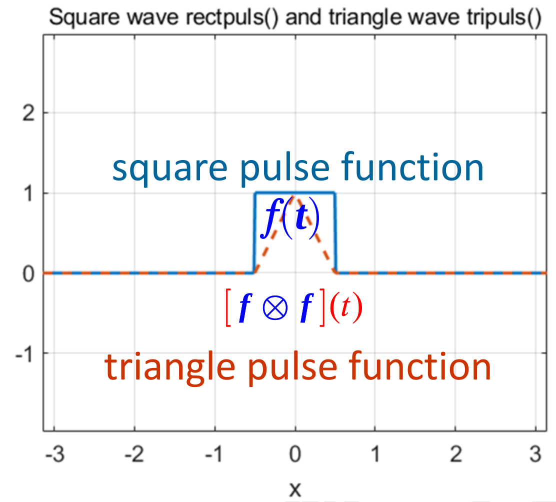




Properties of FCs: examples

Convolution Property: given the Fourier coefficients of two functions, $FC[f]$ and $FC[g]$, then the Fourier coefficients of their product $\boldsymbol{f \cdot g}$ are given by the convolution product $FC[f]*FC[g]$ of their coefficients and vice versa :

$$FC[f \cdot g] = FC[f] * FC[g] \quad \text{e} \quad FC[f * g] = FC[f] \cdot FC[g]$$



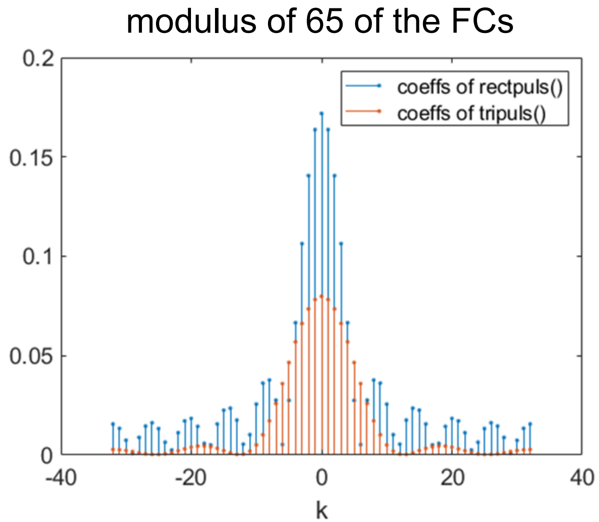
Example

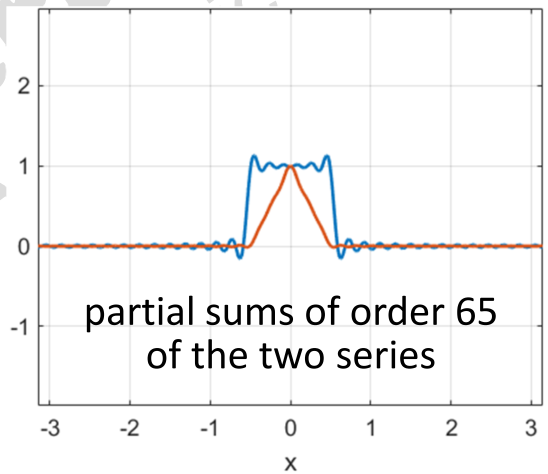
The convolution of 2 square pulses is a triangle pulse

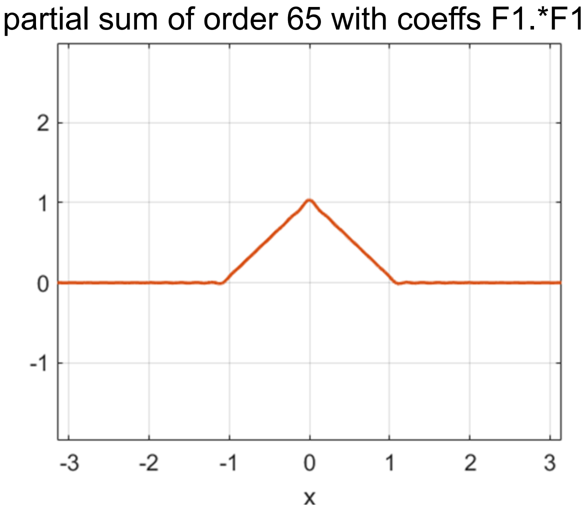
in Signal Processing Toolbox (num)

Compute N+1 FCs for the *square pulse* by **`rectpuls()`**: F1
Compute N+1 FCs for the *triangle pulse* by **`tripuls()`**: F2
Compute the Hadamard product of the FCs: F = F1.*F1
Evaluate the partial sum S with coefficients F
Draw the partial sum S

F. coefficients must be computed numerically







Contents

- Numerical approximation of Fourier coefficients:
  - by quadrature (wrong algorithm).
  - by DFT (right algorithm).
- Examples.
- Windowing error and aliasing error.

Numerical Approximation of Fourier coeffs

In order to compute the coefficients of a partial sum of a FS in $[-T/2, +T/2]$

$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{i\frac{2k\pi}{T}x} \approx \boxed{\sum_{k=-N/2}^{+N/2} \gamma_k e^{i\frac{2k\pi}{T}x}} \quad \text{partial sum}$$

trigonometric polynomial

$\nu_k = \dfrac{|k|}{T}$ is the circular frequency

... the simplest idea would be to use quadrature formulas!

$$\gamma_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{-i\frac{2k\pi}{T}t}\, dt \approx \mathbf{Q}_{N+1}\left[f(t) e^{-i\frac{2k\pi}{T}t} \right]$$

discretization parameters: T e N

However, this approach proves to be very inefficient and, above all, very inaccurate w.r.t. the use of the DFT ...

Example: numerical approximation of Fourer coeffs by means of numerical quadrature

Let us assume a Fourier Series be convergent. What does it happen to its "numerical"* partial sums as its order increases?

* i.e. whose coefficients are approximated by numerical quadrature

$$f(x) = \sum_{k=-\infty}^{+\infty} \gamma_k e^{i\frac{2k\pi}{T}x} \approx \underbrace{\widetilde{S_{N+1}}}_{\text{order}}(x) = \sum_{k=-N/2}^{+N/2} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} \qquad \widetilde{\gamma_k} = \underbrace{\mathbf{Q}_{m+1}\left[f(t)e^{-i\frac{2k\pi}{T}t}\right]}_{\text{quadrature rule}}$$

3 partial sums

$$\left\{\begin{array}{l}
\widetilde{S_{21}}(x) = \sum_{k=-10}^{+10} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} \\
\widetilde{S_{41}}(x) = \sum_{k=-20}^{+20} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} = \sum_{k=-20}^{-11} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} + \underbrace{\boxed{\sum_{k=-10}^{+10} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x}}}_{\widetilde{S_{21}}(x)} + \sum_{k=+11}^{+20} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} \\
\widetilde{S_{61}}(x) = \sum_{k=-30}^{+30} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} = \sum_{k=-30}^{-21} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x} + \underbrace{\boxed{\sum_{k=-20}^{+20} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x}}}_{\widetilde{S_{41}}(x)} + \sum_{k=+21}^{+30} \widetilde{\gamma_k} e^{i\frac{2k\pi}{T}x}
\end{array}\right.$$

To get the three partial sums, simply compute all the coefficients of that of highest order, and then suitably select the coefficients symmetrically w.r.t. the middle index.

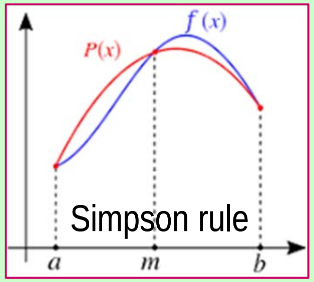
Example: numerical approximation of Fourer coeffs by means of numerical quadrature

$$\gamma_k = \frac{1}{T}\int_{-T/2}^{+T/2} f(\tau)\, e^{-ik\frac{2\pi}{T}\tau}\, d\tau$$

```
T=4; Nmax=60; fun=@sign;  x=linspace(-T/2,T/2,401); ytrue=feval(fun,x);
qfun=@(X,K) fun(X).*exp(-2i*pi/T*K*X); % integrand function
coef=[ ]; Nfun=0; % Nfun: number of function evaluations in the quadrature rule
for k = -Nmax/2 : Nmax/2
    [Q,fcnt] = quad(@(X)qfun(X,k),-T/2,T/2);
    Nfun=Nfun+fcnt; coef=[coef Q/T]; % coef: row-wise vector
end
m=Nmax/2+1;  % middle index
N1=Nmax/3;   % 21
S1=exp(-i*N1*pi/T*x).*polyval( fliplr(coef(m-N1/2:m+N1/2)),exp(2i*pi/T*x) );
N2=Nmax*2/3; % 41
S2=exp(-i*N2*pi/T*x).*polyval( fliplr(coef(m-N2/2:m+N2/2)),exp(2i*pi/T*x) );
N3=Nmax;     % 61
S3=exp(-i*N3*pi/T*x).*polyval( fliplr(coef(m-N3/2:m+N3/2)),exp(2i*pi/T*x) );
plot(x,ytrue,'k',x,real(S1),'b',x,real(S2),'r',x,real(S3),'g')
```

quad is a quadrature routine (Simpson rule)

compute all the coefficients



symmetrical coefficients w.r.t. the middle index

`polyval( coef(m+N1/2:-1:m-N1/2),exp(2i*pi/T*x) )`

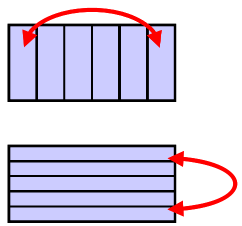
`polyval( coef(m+N2/2:-1:m-N2/2),exp(2i*pi/T*x) )`

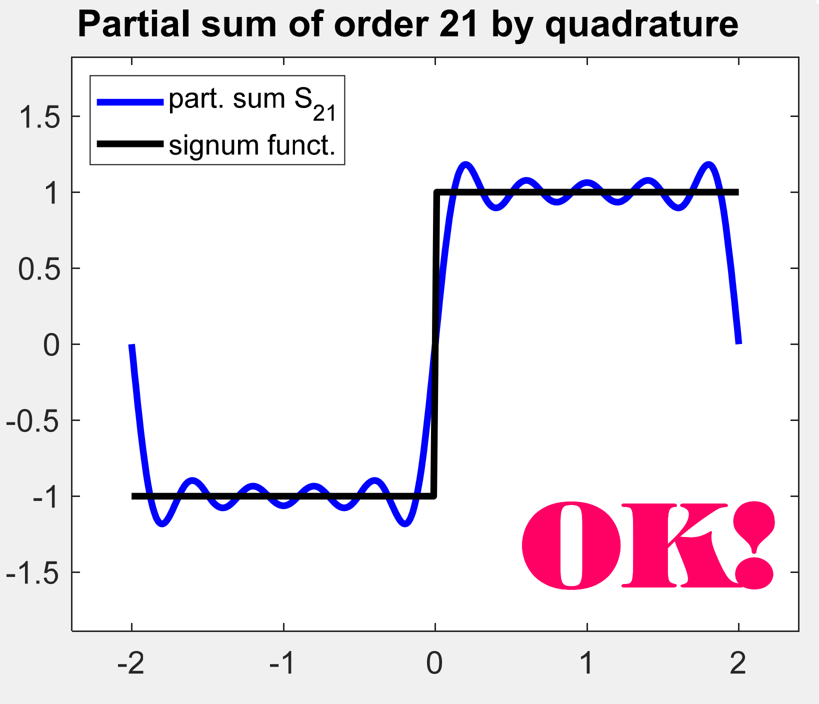
`polyval( coef(m+N3/2:-1:m-N3/2),exp(2i*pi/T*x) )`

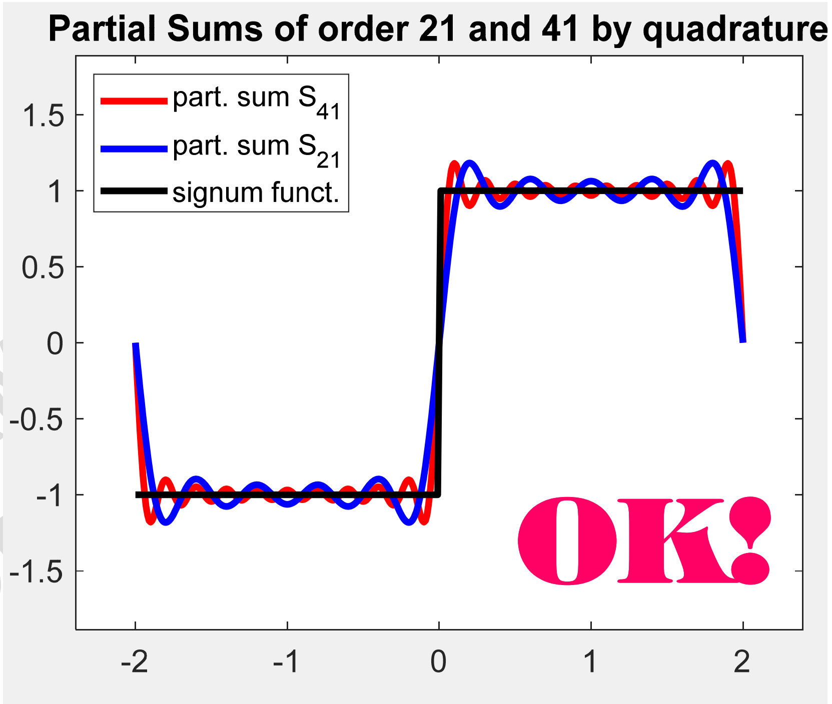
out

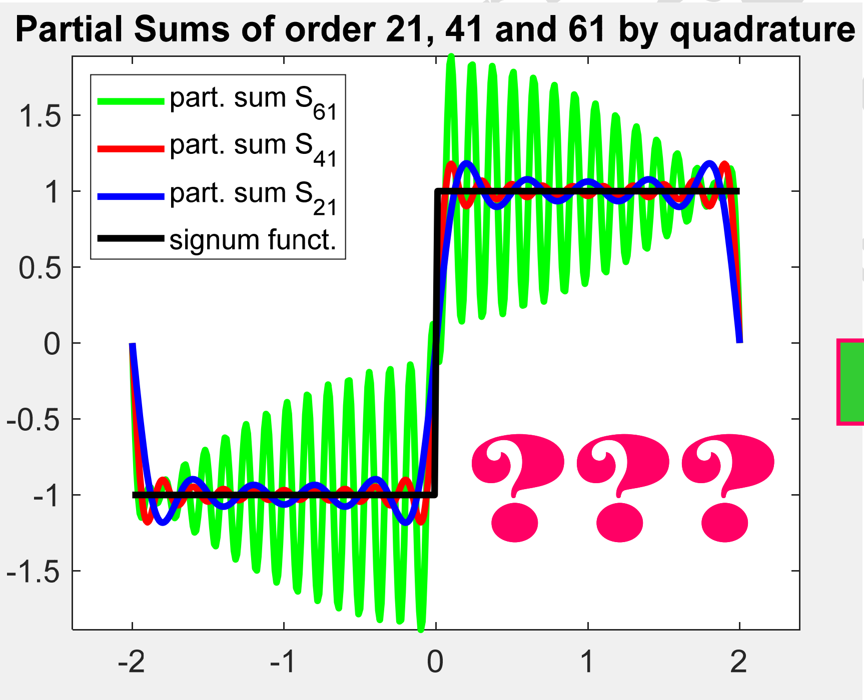
`fliplr(A)` flip left right: flip array left to right

`flipud(A)` flip up down: flip array up to down










```
Nfun
Nfun =
     34857
```

number of function evaluations

<u>INEFFICIENT!</u>

<u>INSTABLE</u> (it amplifies the roundoff error)

If we also use a more accurate MATLAB function [**quadl()**, **quadgk()**] this effect occurs later anyway

Numerical approximation of Fourer coeffs by means of numerical quadrature

If, instead of the expression of the function $f(x)$ (as in the previous example), we only have some of its samples (x_i, y_i), in order to use a quadrature routine, it needs to create an interpolating or approximating function $f^* \approx f$ to be used in the integrand function as a parameter for the quadrature routine.

Therefore we introduce, in addition, an approximation error.
The results will be **worse** than before!

THIS ALGORITHM SHOULD NOT BE USED

(prof. M. Rizzardi)

Example: numerical approximation of Fourer coeffs by means of DFT

$$\gamma_k = \frac{1}{T}\int_{-T/2}^{+T/2} f(\tau)\, e^{-ik\frac{2\pi}{T}\tau}\, d\tau$$

```
T=4; Nmax=60; fun=@sign; x=linspace(-T/2,T/2,401); ytrue=fun(x);
tj=linspace(-T/2,T/2,Nmax+1)'; fj=fun(tj);
f=[.5*(fj(1)+fj(end)); fj(2:end-1)];
c=fftshift(fft(f)); c=[c; c(1)]/Nmax;
c(2:2:end) = -c(2:2:end);
m=Nmax/2+1;   % middle index
N1=Nmax/3;    % 21
S1=exp(-1i*N1*pi/T*x).*polyval(flipud(c(m-N1/2:m+N1/2)),exp(2i*pi/T*x));
N2=Nmax*2/3; % 41
S2=exp(-1i*N2*pi/T*x).*polyval(flipud(c(m-N2/2:m+N2/2)),exp(2i*pi/T*x));
N3=Nmax;      % 61
S3=exp(-1i*N3*pi/T*x).*polyval(flipud(c(m-N3/2:m+N3/2)),exp(2i*pi/T*x));

plot(x,ytrue,'k',x,real(S1),'b',x,real(S2),'r',x,real(S3),'g')
```

(t_j, f_j): 61 samples of $f(x)$

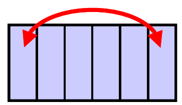
c: column vector

algorithm using DFT

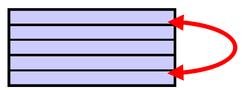
symmetrical coefficients w.r.t. the middle index

out

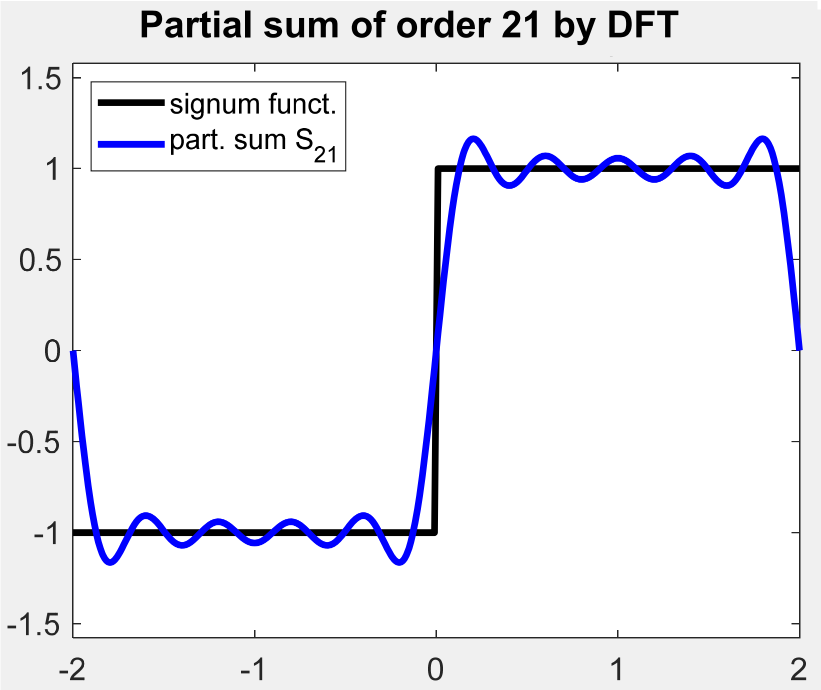
`fliplr(A)` flip left right: flip array left to right

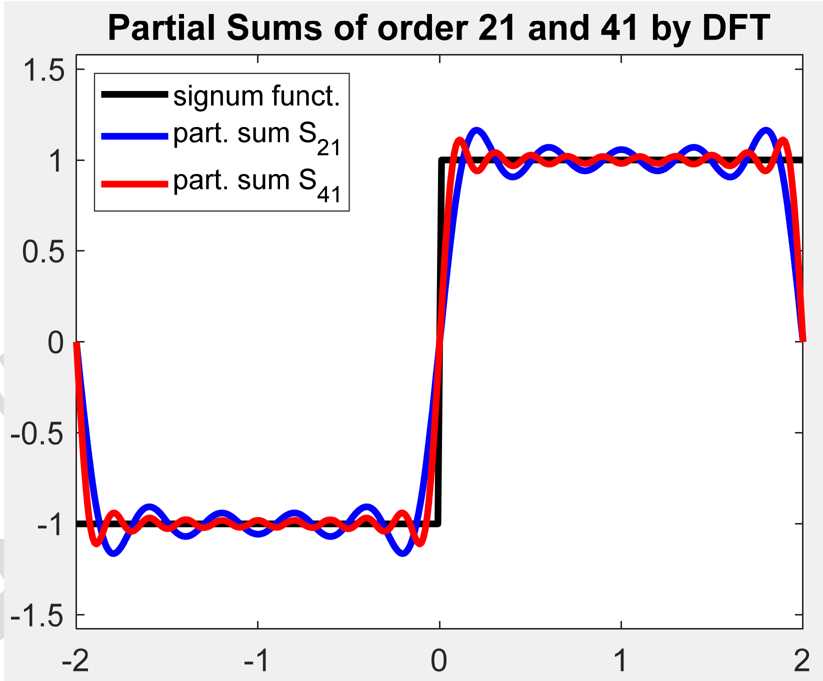


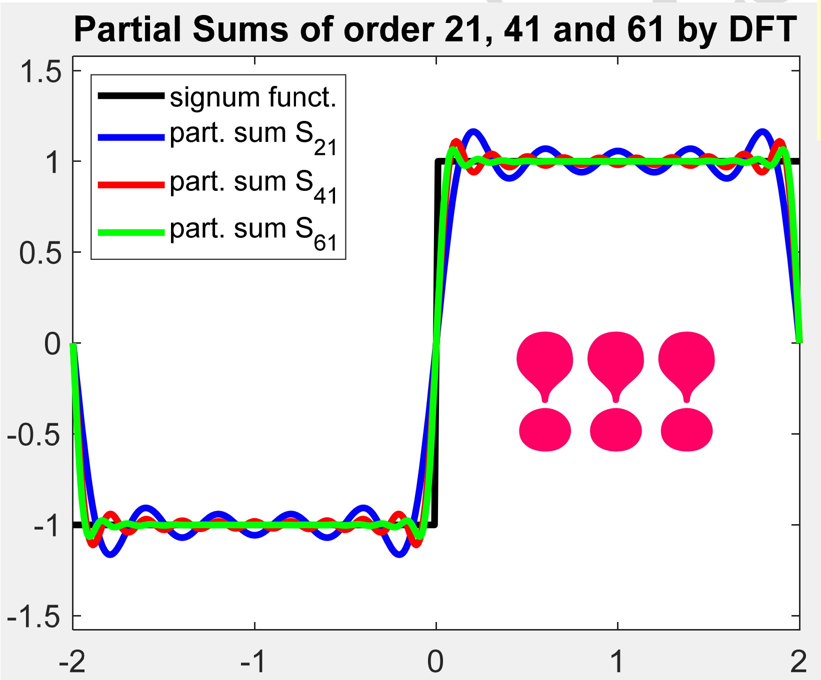
`flipud(A)` flip up down: flip array up to down



(prof. M. Rizzardi)







number of function evaluations

Nfun=61 EFFICIENT!

DFT just requires the input samples

no amplification of roundoff error

(**cond(DFT)=1**)

DFT computed by the **FFT algorithm** gives a **stable**, **accurate** and **efficient algorithm**

Numerical approximation of Fourier coeffs
in $[-T/2, +T/2]$ by means of a DFT

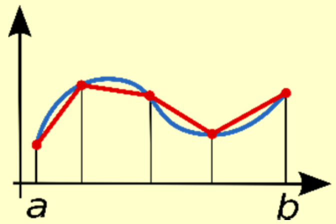
By means of a **DFT** we can **simultaneously** approximate all the coefficients of a partial sum of the Fourier Series of f

$$\sum_{k=-\infty}^{+\infty} \gamma_k e^{i\frac{2k\pi}{T}x} \approx S_N(x) = \sum_{k=-N/2}^{+N/2} \gamma_k e^{i\frac{2k\pi}{T}x} \qquad \gamma_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{-i\frac{2k\pi}{T}t}\,\mathrm{d}t$$

Algorithm idea: we start by applying the *Composite Trapezoidal Rule* T_{N+1} (*) to the integral, with $N=2m$ (even).

Recall

(*) *Composite Trapezoidal Rule* T_{N+1}
with N equally spaced panels



$$\int_a^b \varphi(t)\,\mathrm{d}t \approx T_{N+1}[\varphi] = \frac{(b-a)}{N}\left\{\frac{1}{2}\left[\varphi(a)+\varphi(b)\right] + \sum_{j=1}^{N-1}\varphi(t_j)\right\}$$

where $t_j = a + j\dfrac{(b-a)}{N}, \quad j = 0, 1, \dots, N$

Algorithm derivation: by applying T_{N+1} ($N=2m$) to the integral, we get

$N+1$ knots in $[-T/2,+T/2]$: $\quad t_j = -\frac{T}{2} + j\frac{T}{N}, \quad j = 0,1,\dots,N$

$$\gamma_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \underbrace{f(t)\, e^{-i\frac{2k\pi}{T}t}}_{\varphi(t)}\, dt$$

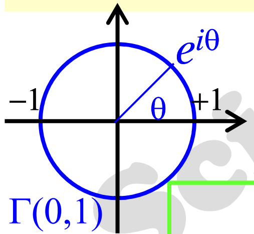
$$T_{N+1}[\varphi] = \frac{(b-a)}{N}\left\{\frac{1}{2}\left[\varphi(a)+\varphi(b)\right] + \sum_{j=1}^{N-1}\varphi(t_j)\right\}$$

$$\gamma_k \approx \tilde{\gamma}_k = \frac{1}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)e^{-i\frac{2k\pi}{T}\left(-\frac{T}{2}\right)} + f\left(+\tfrac{T}{2}\right)e^{-i\frac{2k\pi}{T}\left(+\frac{T}{2}\right)}\right] + \sum_{j=1}^{N-1} f(t_j)\, e^{-i\frac{2k\pi}{T}\left(-\frac{T}{2}+j\frac{T}{N}\right)}\right\} =$$

$$= \frac{1}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)e^{ik\pi} + f\left(+\tfrac{T}{2}\right)e^{-ik\pi}\right] + \sum_{j=1}^{N-1} f(t_j)\, e^{ik\pi+\frac{2\pi i}{N}(-kj)}\right\} =$$

$$= \frac{\left(e^{i\pi}\right)^k}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right) + f\left(+\tfrac{T}{2}\right)e^{-ik2\pi}\right] + \sum_{j=1}^{N-1} f(t_j)\, e^{i\frac{2\pi}{N}(-kj)}\right\}$$

$e^{-i\pi} = e^{+i\pi} = -1 \qquad e^{-ik2\pi} = e^0 = 1$



$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$: periodic function of period $= 2\pi$

Euler's formula

$$\boxed{\gamma_k \approx \tilde{\gamma}_k = \frac{(-1)^k}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right) + f\left(+\tfrac{T}{2}\right)\right] + \sum_{j=1}^{N-1} f(t_j)\, e^{i\frac{2\pi}{N}(-kj)}\right\}}$$

numerical approximation of the k^{th} Fourier coefficient

In particular, the first and the last coefficients are equal

$$\gamma_{-\frac{N}{2}} = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{+i\pi\frac{N}{T}t}\, dt$$

$$\gamma_{-\frac{N}{2}} \approx \tilde{\gamma}_{-\frac{N}{2}} = \frac{(-1)^{-\frac{N}{2}}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{i\frac{2\pi}{N}\left(+\frac{N}{2}j\right)}\right\} =$$

$$= \frac{(-1)^{-\frac{N}{2}}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{+i\pi j}\right\}$$

equal

$$\gamma_{+\frac{N}{2}} \approx \tilde{\gamma}_{+\frac{N}{2}} = \frac{(-1)^{+\frac{N}{2}}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{i\frac{2\pi}{N}\left(-\frac{N}{2}j\right)}\right\} =$$

$$\gamma_{+\frac{N}{2}} = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\, e^{-i\pi\frac{N}{T}t}\, dt$$

$$= \frac{(-1)^{+\frac{N}{2}}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{-i\pi j}\right\}$$

=

$$\boxed{\gamma_k \approx \tilde{\gamma}_k = \frac{(-1)^{k}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{i\frac{2\pi}{N}(-kj)}\right\}}$$

for $\underbrace{k = -\frac{N}{2},\dots,0,\dots,\frac{N}{2}-1}_{N}$ since $\tilde{\gamma}_{-\frac{N}{2}} = \tilde{\gamma}_{+\frac{N}{2}}$

Then, N coefficients are computed instead of $N+1$: $\underbrace{k = \boxed{-\frac{N}{2}},\dots,0,\dots,\frac{N}{2}-1,\boxed{+\frac{N}{2}}}_{N+1}$

Indeed, given $N+1$ samples (N is even), you can only get N different coefficients. If you try to compute more of them (outside the indices $\{-N/2, \dots, +N/2-1\}$) you get the same coefficients again. (this is a consequence of the $\mathbf{DFT}_N$ periodicity)

$$\tilde{\gamma}_{\frac{N}{2}+1} = \frac{(-1)^{\frac{N}{2}+1}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\,\boxed{e^{i\frac{2\pi}{N}\left[-\left(\frac{N}{2}+1\right)j\right]}}\right\} =$$

equal

$$e^{i\frac{2\pi}{N}\left[-\left(\frac{N}{2}+1\right)j\right]} = e^{-i\pi j}\,e^{-i\frac{2\pi}{N}j}$$

equal

$$e^{i\frac{2\pi}{N}\left[-\left(-\frac{N}{2}+1\right)j\right]} = e^{+i\pi j}\,e^{-i\frac{2\pi}{N}j}$$

$$\tilde{\gamma}_{-\frac{N}{2}+1} = \frac{(-1)^{-\frac{N}{2}+1}}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\,\boxed{e^{i\frac{2\pi}{N}\left[-\left(-\frac{N}{2}+1\right)j\right]}}\right\} =$$

For instance, given 5 samples ($N=4$) we can only get: $\tilde{\gamma}_{-2}, \tilde{\gamma}_{-1}, \tilde{\gamma}_{0}, \tilde{\gamma}_{+1}, \tilde{\gamma}_{+2}$ =

They are repeated periodically

$$\dots, \tilde{\gamma}_{-2}, \tilde{\gamma}_{-1}, \tilde{\gamma}_{0}, \tilde{\gamma}_{+1}, \tilde{\gamma}_{-2}, \tilde{\gamma}_{-1}, \tilde{\gamma}_{0}, \tilde{\gamma}_{+1}, \tilde{\gamma}_{-2}, \tilde{\gamma}_{-1}, \tilde{\gamma}_{0}, \tilde{\gamma}_{+1}, \tilde{\gamma}_{-2}, \dots$$

in $[-T/2,+T/2]$ $\quad S_{N+1}(x) \approx \widetilde{S}_{N+1}(x) = \sum_{k=-N/2}^{+N/2} \tilde{\gamma}_k e^{i\frac{2k\pi}{T}x}$

$$\gamma_k \approx \widetilde{\gamma}_k = \frac{(-1)^k}{N}\left\{\frac{1}{2}\left[f\left(-\tfrac{T}{2}\right)+f\left(+\tfrac{T}{2}\right)\right]+\sum_{j=1}^{N-1} f(t_j)\, e^{-\frac{2\pi i}{N}kj}\right\}$$

$$\omega_N = e^{-\frac{2\pi i}{N}}$$

$$\widetilde{\gamma}_k = \frac{(-1)^k}{N}\boxed{\sum_{j=0}^{N-1}\mathbf{f}_j\,\omega_N^{jk}} \quad \text{for} \quad \boxed{k=-\frac{N}{2},\ldots,0,\ldots,\frac{N}{2}-1}$$

it looks like a DFT $F_k = \boxed{\sum_{j=0}^{N-1} f_j\,\omega_N^{jk},}$ $\boxed{k=0,1,\ldots,N-1}$

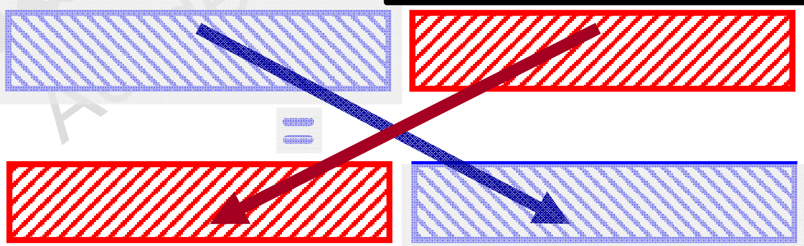
but indices differ from those in a DFT

$$k=\boxed{-\frac{N}{2},\ldots,-2,-1} \quad e^{i\left(-\frac{2\pi}{N}k+2\pi\right)j} = e^{i\left(2\pi\frac{N-k}{N}\right)j} = e^{\frac{2\pi i}{N}(N-k)j} = e^{\frac{2\pi i}{N}(h)j} \quad h=N-k=\boxed{\frac{N}{2},\ldots,N-2,N-1}$$

Rearrangement of components

$$h = 0,1,\cdots,\frac{N}{2}-1,\ \boxed{\frac{N}{2},\frac{N}{2}+1,\cdots,N-1}$$

$\mathrm{DFT}_N[\underline{\mathbf{f}}]$



=

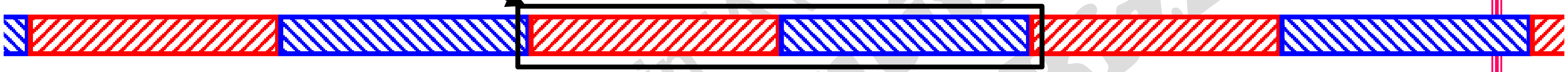
$\underline{\widetilde{\gamma}}$

$$k \Rightarrow \boxed{-\frac{N}{2},\ \cdots,\ -1,}\ \ 0,\ 1,\ \cdots,\ \frac{N}{2}-1$$

Quiz
and in $[0, T]$?

$$\widetilde{\gamma}_k = \frac{(-1)^k}{N} \boxed{\sum_{j=0}^{N-1} \mathbf{f}_j \ \omega_N^{jk}} \qquad \text{per} \quad k = -\frac{N}{2}, \ldots, 0, \ldots, \frac{N}{2}-1$$

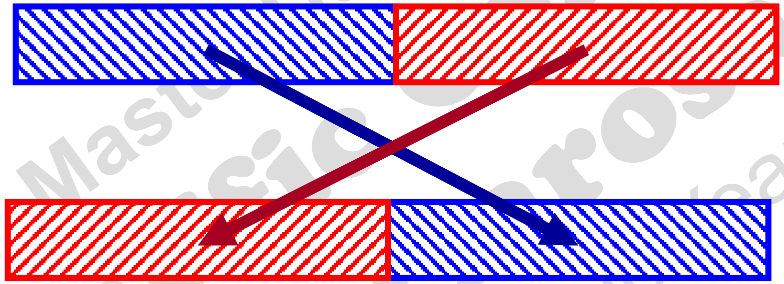
The summation can be computed using a $\mathbf{DFT}_N[\underline{\mathbf{f}}]$ as long as a rearrangement of the N components is carried out (by the DFT$_N$ periodicity):



Fourier coefficients

rearrangement of components

$$k = 0, 1, \cdots, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1, \cdots, N-1$$



$\mathbf{DFT}_N[\underline{\mathbf{f}}]$ $\qquad$ $\underline{\widetilde{\gamma}}$

$$k = -\frac{N}{2}, \ \cdots, \ -1, \ \ 0, \ 1, \ \cdots, \ \frac{N}{2}-1$$

$$F_k = \boxed{\sum_{j=0}^{N-1} \mathbf{f}_j \, e^{-i\frac{2\pi}{N}kj}}, \qquad k = 0, 1, \ldots, N-1$$

equal $\qquad$ different

$$\widetilde{\gamma}_k = \frac{(-1)^k}{N} \boxed{\sum_{j=0}^{N-1} \mathbf{f}_j \, e^{-i\frac{2\pi}{N}kj}}, \qquad k = -\frac{N}{2}, \ldots, 0, \ldots, \frac{N}{2}-1$$

At last, to compute $\underline{\widetilde{\gamma}}$ we add: the **last coefficient** $\widetilde{\gamma}_{+\frac{N}{2}} = \widetilde{\gamma}_{-\frac{N}{2}}$

and the **scale factors** $(-1)^k / N, \quad k = -N/2, \ldots, N/2$

Algorithm for $\tilde{\gamma}_k$

Input: $N+1$ equispaced samples $f_j = f(t_j)$ (N even)
in $[-T/2, +T/2]$

1. Define the sample vector $\underline{\mathbf{f}}$: $\begin{cases} \mathbf{f}_0 = \frac{1}{2}\left[f(t_0) + f(t_N)\right] \\ \mathbf{f}_j = f(t_j), \qquad j = 1, \dots, N-1 \end{cases}$

2. Compute the **DFT** (MATLAB `fft()`)

3. Reorder the vector (MATLAB `fftshift()`)

4. Add the last component* and the scale factors
   * the same as the first
   $\left((-1)^k / N, \quad k = -N/2, \dots, +N/2\right)$.

$(-1)^k$ means that we change the sign of the even or odd place components according to the value of $N/2$.

What are the changes in the algorithm for Fourier coefficients computed in $[0, 2\pi]$ or in $[0, T]$?

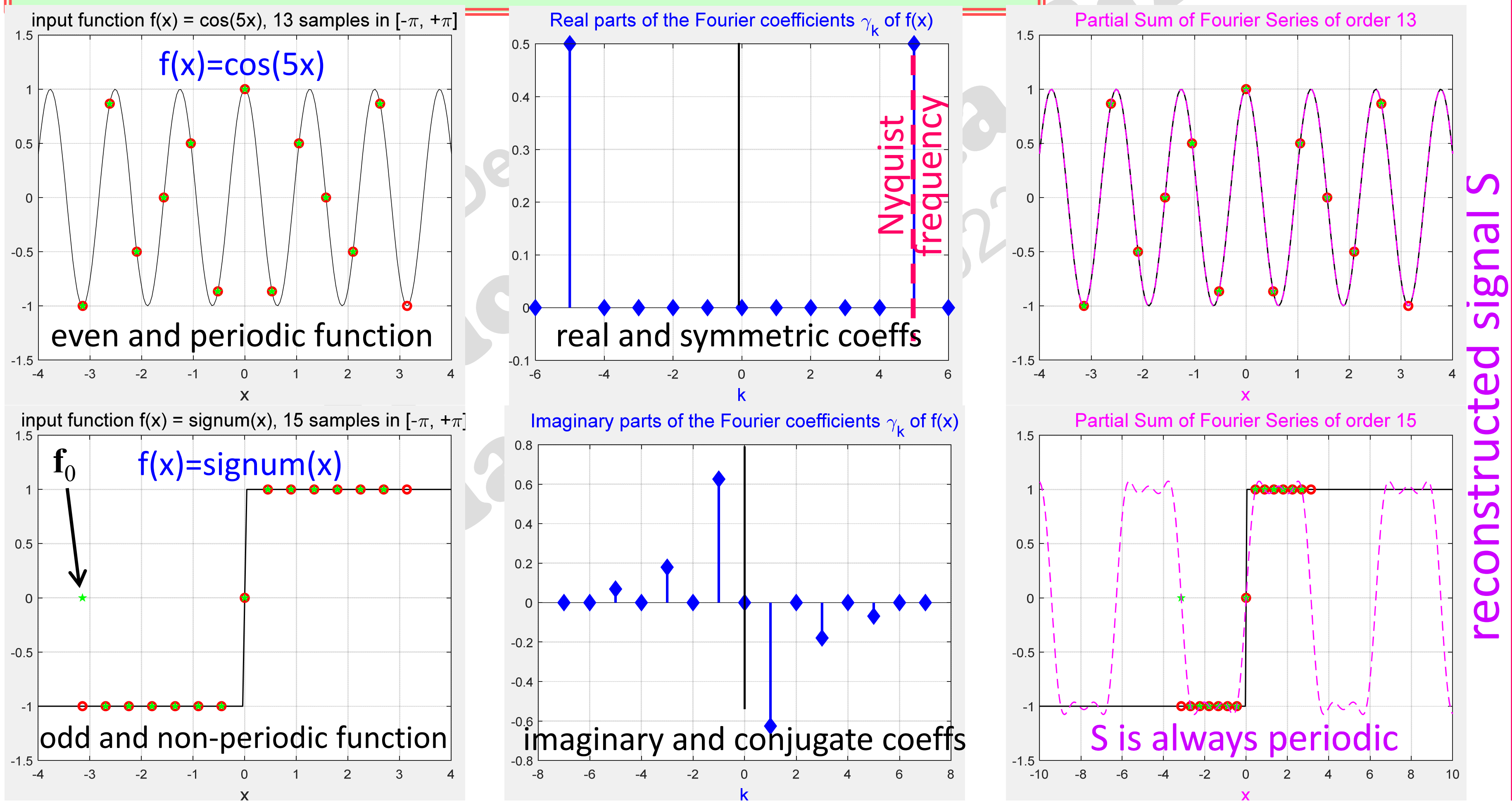
Exercise

Derive the formulas for the Fourier coefficients in the interval $[0, 2\pi]$ and in $[0, T]$ as a consequence of the Time-Shift Property. What is the difference with the intervals $[-\pi, +\pi]$ and $[-T/2, +T/2]$.

MATLAB **examples:** partial sum of Fourier Series

```
pf=@(x) …; % f(x)
x=linspace(-2*pi,2*pi,499); y=pf(x); % for graphics
N=14; T=2*pi; tj=T/N*(-N/2:N/2)'; fj=pf(tj); % samples
f=[.5*(fj(1)+fj(end));fj(2:end-1)]; % vector f
F=fftshift(fft(f)); F=[F;F(1)]/N; F(1:2:end)=-F(1:2:end);
plot(x,y,tj,fj,'ro',tj(1:end-1),f,'gp')
stem(-N/2:N/2,imag(F),'b-d')
S=exp(-1i*N*pi/T*x).*polyval(flipud(F),exp(2i*pi/T*x));
plot(x,y,'b',x,real(S),'r-.',tj,fj,'pr')
```

```
… N=12; T=2*pi; …
… F(2:2:end)=-F(2:2:end);
```



Fourier Analysis of f(x):

decomposition into elementary waves

Fourier Synthesis of f(x):

superposition of elementary waves

Algorithm ???

Fourier Synthesis in [-π, +π] (signal reconstruction)

$$x \in [-\pi, +\pi] \qquad f(x) \approx \tilde{S}_{N+1}(x) = \sum_{k=-N/2}^{+N/2} \tilde{\gamma}_k e^{ikx}$$

Given the Fourier coefficients of $f(x)$, the computing of a **partial sum of its FS** at sample points can be traced back to an **IDFT** in $[-\pi,+\pi]$.

In facts, by evaluating $f(x)$ at x_j, where

$$x_j = j\frac{2\pi}{N} - \pi, \quad j = 0,1,\dots,N$$

$$f(x_j) \approx \tilde{S}_{N+1}(x_j) = \sum_{k=-N/2}^{+N/2} \tilde{\gamma}_k \, e^{ikx_j} = \sum_{k=-N/2}^{+N/2} \tilde{\gamma}_k \, e^{ik\left(j\frac{2\pi}{N}-\pi\right)} =$$

$$= \sum_{k=-N/2}^{+N/2} \tilde{\gamma}_k \, e^{i\frac{2\pi}{N}kj} \underbrace{e^{ik(-\pi)}}_{=(-1)^k} = \sum_{k=-N/2}^{+N/2} \boxed{(-1)^k \tilde{\gamma}_k} \, e^{i\frac{2\pi}{N}jk}, \quad j = 0,1,\dots,N-1$$

$$\mathrm{IDFT}_N[\underline{\psi}] = \frac{1}{N}\sum_{k=0}^{N-1} \psi_k \, e^{i\frac{2\pi}{N}kj}, \quad j = 0,1,\dots,N-1$$

we can use the same algorithm as for FCs, with a few changes

Algorithm steps in $[-\pi, +\pi]$

Input: $N+1$ coefficienti di Fourier γ_k (N even)

Output: $N+1$ campioni equispaziati $f_j \approx f(t_j)$

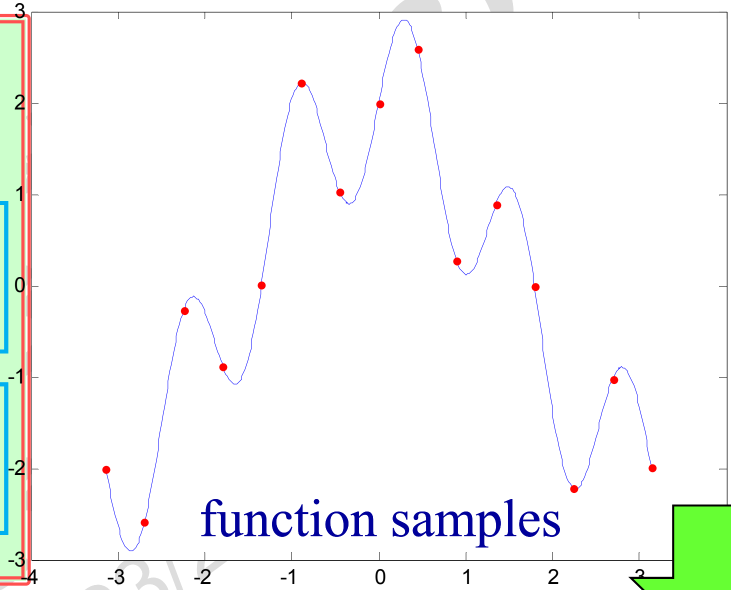
1. Define the vector $\underline{\Phi}$: $\Phi_k = \widetilde{\gamma_k}$, $k = -N/2, \ldots, +N/2 - 1$
   except for the last coefficient being the same as the first
2. $\boxed{\text{Change sign to alternating components}}$
   (even or odd place components according to the value of $N/2$)
3. Reordering (`fftshift()`)
4. Compute the **IDFT** (`ifft()`)
5. Add the last component and the scale factor
   (N, $k=-N/2, \ldots, +N/2$)

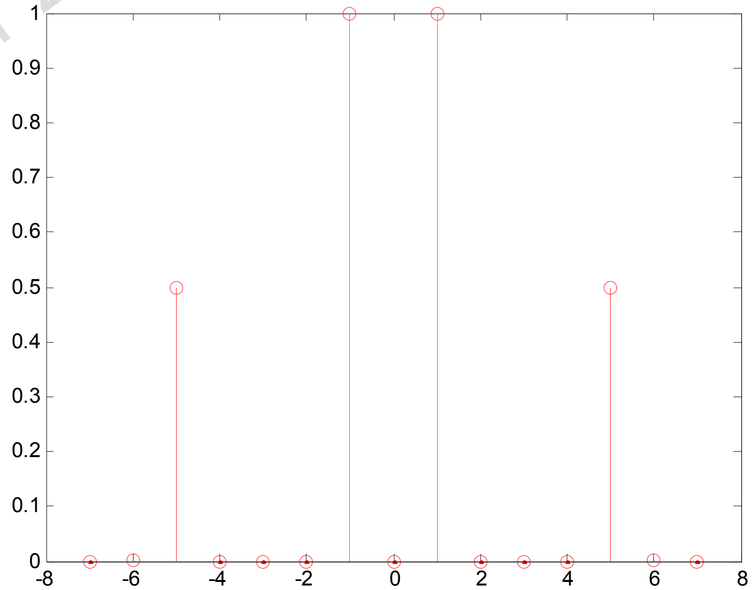
Example in $[-\pi, +\pi]$

```
pf=@(x) 2*cos(x)+sin(5*x);
x=linspace(-pi,pi,400);  y=pf(x);
N=14; j=(-N/2:N/2)'; tj=2*pi/N*j; fj=pf(tj);
figure; plot(x,y,tj,fj,'r.')
f=[.5*(fj(1)+fj(end)); fj(2:end-1)];
F=fftshift(fft(f));  F(1:2:end)=-F(1:2:end);
F=[F;F(1)]/N;
figure;  stem(j, abs(F),'r-o');
G=F(1:end-1);
G(1:2:end)=-G(1:2:end);
g=ifft(fftshift(G));    g=[g;g(1)]*N;
figure; plot(x,y,'b',tj,fj,'r.',tj,real(g),'.m-.')
```

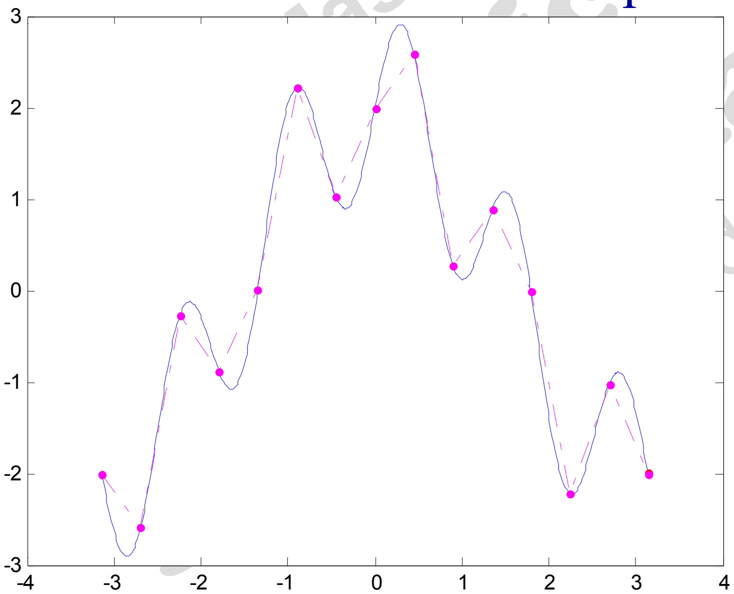
DFT

IDFT





reconstructed function samples



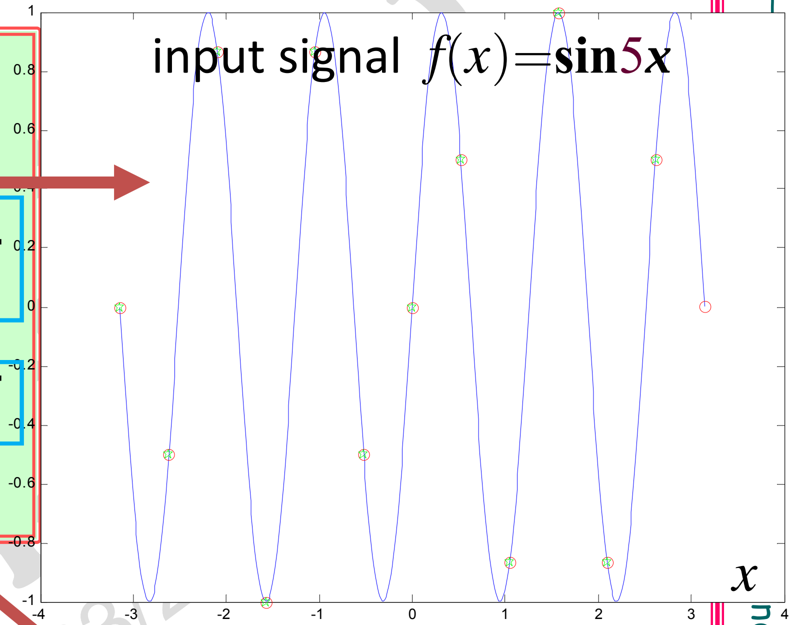
(prof. M. Rizzardi)

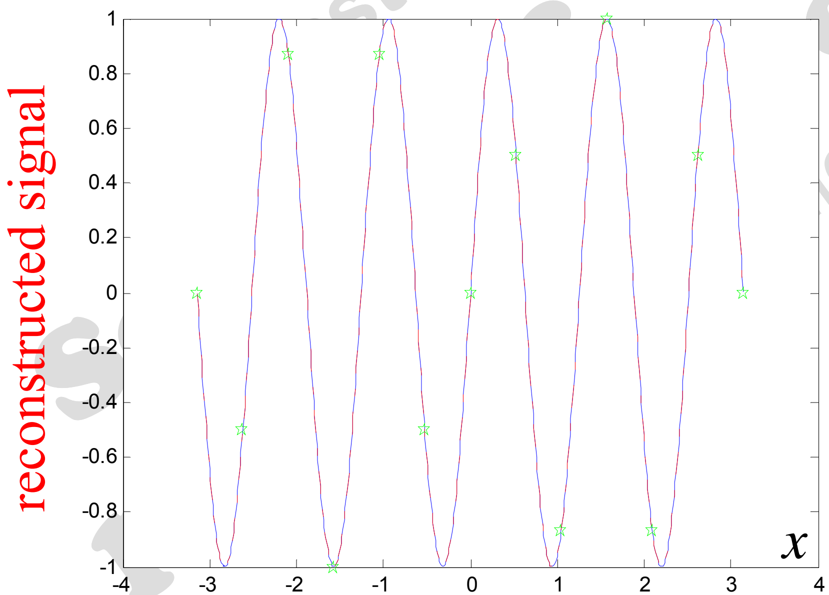
Signal reconstruction by a **trigonometric polynomial**

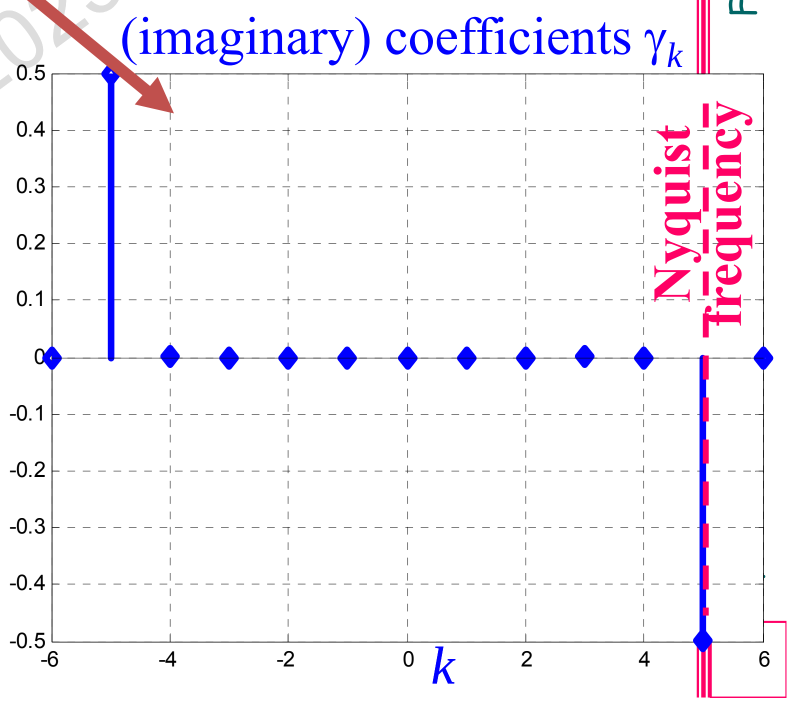
```
pf=@(x) sin(5*x);                          % sin(5x)
x=linspace(-pi,pi,499); y=feval(pf,x);
N=12; tj=2*pi/N*(-N/2:N/2)'; fj=feval(pf,tj);
figure; plot(x,y,tj,fj,'ro',tj(1:end-1),f,'gp')
f=[.5*(fj(1)+fj(end)); fj(2:end-1)];
F=fftshift(fft(f)); F=[F;F(1)]/N;
F(2:2:end)= -F(2:2:end);
figure; stem(-N/2:N/2,imag(F),'b-d')
G=F(1:end-1);  G(2:2:end)=-G(2:2:end);
g=ifft(fftshift(G));    g=[g;g(1)]*N;
S=exp(-i*N/2*x).*polyval(F(end:-1:1),exp(i*x));
figure; plot(x,y,'b',x,real(S),'r-.',tj,real(g),'pg')
```

DFT

IDFT





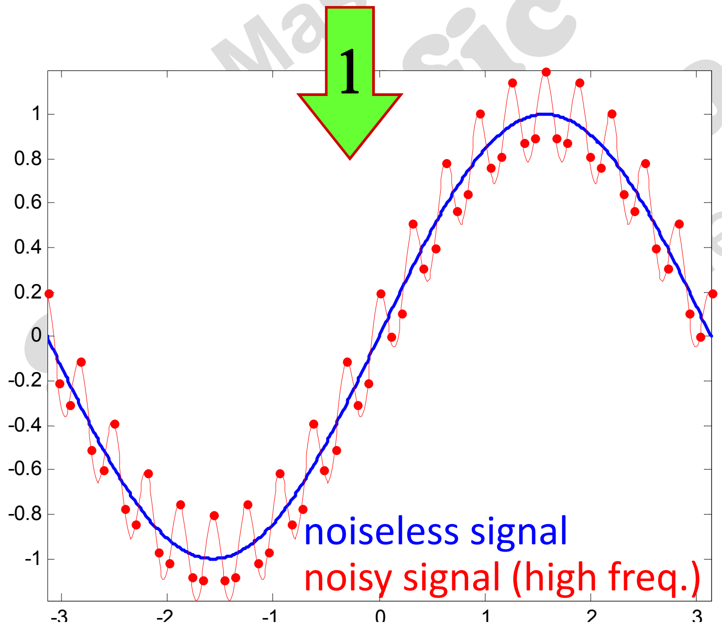


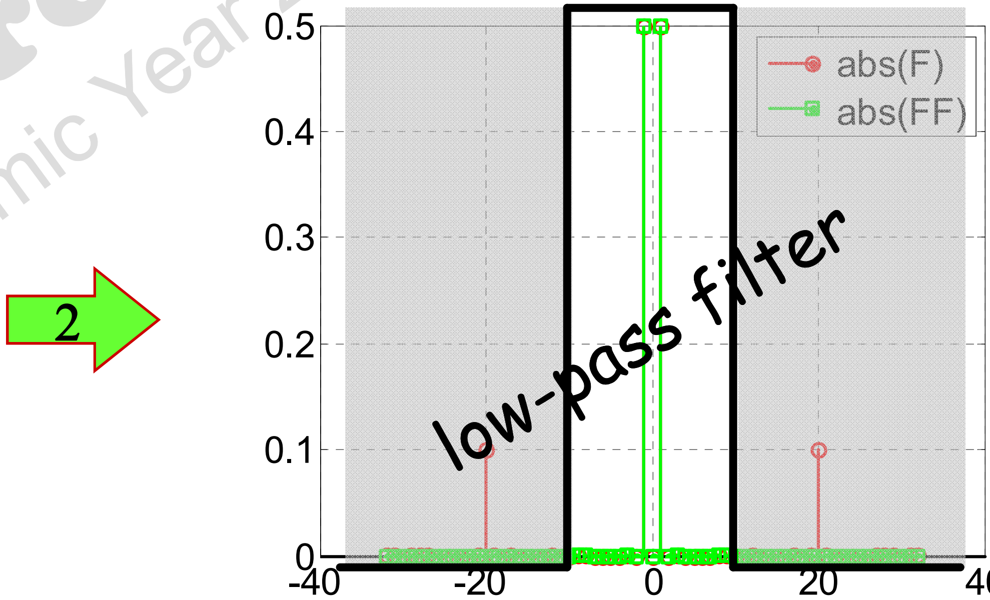
Application: signal filtering

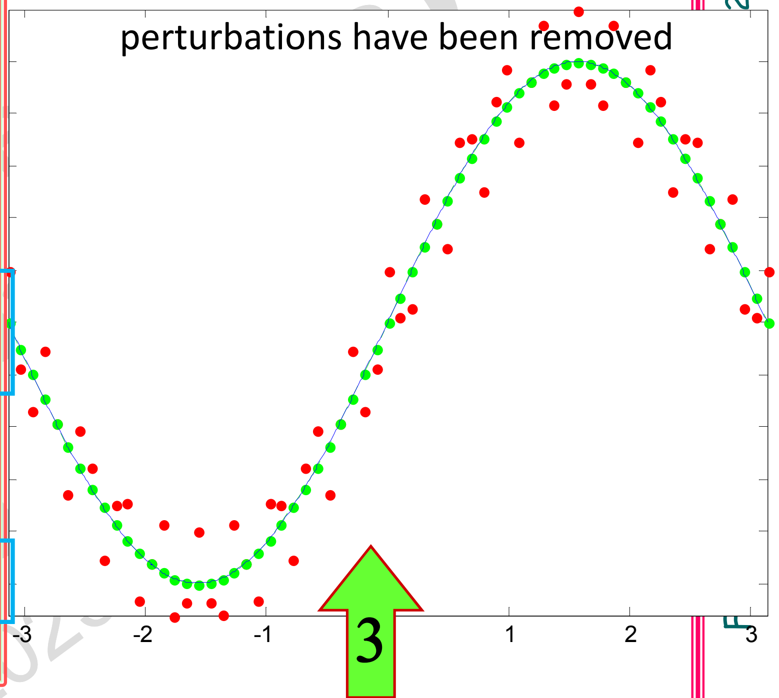
```
pf=@(x)sin(x);    pNoise=@(x)0.2*cos(20*x);
x=linspace(-pi,pi,499); y=pf(x); % noiseless signal
yp=pf(x) + pNoise(x);            % noisy signal
N=64; T=2*pi; nu=-N/2:N/2; tj=T/N*nu';
fj=pf(tj) + pNoise(tj);          % noisy samples
figure; plot(x,y,x,yp,'r',tj,fj,'r.')
f=[.5*(fj(1)+fj(end)); fj(2:end-1)];
F=fftshift(fft(f));    F=[F;F(1)]/N;
F(2:2:end)=-F(2:2:end);
FF=zeros(size(F)); k=find(abs(nu)<10); FF(k)=F(k);
figure; stem(nu, abs(F),'r-o'); hold on
stem(nu, abs(FF),'g--s')
G=FF(1:end-1);  G(2:2:end)=-G(2:2:end);
g=ifft(fftshift(G));      g=[g;g(1)]*N;
figure; plot(x,y,'b',tj,fj,'r.',tj,real(g),'.g-.')
```

DFT

IDFT

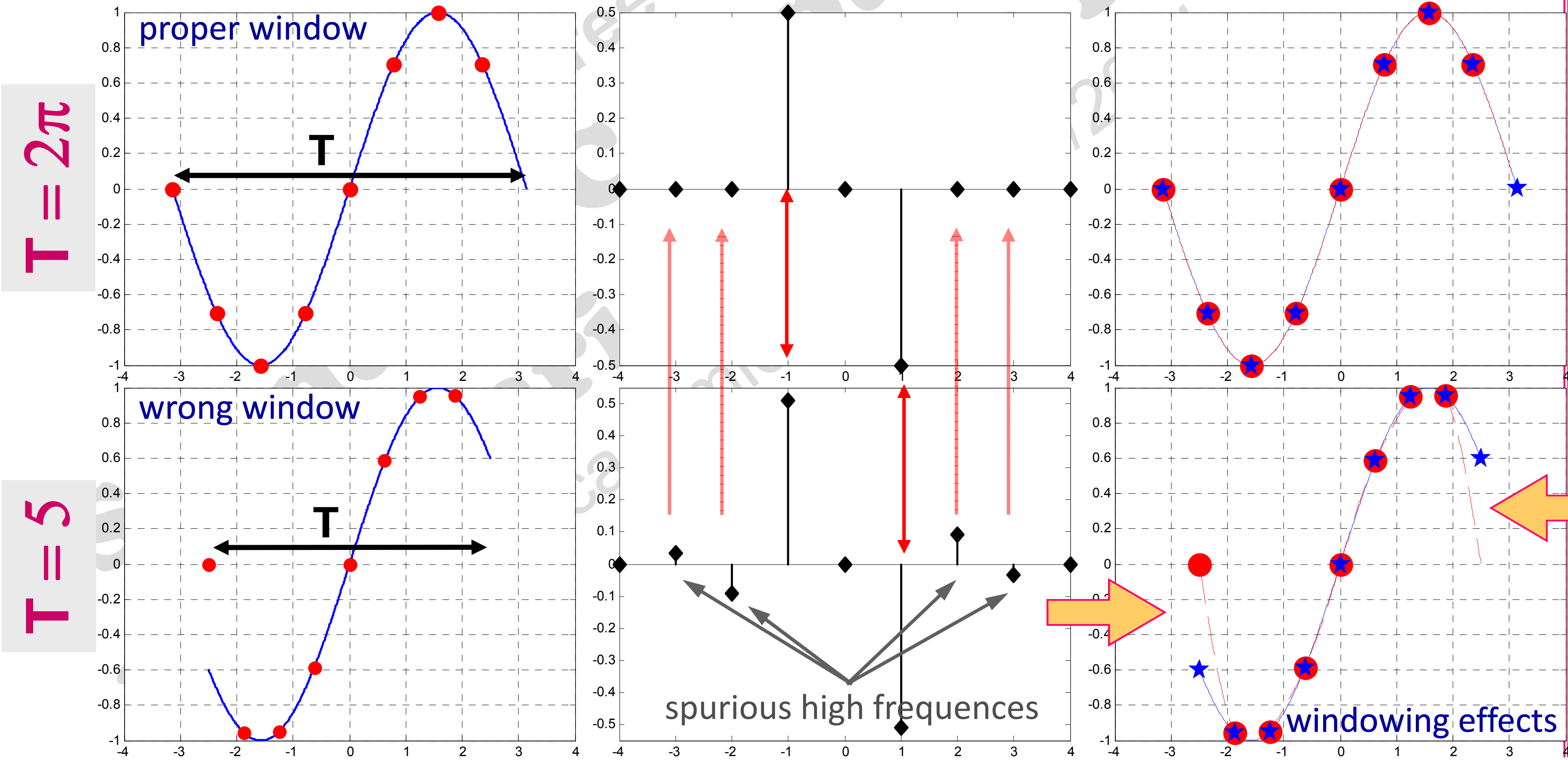






Example: windowing error

```
pf=@sin; T=5; x=linspace(-T/2,T/2,499); y=pf(x);
N=8; tj=T/N*(-N/2:N/2)'; fj=pf(tj); figure; plot(x,y, tj(1:end-1),f,'ro'); grid on
f=[.5*(fj(1)+fj(end));fj(2:end-1)]; F=fftshift(fft(f)); F=[F;F(1)]/N;
F(2:2:end)=-F(2:2:end);
figure; stem(-N/2:N/2,imag(F),'k-d') % FC
S=exp(-i*N*pi/T*x).*polyval(F(end:-1:1),exp(i*2*pi/T*x)); % partial sum of FS
figure; plot(x,y,'b',x,real(S),'r--',tj(1:end-1),f,'ro',tj,fj,'bp')
```



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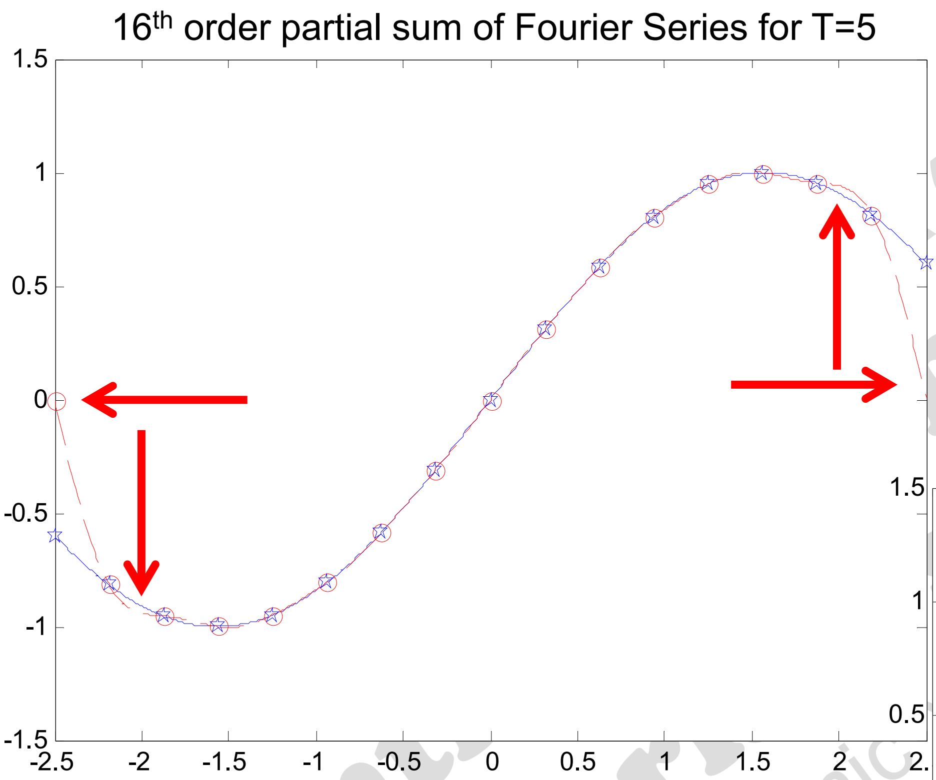
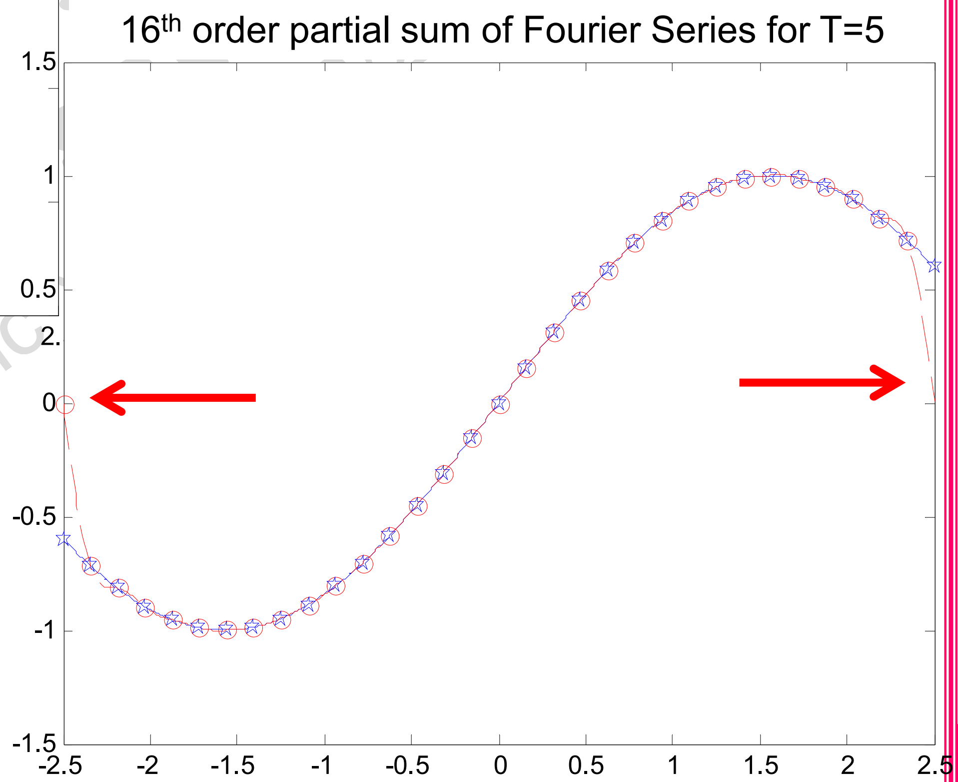
Example: windowing error



The higher-order partial sums also oscillate and deviate from the function generating the samples at endpoints of the interval of width T=5

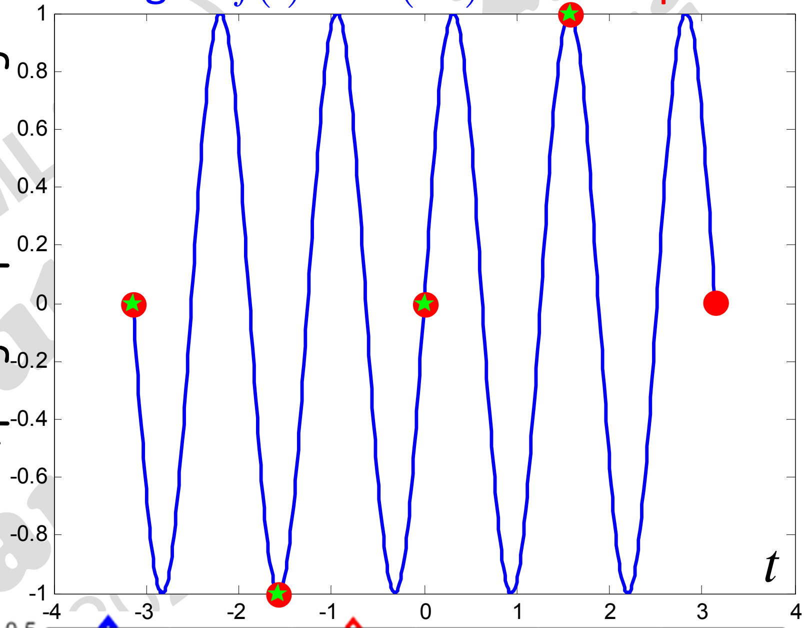


Esempio: aliasing error

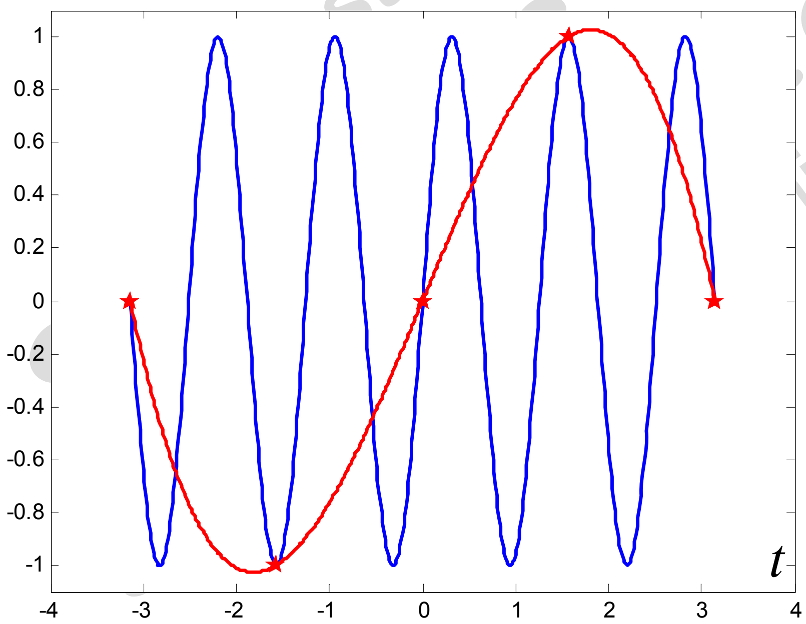
```
pf=@(x) sin(5*x); T=2*pi; N=4; % N is small!
x=linspace(-T/2,T/2,499);y=pf(x);
tj=T/N*(-N/2:N/2)'; fj=pf(tj);
f=[.5*(fj(1)+fj(end));fj(2:end-1)];
plot(x,y,tj,fj,'ro',tj(1:end-1),f,'gp')
F=fftshift(fft(f));F=[F;F(1)]/N;
F(2:2:end)=-F(2:2:end);
h=stem(-N/2:N/2,imag(F),'r-d'); …
S=exp(-i*N*pi/T*x).*polyval(F(… …));
plot(x,y,'b',x,real(S),'r',tj,real(g),'pr')
```

signal $f(t)=\sin(5x)$ and samples

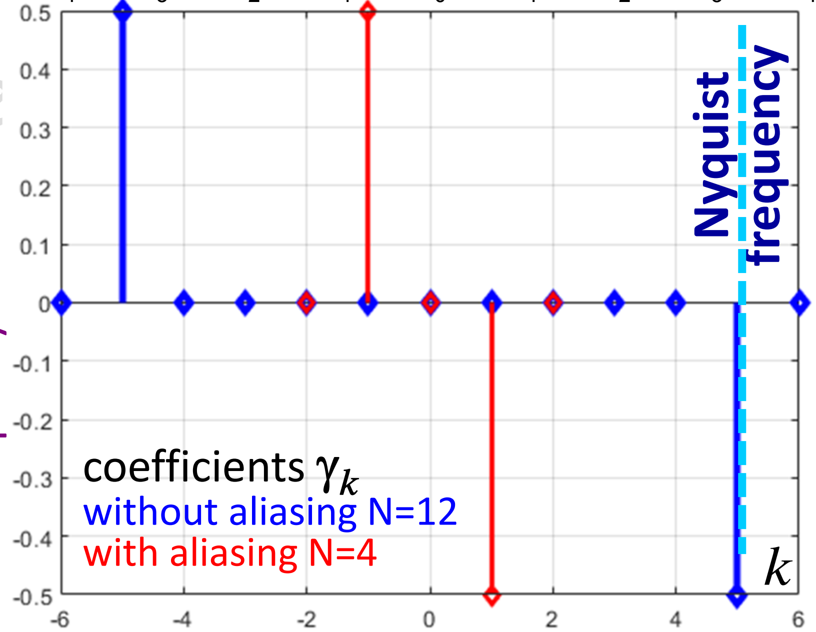
the sampling step is too large



reconstructed signal: $\sin(x)$ instead of $\sin(5x)$



if the sampling frequency is not appropriate, a function with a lower frequency is reconstructed



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How to remove the Windowing error?

Just "observe" the whole signal if possible.

For periodic functions, just choose the window width equal to the function period or a multiple thereof

How to remove the Aliasing error?

Just "sample" all the frequencies.

For band limited functions, just choose a "suitable" sample rate (sampling frequency) according to …

the Sampling Theorem* ⇨ "appropriate" frequency

* later (in Fourier Transform)