SIS Scuola Interdipartimentale
delle **Scienze**, dell'**Ingegneria**
e della **Salute**

DiST
DIPARTIMENTO DI SCIENZE
E TECNOLOGIE

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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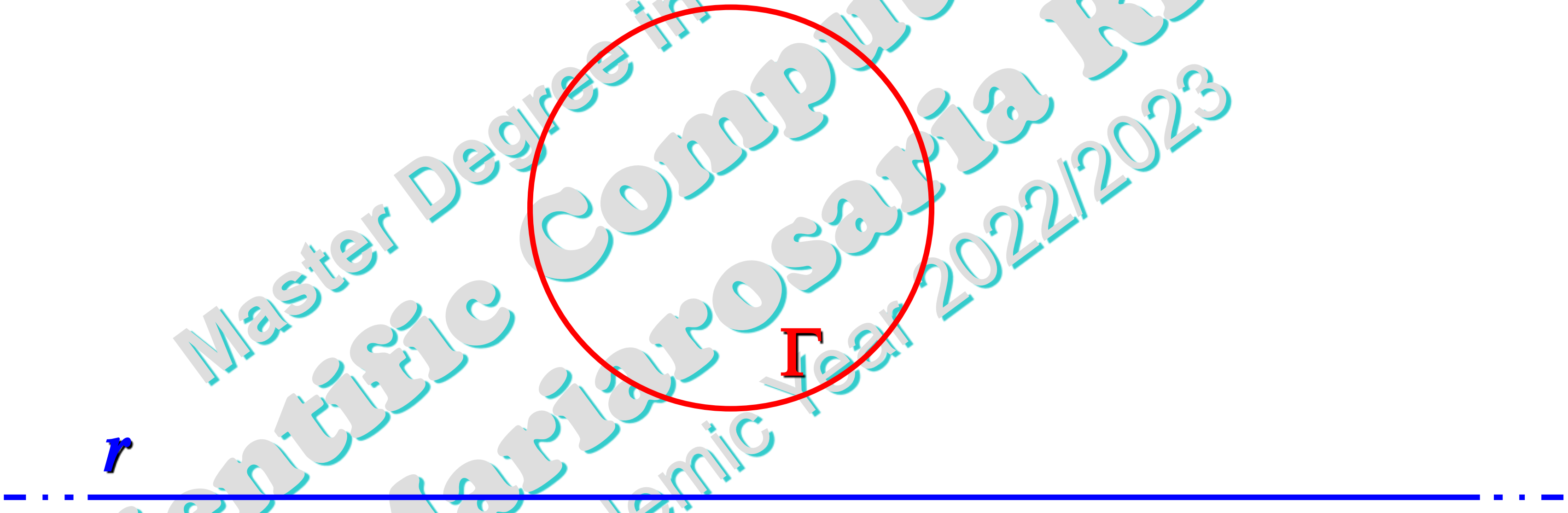
Contents

- **Extended complex plane $\mathbb{C}^*$.**
- **Stereographic projection.**
- **Moebius mappings.**

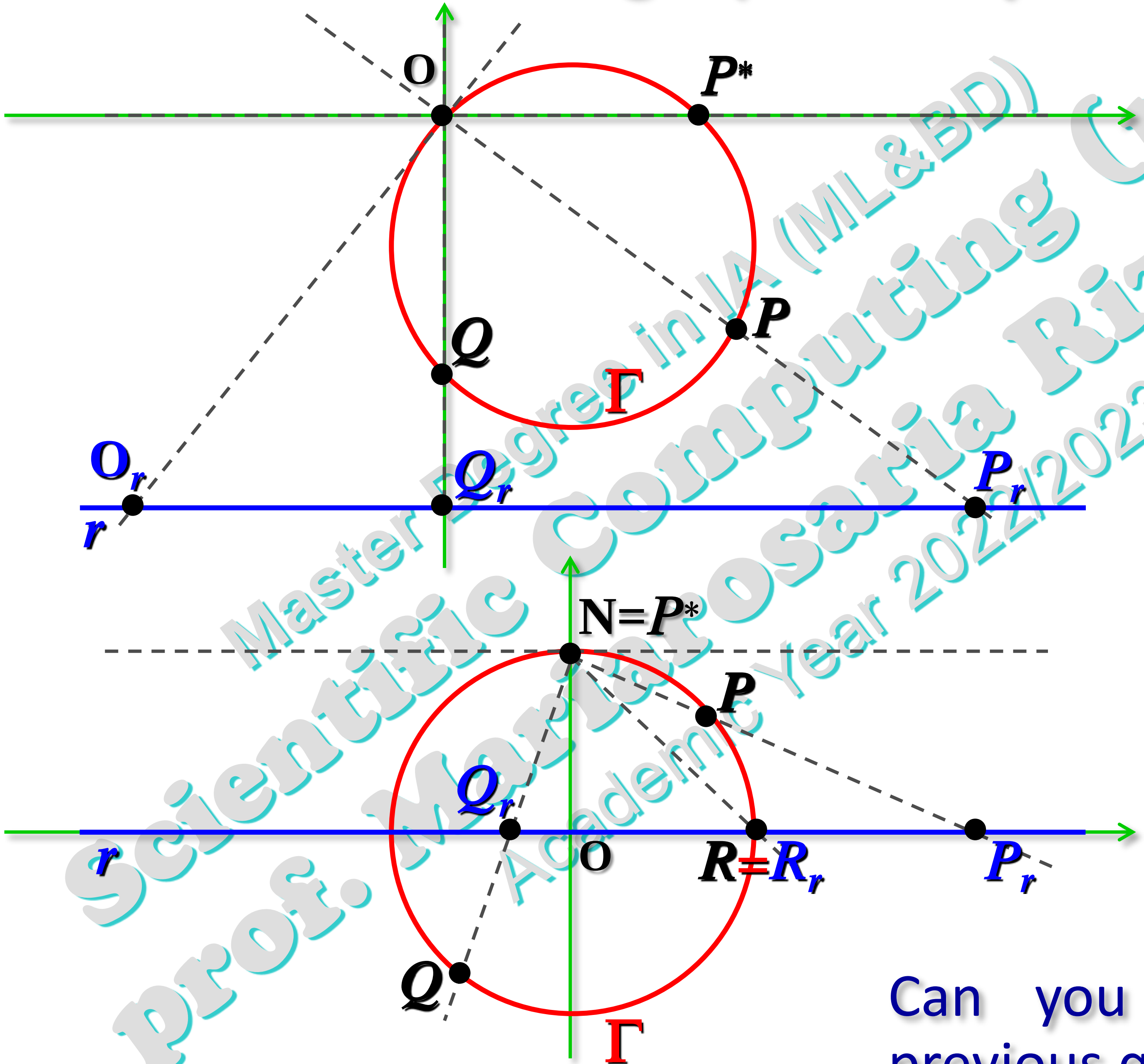
Quiz

Let us consider the following plane curves: a circle Γ and a straight line r.



Are there more points on the circle Γ or on the line r?

2D Stereographic Projection



$P \longleftrightarrow P_r$

$Q \longleftrightarrow Q_r$

$P^* \longleftrightarrow \infty$

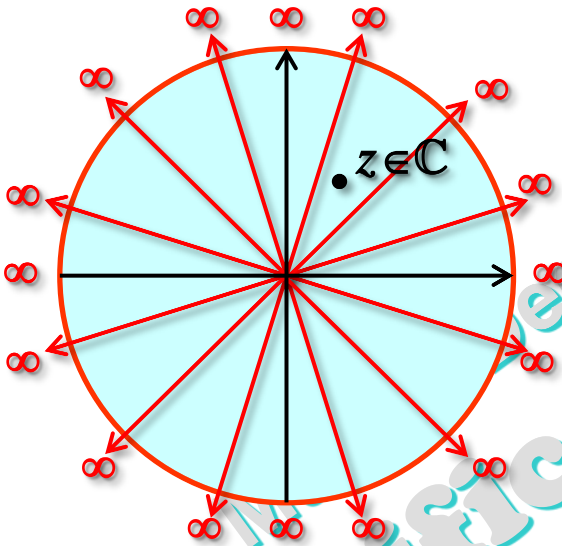
...

Can you answer the previous question now?

Extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

$\mathbb{C}^*$ or $\mathbb{C}_\infty$ denotes the extended complex plane, that contains all the complex numbers and also the point at ∞

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$$



There is a single point at ∞, which lies on the boundary of the complex plane $\mathbb{C}$.

It can be imagined as a point where $|z| \to \infty$ in all directions (its argument is undefined).

In the extended complex plane $\mathbb{C}^*$ the following operations are defined (as limits):

$z + \infty = \infty$, $\quad \infty + \infty = \infty$,

$z \times \infty = \infty \ (z \neq 0)$, $\quad \infty \times \infty = \infty$,

$z / \infty = 0$, $\quad 0 / \infty = 0$,

$z / 0 = \infty$, $\quad \infty / 0 = \infty$.

But the following operations are undefined:

$+\infty - \infty$,

$0 \times \infty$,

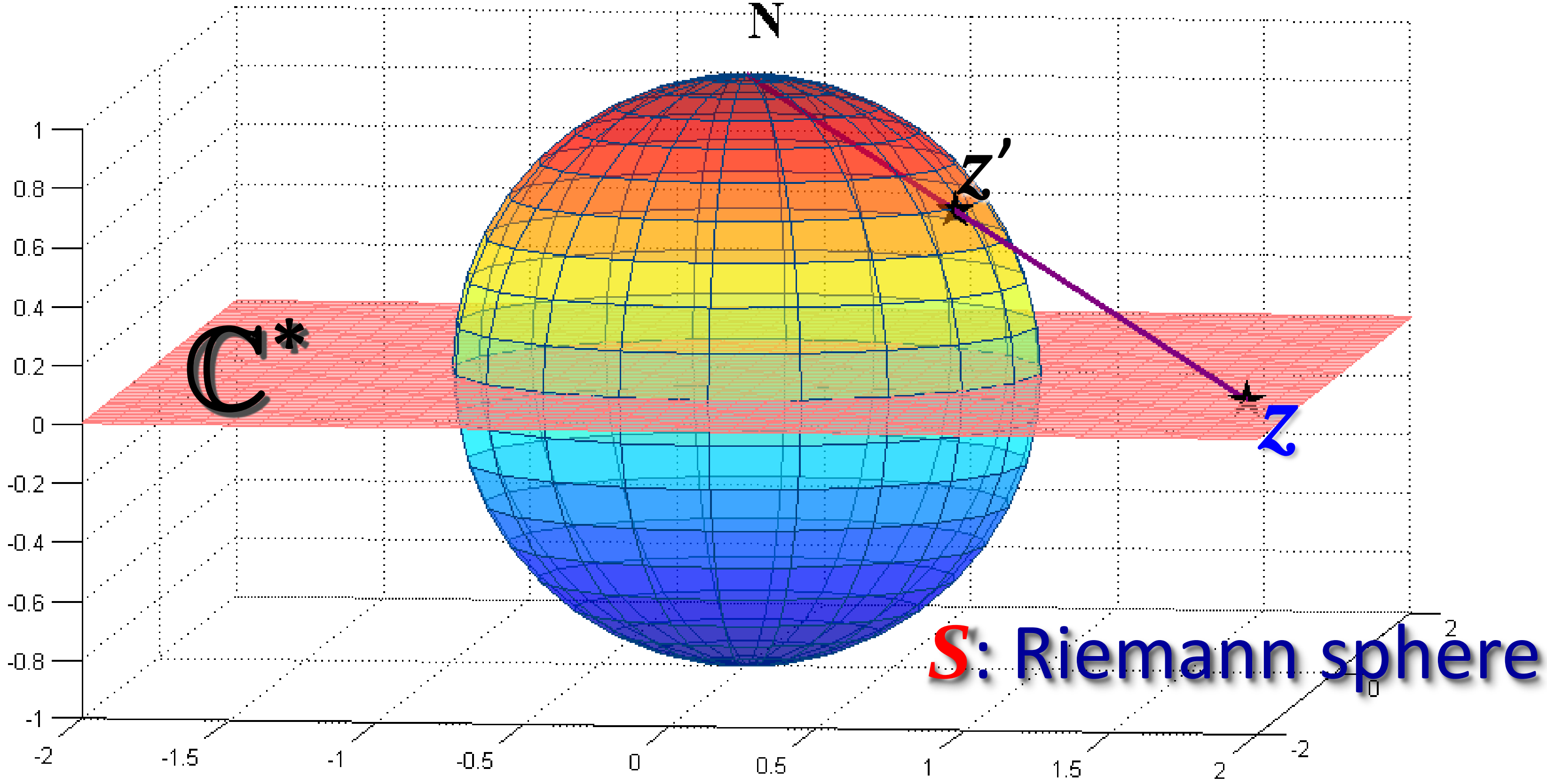
$0 / 0$,

∞ / ∞

3D Stereographic projection (1/6)

It is a one-to-one mapping between the extended complex plane $\mathbb{C}^*$ and all the points on the unit sphere $S(0,1)$. Each complex number z is mapped to z', that is the intersection between the sphere and the half-line passing through the north pole N and z.

$$S(0,1)=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\right\}$$



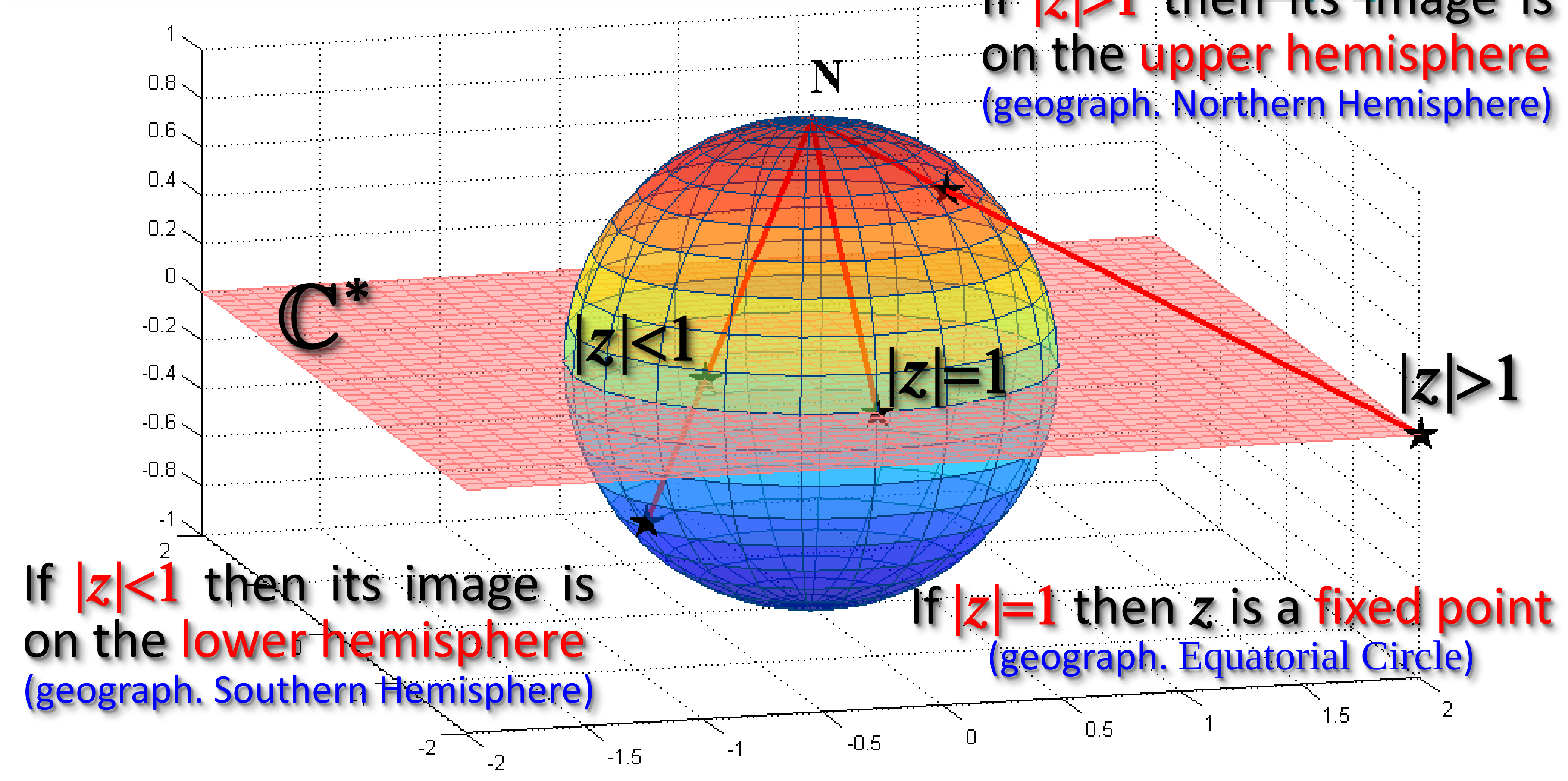
S: Riemann sphere

SCp2_08e.5

Conformal Mappings

(prof. M. Rizzardi)

Stereographic projection (2/6)



If $|z|>1$ then its image is on the upper hemisphere (geograph. Northern Hemisphere)

If $|z|<1$ then its image is on the lower hemisphere (geograph. Southern Hemisphere)

If $|z|=1$ then z is a fixed point (geograph. Equatorial Circle)

There is no complex number z corresponding to the North Pole $N(0,0,1)$.

The map between $\mathbb{C}^*$ and S is completed by setting:

$$N(0,0,1) \longleftrightarrow \infty$$

and now, on S, the point at ∞ is as visible as any other point.

Stereographic projection (3/6)

The (one-to-one) map is defined as:

$$z \in \mathbb{C} \longrightarrow z' \in S_{(0,1)} - \{\mathrm{N}\} \subset \mathbb{R}^3$$

but it can be completed to the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ if the north pole $\mathrm{N}(0,0,1)$ is mapped to ∞.

$S_{(0,1)}$: Riemann sphere

$$z \in \mathbb{C}^* \longleftrightarrow z' \in S_{(0,1)} \subset \mathbb{R}^3$$

from $\mathbb{C}^*$ to S

$$z' = \begin{cases} z_1' = 2\dfrac{\operatorname{Re} z}{|z|^2+1} \\ z_2' = 2\dfrac{\operatorname{Im} z}{|z|^2+1} \\ z_3' = \dfrac{|z|^2-1}{|z|^2+1} \end{cases} \quad \Longleftrightarrow \quad z = \frac{z_1' + i z_2'}{1 - z_3'}$$

from S to $\mathbb{C}^*$

Stereographic Projection (4/6)

It makes visible the point ∞ as any other point.

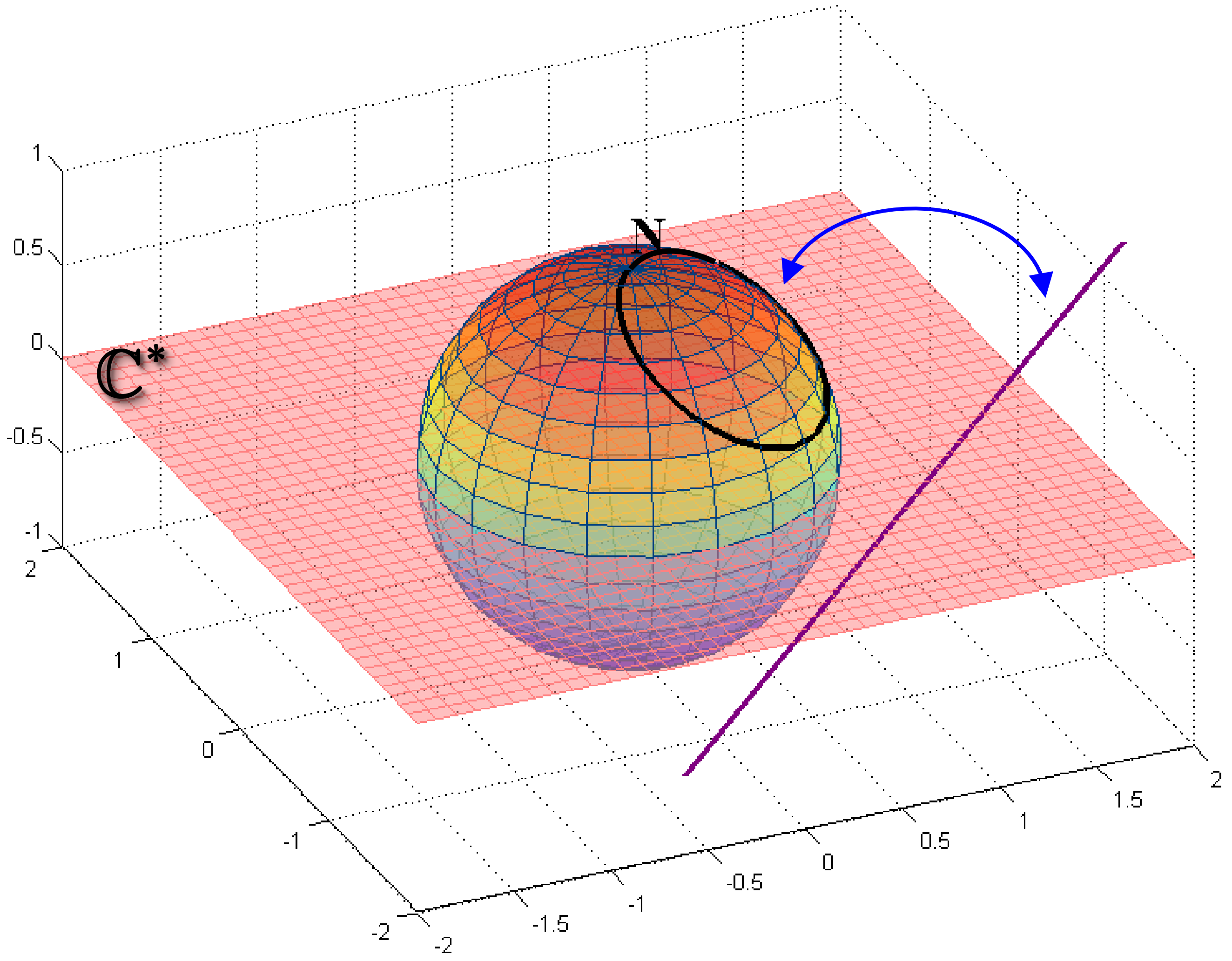
Property

The Stereographic Projection maps all the lines and circles, in the complex plane, onto all the circles on the sphere.
The images of the lines in the complex plane are just the circles on the sphere passing through the North Pole N.

DEF

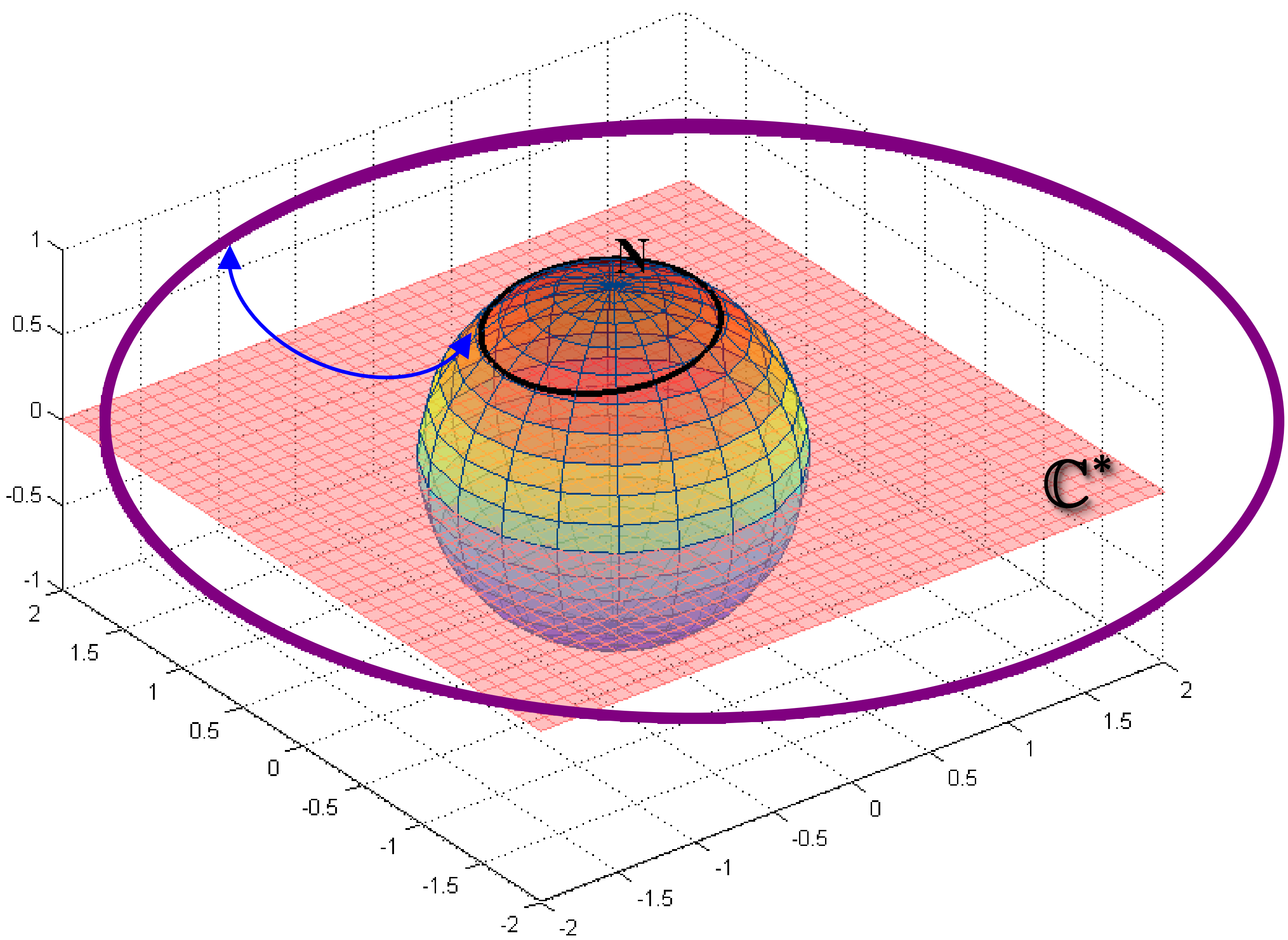
A **generalized circle** (also named as **cline** or **circline**) is a **straight line** $\cup \{\infty\}$ or a **circle**

Stereographic projection of a line (5/6)



A line in the complex plane is mapped to a circle on *S* passing through N

Stereographic projection of a circle (6/6)



A circle in the complex plane is mapped to a circle on *S* not passing through N

Moebius mappings

$$T_M : z \in \mathbb{C}^* \longrightarrow w \in \mathbb{C}^* \qquad w = T_M(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}$$

$\mathbb{C}^*$ is the extended complex plane

main correspondences

$$z \neq -\frac{d}{c} \xrightarrow{T_M} w = \frac{az+b}{cz+d}$$

$$z = -\frac{d}{c} \xrightarrow{T_M} w = \infty$$

$$z = \infty \xrightarrow{T_M} w = \frac{a}{c}$$

We can also use the homogeneous coordinates for complex numbers

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} u \\ v \end{pmatrix} \longleftrightarrow \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} w \\ 1 \end{pmatrix}$$

$$\infty = (\zeta, 0)^T, \ \zeta \in \mathbb{C}^* : \zeta \neq 0$$

M: transformation matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} w \\ 1 \end{pmatrix} = M \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \equiv \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

$$T_M(z) = \frac{az+b}{cz+d}$$

det M ⇓

$$T'_M(z) = \frac{\boxed{ad-bc}}{(cz+d)^2}$$

```
syms a b c d z
M=[a b;c d]; Z=[z;1];
w=M*Z; T=w(1)/w(2);
dT=simplify(diff(T,z))
dT =
(a*d - b*c)/(d + c*z)^2
```

$$\lim_{z\to\infty} T'_M(z) = 0$$

T_M conformal if det$M \neq 0$,
non-conformal at ∞.

If det$M=0$, what does it mean for T_M?

Moebius mappings

$$w = T_M(z) = \frac{az+b}{cz+d}$$

invertible

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det(M) \neq 0$$

$$w \in \mathbb{C}^* \longrightarrow z \in \mathbb{C}^*$$

inverse mapping

$$z = T_M^{-1}(w) = \frac{dw-b}{-cw+a}$$

its inverse is of the same kind as T_M

inverse transformation matrix $M^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \text{adj } M \; : \; M \cdot \boxed{M^*} = \det M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The matrix M^* of the inverse mapping is the **adjoint of M**, defined as: $M^* = \det M \cdot M^{-1}$

i.e., M^* is the same as the inverse, with the addition of the scale factor $\det(M)$

Properties of a Moebius map (1/2)

☑ The inverse map of a T_M is a Moebius map too.

☑ A T_M (≠ identity) has 2 fixed points at most. If a T_M has 3 fixed points, then it is the identity map.

☑ Every T_M consists of at most two translations, a homothety, a rotation and an inversion:

$$w = T_M(z) = \frac{az+b}{cz+d} = f_4 \circ f_3 \circ f_2 \circ f_1$$

$$\underbrace{f_1(z) = z + d/c}_{\text{translation}}, \quad \underbrace{f_2(z) = 1/z}_{\text{inversion}}, \quad \underbrace{f_3(z) = \frac{bc-ad}{c^2}z}_{\text{homothety + rotation}}, \quad \underbrace{f_4(z) = z + a/c}_{\text{translation}}$$

☑ A T_M maps "generalized circles" to "generalized circles".

A generalized circle is a straight line or a circle.

Fixed points of a Moebius map

How many and what are the fixed points of a T_M?

z fixed point of $T_M(z)$ $\overset{\text{def}}{\Longleftrightarrow}$ $T_M(z) = z$

☑ If $z=\alpha/\beta$ is a fixed point of T_M, then $(\alpha,\beta)^T$ is an eigenvector of M

But the eigenvalues of M have no connection with the fixed points of T_M

Example 1: inversion
(or reciprocal map)

$$T_M(z) = \frac{1}{z} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

```
syms z;  fz=1/z;
zu=solve(fz == z)
zu =
-1
 1
```

2 fixed points

```
M=sym([0 1;1 0]); [V,D]=eig(sym(M))
V =
[1, -1]
[1,  1]
```

```
D =
[1,  0]
[0, -1]
```

Example 2:

$$T_M(z) = 4z + 8i = \frac{4z+8i}{1} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 8i \\ 0 & 1 \end{pmatrix}$$

```
syms z;  fz=4*z+8i;
zu=solve(fz == z)
zu =
-8i/3
```

1 fixed point

```
M=sym([4 8i;0 1]); [V,D]=eig(sym(M))
V =
[-8i/3, 1]
[    1, 0]
```

```
D =
[1, 0]
[0, 4]
```

Fixed points of a Moebius map

How many and what are the fixed points of a T_M?

z fixed point of $T_M(z)$ $\overset{\text{def}}{\Longleftrightarrow}$ $T_M(z) = z$

☑ If $z=\alpha/\beta$ is a fixed point of T_M, then $(\alpha,\beta)^T$ is an eigenvector of M

But the eigenvalues of M have no connection with the fixed points of T_M

Example 3:

$$T_M(z) = \frac{z-i}{z+i} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

```
syms z;  fz=(z-1i)/(z+1i);
zu=solve(fz == z)
zu =
-(-6i)^(1/2)/2 + (1/2 - 1i/2)
 (-6i)^(1/2)/2 + (1/2 - 1i/2)
```

2 fixed points

```
M=sym([1 -i;1 i]); [V,D]=eig(sym(M)); V
V =
[-(-6i)^(1/2)/2+(1/2-1i/2), (-6i)^(1/2)/2+(1/2-1i/2)]
[                        1,                        1]
```

Example 4:

$$T_M(z) = \frac{z-2}{z-1} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

```
syms z;  fz=(z-2)/(z-1);
zu=solve(fz == z)
zu =
1 - 1i
1 + 1i
```

2 fixed points

```
M=sym([1 -2;1 -1]); [V,D]=eig(sym(M))
V =
[1 - 1i, 1 + 1i]
[     1,      1]
```

```
D =
[-1i,  0]
[  0, 1i]
```

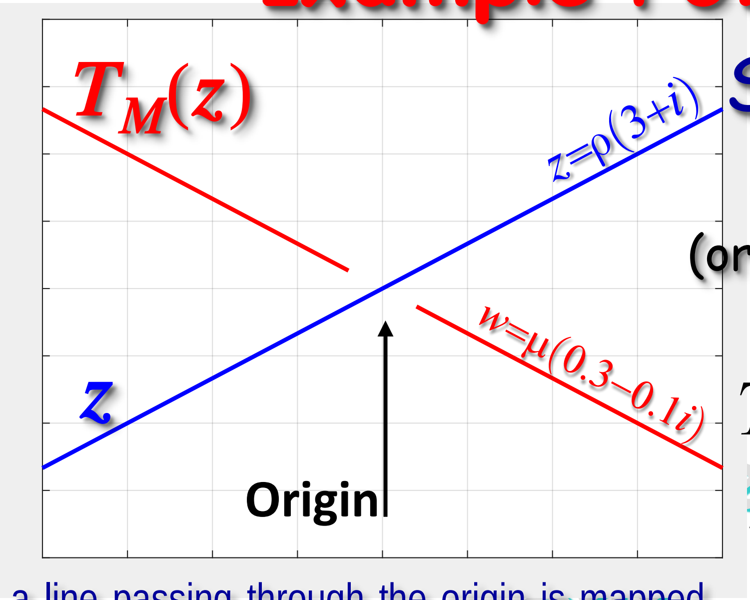
Example 1 of a Moebius mapping

Study the inversion

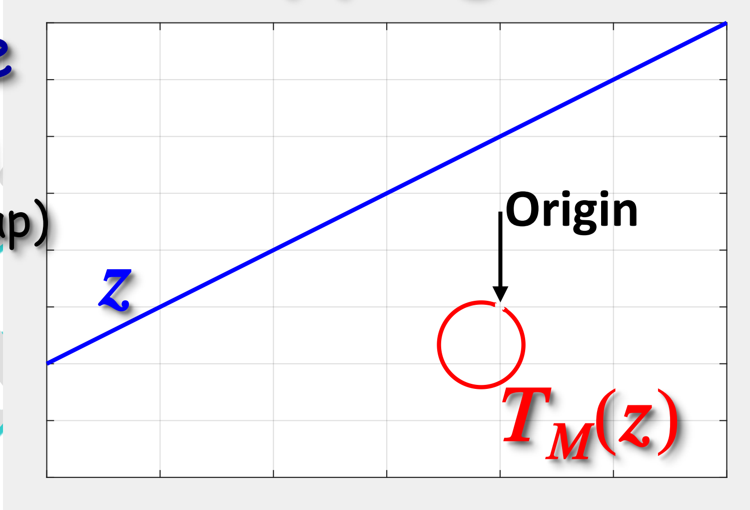
(or reciprocal map)

$$T_M(z) = \frac{1}{z}$$

$$T_M(z) = \frac{\overline{z}}{|z|^2}$$



a line passing through the origin is mapped to a line passing through the origin



a line not passing through the origin is mapped to a circle passing through the origin

origin curve ━━

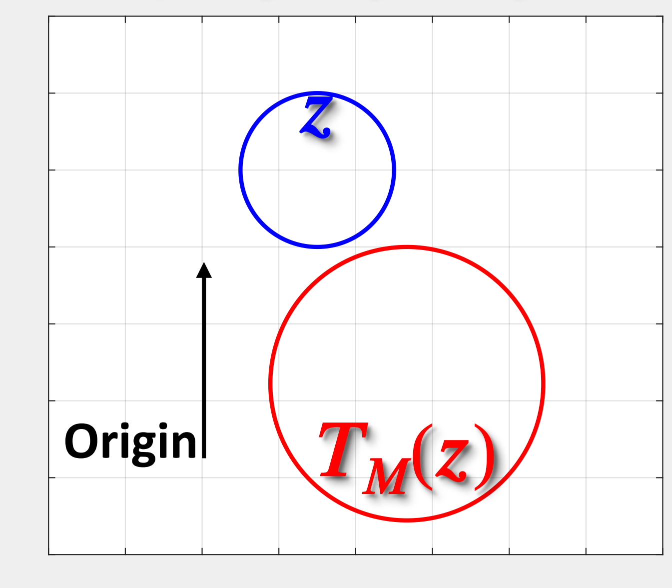
image curve ━━

$$z \neq 0 \xrightarrow{T_M} w = \frac{1}{z}$$

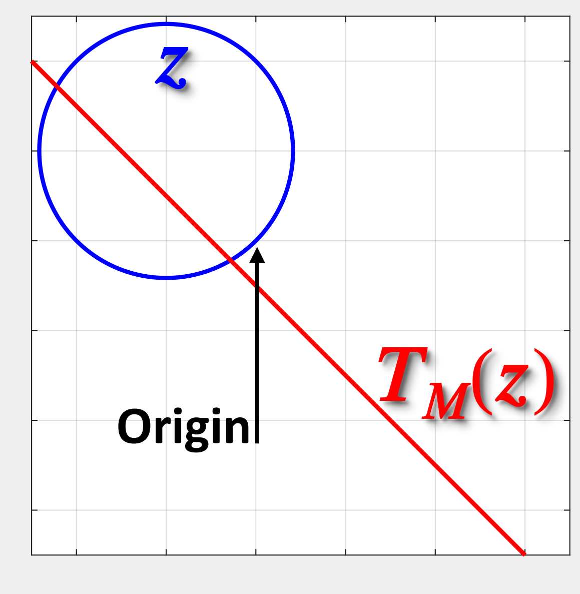
$$z = 0 \xrightarrow{T_M} w = \infty$$

$$z = \infty \xrightarrow{T_M} w = 0$$

$$0 \longleftrightarrow \infty$$

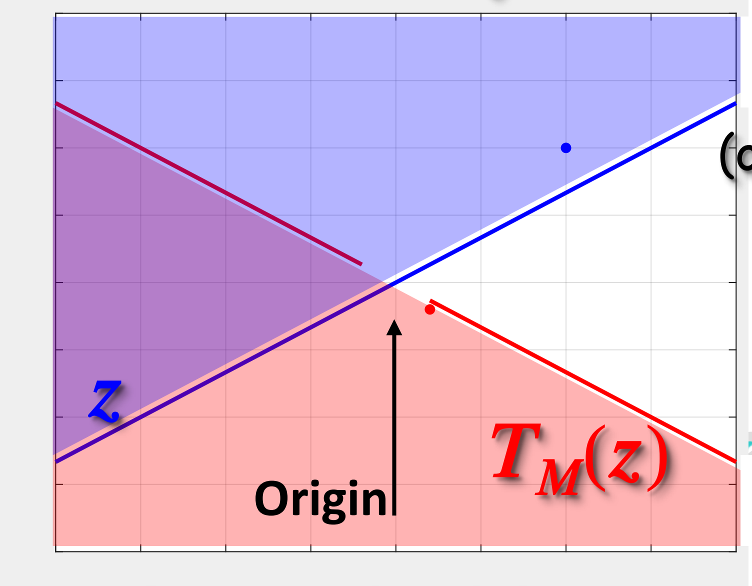


a circle not passing through the origin is mapped to a circle not passing through the origin



a circle passing through the origin is mapped to a line

Example 1 of a Moebius mapping

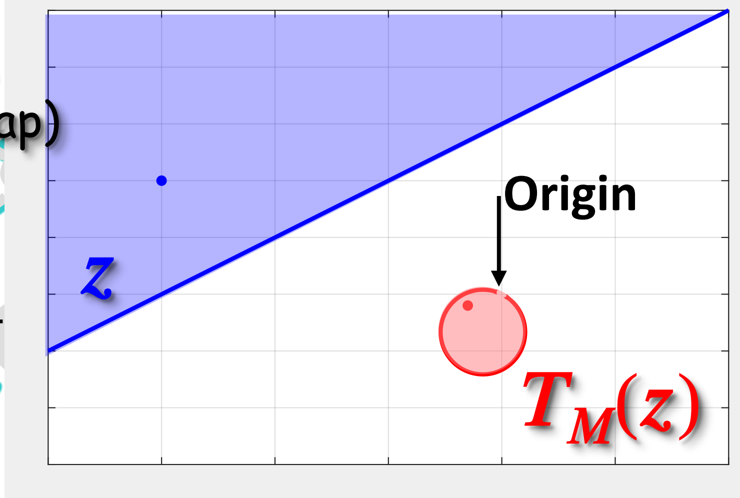


a line passing through the origin

inversion

(or reciprocal map)

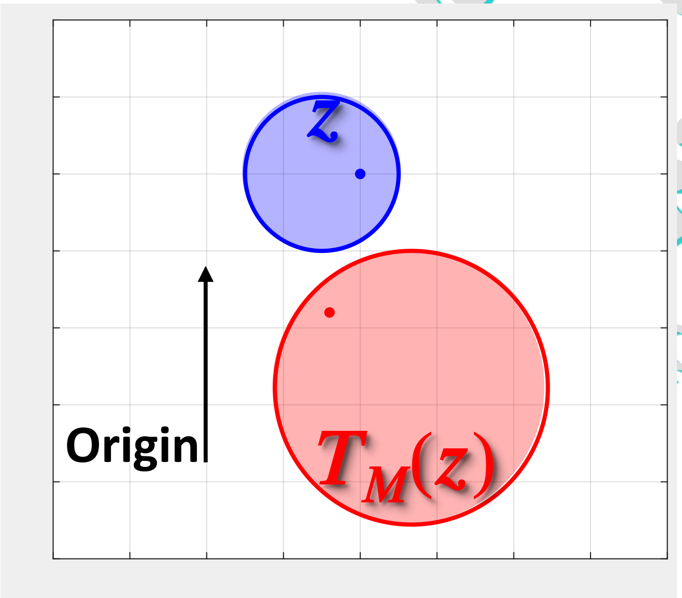
$$T_M(z)=\frac{1}{z}$$



a line not passing through the origin

origin domain ——

image domain ——



a circle not passing through the origin

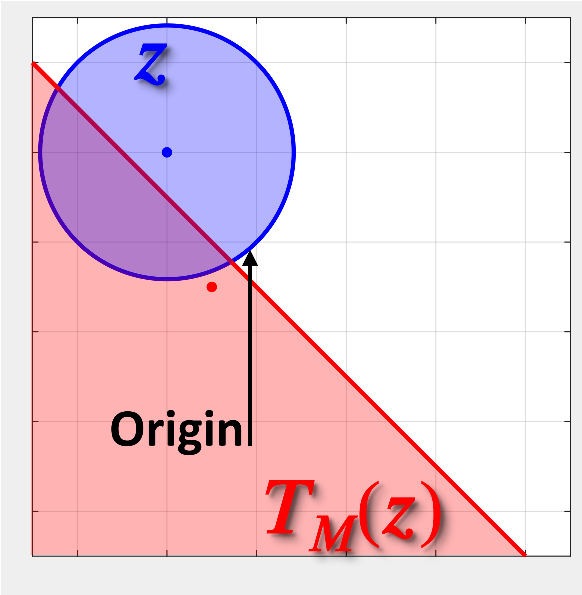
main correspondences

$$z\neq 0 \xrightarrow{T_M} w=\frac{1}{z}$$

$$z=0 \xrightarrow{T_M} w=\infty$$

$$z=\infty \xrightarrow{T_M} w=0$$

$$0 \longleftrightarrow \infty$$



a circle passing through the origin

Example 2 of a Moebius mapping

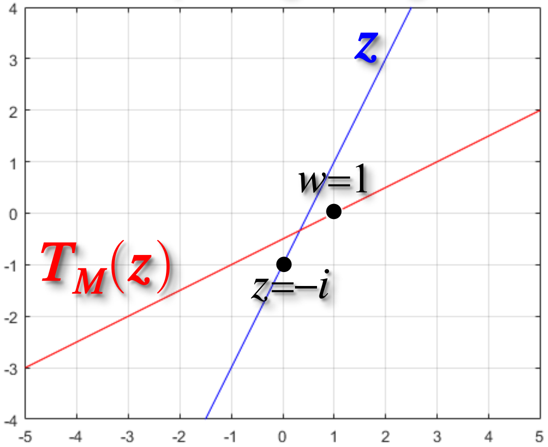
$$w = T_M(z) = \frac{z-i}{\boxed{z+i}} \longrightarrow \boxed{z=-i}$$

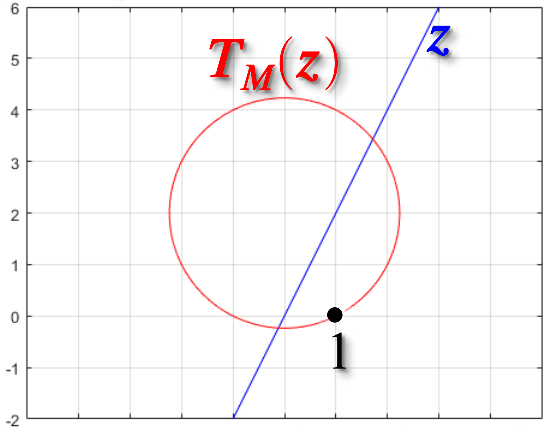
$$z=-i \iff w=\infty$$

$$z \neq -i \iff w = T_M(z)$$

$$z=\infty \iff w=1$$

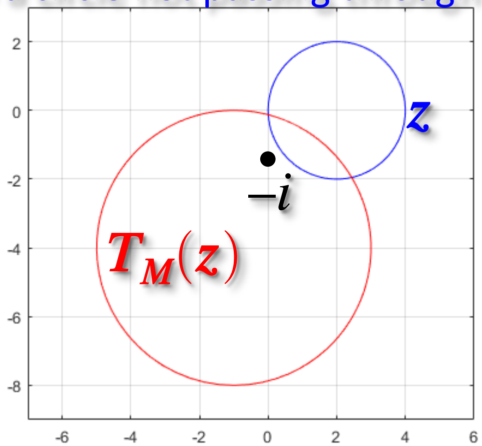
a line passing trough $z=-i$ is mapped to a line passing through $w=1$

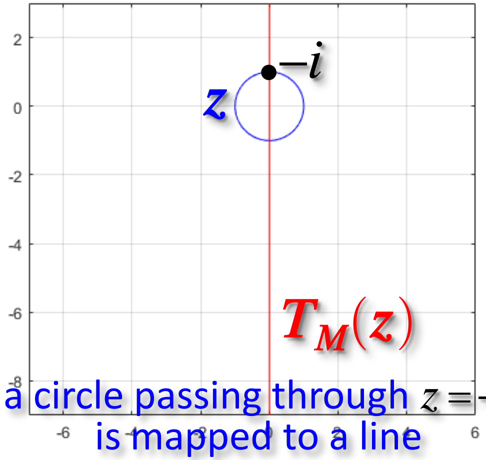




a line not passing through $z=-i$ is mapped to a circle passing through $w=1$

a circle not passing through $z=-i$ is mapped to a circle not passing through $w=1$





a circle passing through $z=-i$ is mapped to a line

Example 2 of a Moebius mapping

origin domain

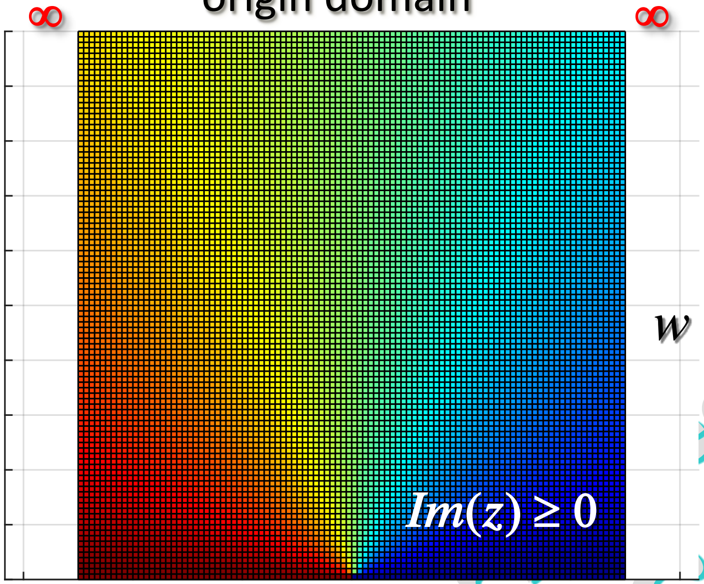
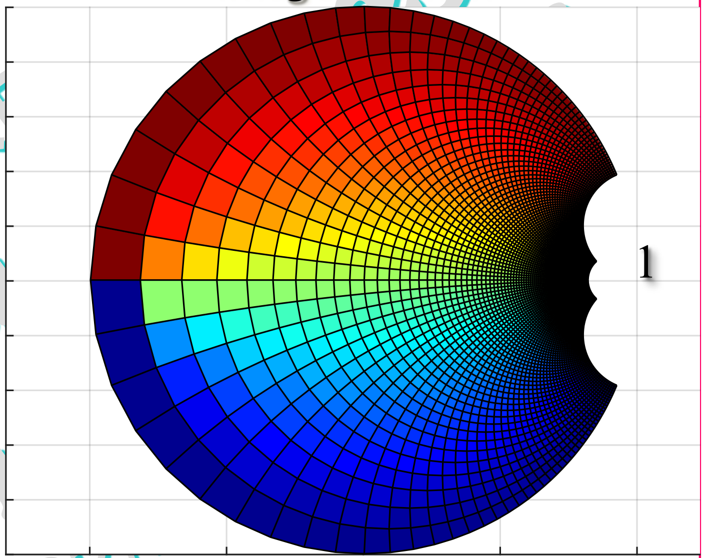


image domain

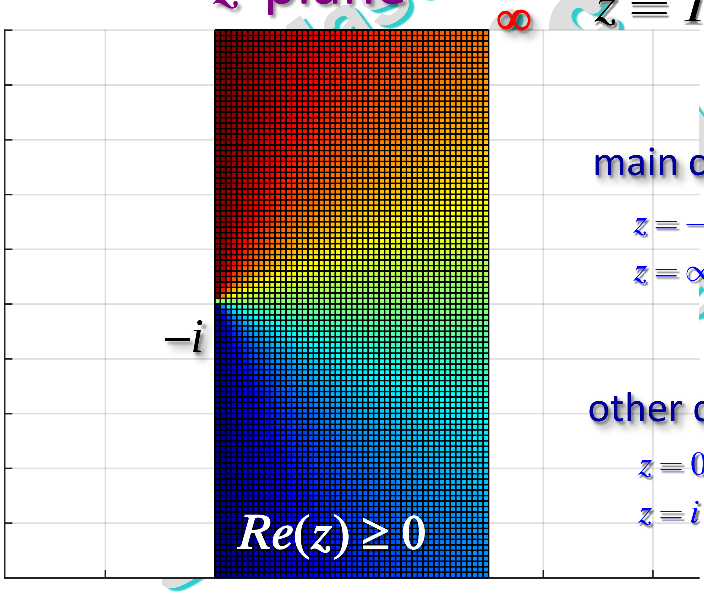


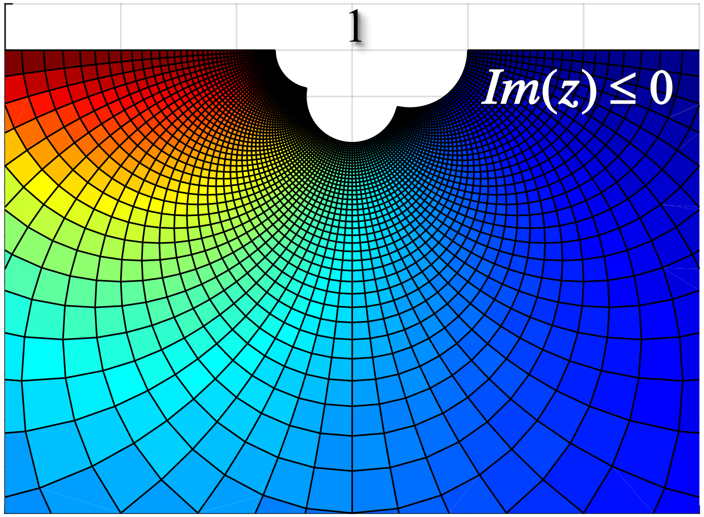
$$w = T_M(z) = \frac{z-i}{z+i}$$

z-plane

$$z = T_M^{-1}(w) = \frac{-i(w+1)}{w-1}$$

w-plane





main correspondences

$z = -i \longleftrightarrow w = \infty$

$z = \infty \longleftrightarrow w = 1$

other correspondences

$z = 0 \longleftrightarrow w = -1$

$z = i \longleftrightarrow w = 0$

Properties of a Moebius map (2/2)

☑ There exists a single T_M that maps 3 different points z_1, z_2, z_3 to 3 different points w_1, w_2, w_3. This T_M is implicitly defined by the following equation

$$\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}$$

If one of the points is ∞, then the ratio containing ∞ must be replaced by 1.

Examples: how to find a T_M

A point at ∞

$z_1 = 0 \longleftrightarrow w_1 = -1$

$z_2 = i \longleftrightarrow w_2 = 0$

$\boxed{z_3 = \infty} \longleftrightarrow w_3 = 1$

$$\frac{w+1}{w-1}\cdot\frac{-1}{+1}=\frac{z}{i}\cdot\boxed{\xcancel{\frac{z_2-z_3}{z-z_3}}}=\frac{z}{i}$$

$$w=T_M(z)=\frac{z-i}{z+i}$$

No point at ∞

Let: $W_k=(w_2-w_3)/(w_2-w_1)$, $Z_k=(z_2-z_3)/(z_2-z_1)$

$$w=T_M(z)=\frac{z\,(w_1\,W_k-w_3\,Z_k)\;+z_1\,w_3\,Z_k-z_3\,w_1\,W_k}{z\,(W_k-Z_k)\;+z_1\,Z_k-z_3\,W_k}$$

Example MATLAB

Find, with the Symbolic Math Toolbox, T_M and T_M^{-1} by the 3-point property

No point at ∞

$$\frac{w-w_1}{w_2-w_1}\cdot\frac{w_2-w_3}{w-w_3}=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}$$

```
zk=[1;1i;-1]; wk=[1;-1i;-1];
syms W Z
Eqn=(W-wk(1))*(wk(2)-wk(3))/((W-wk(3))*(wk(2)-wk(1))) == ...
    (Z-zk(1))*(zk(2)-zk(3))/((Z-zk(3))*(zk(2)-zk(1)));
S=solve(Eqn,W,'ReturnConditions',true)
S = struct with fields:
             W: 1/Z
    parameters: [1×0 sym]
    conditions: Z ~= 0 & Z ~= -1
TM=matlabFunction(simplify(S.W)) % TM: Moebius mapping
TM = function_handle with value:
    @(Z)1.0./Z
S=solve(Eqn,Z,'ReturnConditions',true)
S = struct with fields:
             Z: 1/W
    parameters: [1×0 sym]
    conditions: W ~= 0 & W ~= -1
TM1=matlabFunction(simplify(S.Z)) % TM1: inverse TM
TM1 = function_handle with value:
    @(W)1.0./W
```

Example MATLAB (cont.)

Find, with the Symbolic Math Toolbox, T_M and T_M^{-1} by the 3-point property

z_3 is the point at ∞

$$\frac{w-w_1}{w_2-w_1}\cdot\frac{w_2-w_3}{w-w_3}=\frac{z-z_1}{z_2-z_1}\cdot\cancel{\frac{z_2-z_3}{z-z_3}}$$

```
zk=[0;1i;Inf]; wk=[-1;0;1];
syms W Z
Eqn=(W-wk(1))*(wk(2)-wk(3))/((W-wk(3))*(wk(2)-wk(1))) == ...
    (Z-zk(1))/(zk(2)-zk(1));
S=solve(Eqn,W,'ReturnConditions',true)
S = struct with fields:
            W: (1 + Z*1i)/(-1 + Z*1i)
   parameters: [1×0 sym]
   conditions: Z ~= -1i
TM=matlabFunction(simplify(S.W)) % TM: Moebius mapping
TM = function_handle with value:
    @(Z)(Z.*1i+1.0)./(Z.*1i-1.0)
S=solve(Eqn,Z,'ReturnConditions',true)
S = struct with fields:
            Z: -((W + 1)*1i)/(W - 1)
   parameters: [1×0 sym]
   conditions: symtrue
TM1=matlabFunction(simplify(S.Z)) % TM1: inverse TM
TM1 = function_handle with value:
     @(W)((W+1.0).*-1i)./(W-1.0)
```

$$w=T_M(z)=\frac{z-i}{z+i}$$

Example MATLAB (cont.)

Find, with the Symbolic Math Toolbox, T_M and T_M^{-1} by the 3-point property

w_3 is the point at ∞

$$\frac{w-w_1}{w_2-w_1}\cdot\cancel{\frac{w_2-w_3}{w-w_3}}=\frac{z-z_1}{z_2-z_1}\cdot\frac{z_2-z_3}{z-z_3}$$

```
zk=[.75;.55;.25]; wk=[3;2;Inf];
syms W Z
Eqn=(W-wk(1)) /(wk(2)-wk(1)) == ...
    (Z-zk(1))*(zk(2)-zk(3))/((Z-zk(3))*(zk(2)-zk(1)));
S=solve(Eqn,W,'ReturnConditions',true)
S = struct with fields:
             W: 1/Z
    parameters: [1×0 sym]
    conditions: symtrue
TM=matlabFunction(simplify(S.W)); % TM: Moebius mapping
```

```
S=solve(Eqn,Z,'ReturnConditions',true)
S = struct with fields:
             Z: 1/W
    parameters: [1×0 sym]
    conditions: W ~= 0
TM1=matlabFunction(simplify(S.Z)); % TM1: inverse TM
```

Exercise

Write a single MATLAB function

`W = TM(zk,wk)`

that includes the 3 cases [hint: `isinf`]. Also write a `TM1()` function for the inverse transformation.