SIS Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST Dipartimento di Scienze e Tecnologie

# L. Magistrale in IA (ML&BD)

# Scientific Computing (part 2 – 6 credits)


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# Contents

- **Conformal Mappings.**
- **Law on local arc transformation** (geometric interpretation of derivative of a complex function).
- **Theorem**: $f$ **holomorphic at** $z_0$ **and** $f'(z_0) \neq 0 \Rightarrow w = f(z)$ **conformal.**

# Complex Functions and plane transformations

$$f \;:\; z \in \boldsymbol{A} \longrightarrow w = f(z) = u(x,y) + iv(x,y) \in \boldsymbol{B}$$

$z = x + iy$

$w = u + iv$

$$w = f(z) \iff \mathbf{T}\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

***complex function***

***transformation* T** between $(x, y)$-plane and $(u, v)$-plane

**T** locally invertible $\iff$ $\det[J(x_0,y_0)] = \left|\dfrac{\partial(u,v)}{\partial(x,y)}(x_0,y_0)\right| \neq 0$ $\iff$ $f'(z_0) \neq 0$

$\det[J(x_0,y_0)] \neq 0$

***A*** and ***B*** simply connected domains $\Rightarrow$ **T** globally invertible in ***A***

# **Example:** quadratic function

$$w = f(z) = z^2$$

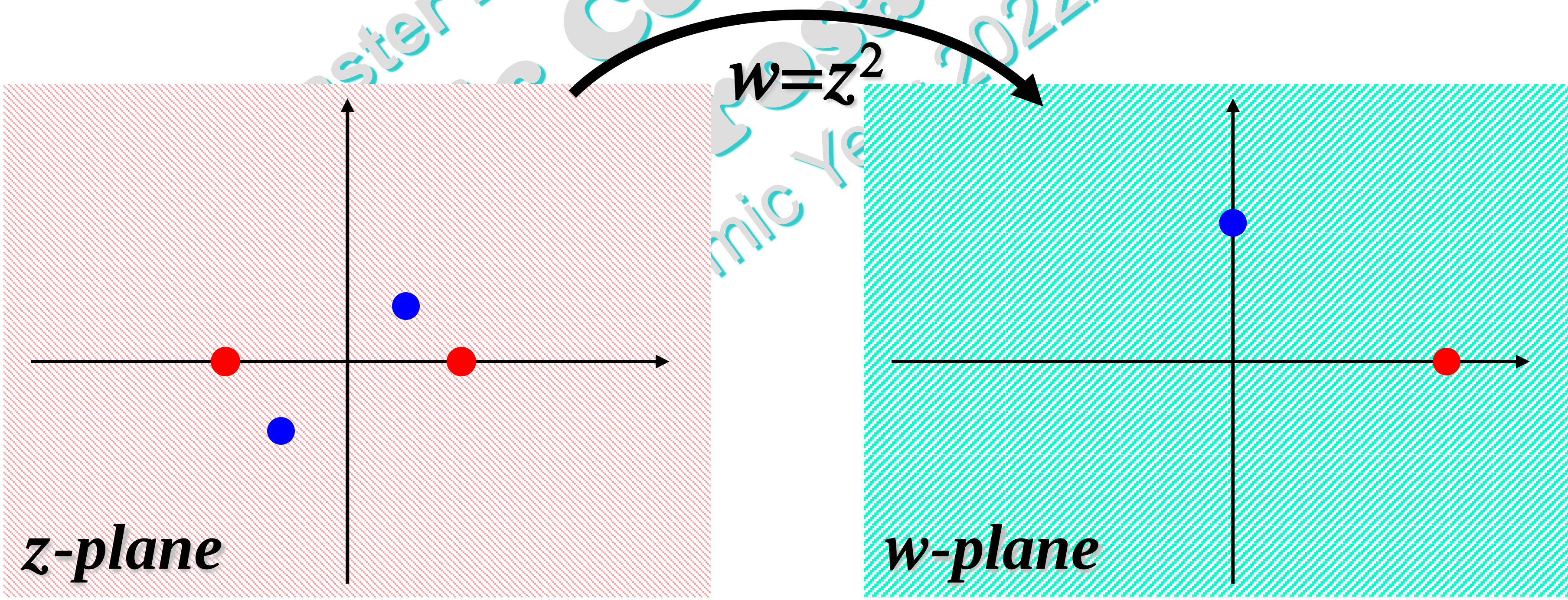
$$z = x + iy$$

$$w = u + iv$$

$$T \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

locally invertible in the whole complex plane except at origin

$$\left|J(x,y)\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}(x,y)\right| = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4\left(x^2+y^2\right) \neq 0 \iff (x,y) \neq (0,0)$$



# Example: complex conjugate function

$w = f(z) = \overline{z}$

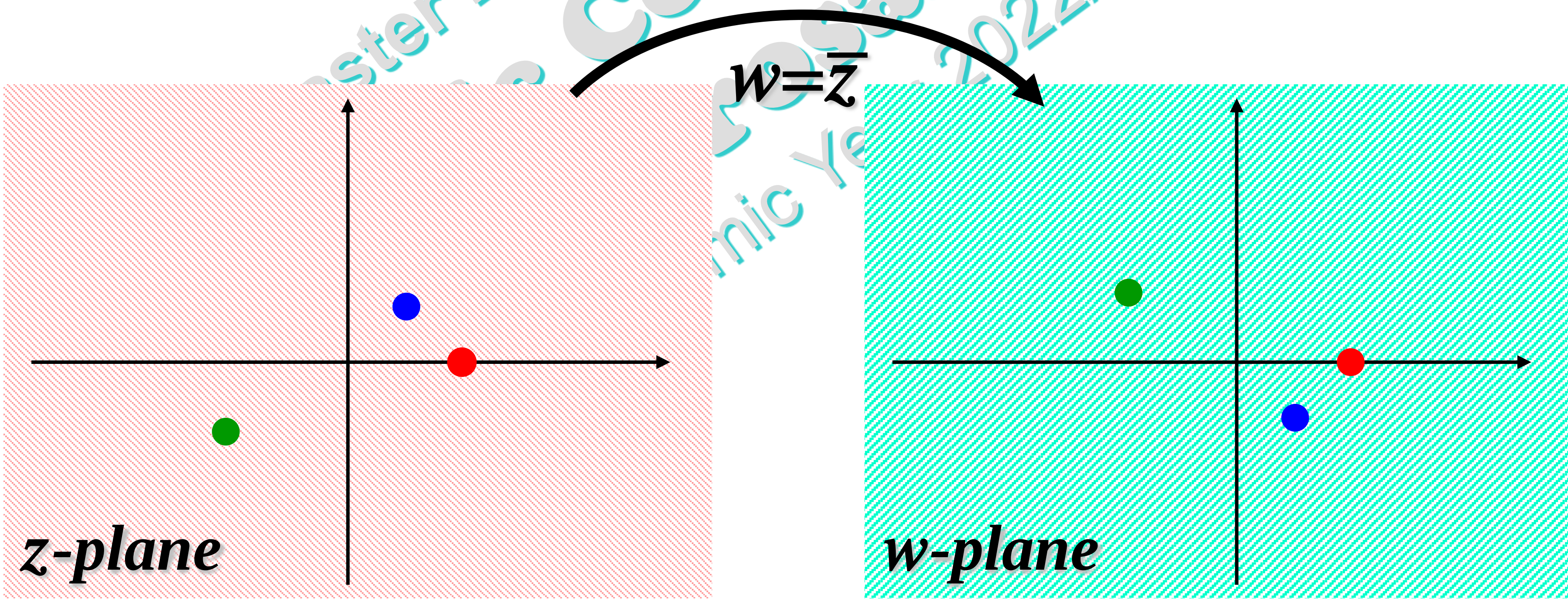
$z = x + iy$

$w = u + iv$

$$T\begin{cases} u = x \\ v = -y \end{cases}$$

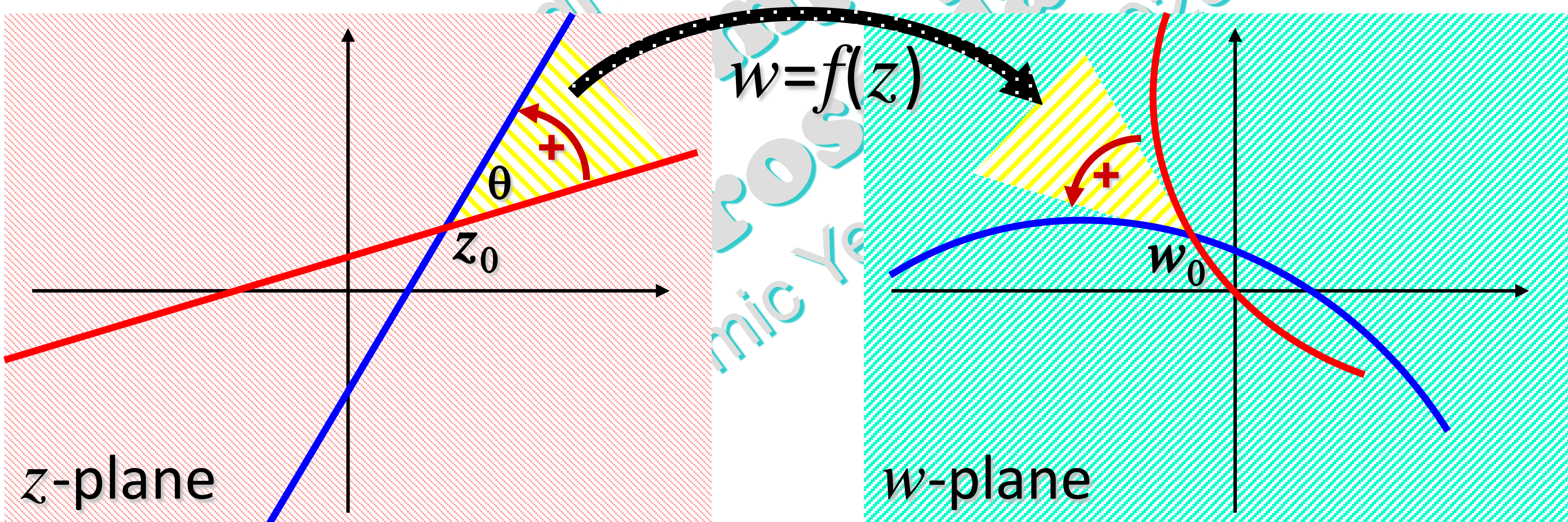
globally invertible in the whole complex plane

$$|J(x,y)| = \left|\frac{\partial(u,v)}{\partial(x,y)}(x,y)\right| = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0 \iff \forall (x,y)$$



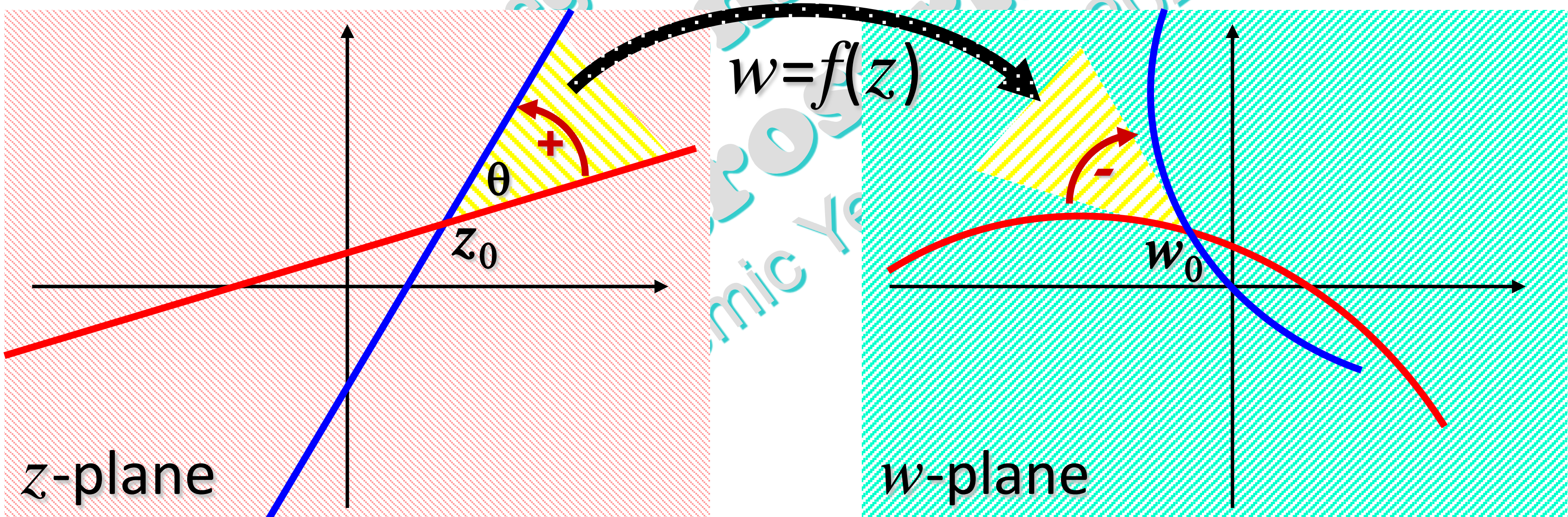
# Conformal map

A conformal map $w=f(z)$ preserves the magnitude of local angles and their orientation.



# Anticonformal map

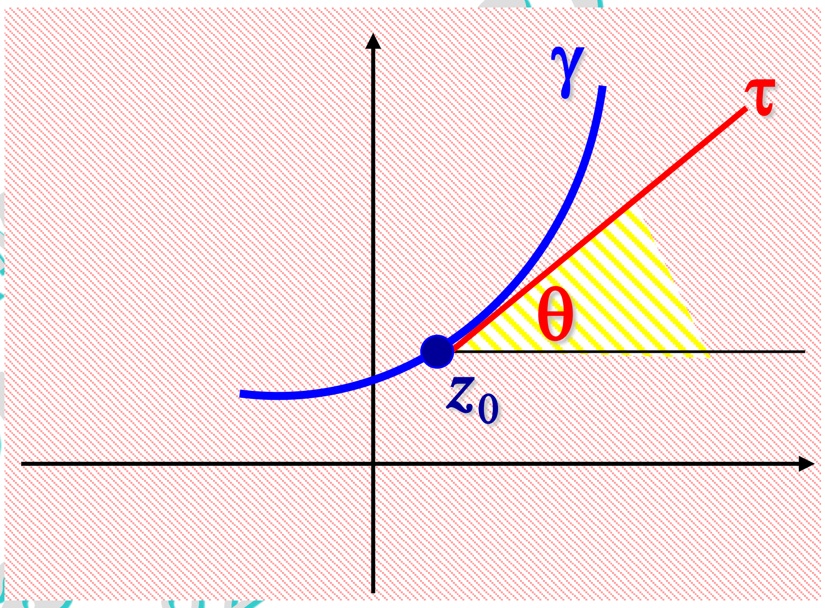
An anticonformal map $w=f(z)$ preserves the magnitude of local angles but it inverts their orientation.



# Transformation of local arcs

Let $\gamma$ be a local arc of a curve:

- contained in a neighbourhood of $z_0$,
- passing through the point $z_0$,
- regular (i.e. equipped with a continuous tangent $\tau$ at each point):




$$\gamma \;:\; z=z(t)=x(t)+iy(t), \quad t\in[a,b]$$
$$z_0=z(t_0)$$

The parametric equation of the tangent line $\tau$ to $\gamma$ at $z_0$ is:

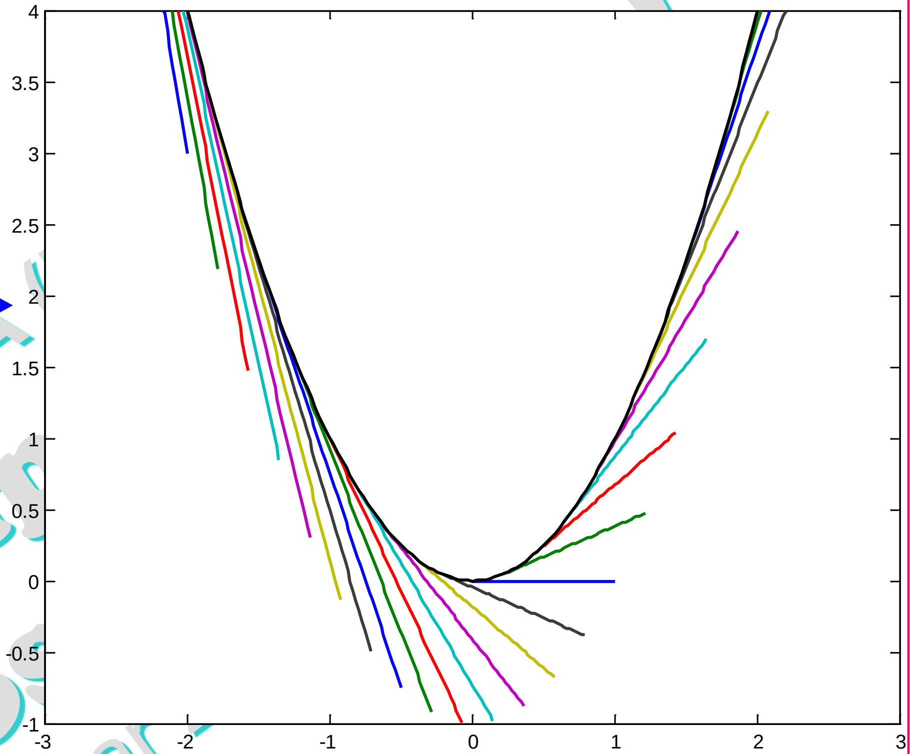
$$\tau \;:\; z=z(\rho)=z_0+\rho z'(t_0), \qquad \rho\in]-\infty,+\infty[.$$

By definition, the tangential angle $\theta$ of the arc $\gamma$ at $z_0$ is the angle between the tangent $\tau$ and the $x$-axis:

$$\boxed{\theta = \arg z'(t_0)}$$

**Example** $\gamma$ is an arc of parabola of eq.: $z=z(t)=t+it^2,\ t\in[-3,+3]$

an arc of parabola and some tangent segments



```
syms t real;  N=197;  tt=linspace(-3,3,N);
z=@(t) t+i*t^2;  zt=subs(z(t),tt);
zprime=(diff(z(t)));  zprimet=subs(zprime,tt);
rho=linspace(0,1,50)'; j=0; step=7;
for h=1:step:N
        j=j+1; tanz(:,j)=zt(h)+rho*zprimet(h);
end
plot(real(tanz),imag(tanz),real(zt),imag(zt),'k');
axis([-3 3 -1 4]); axis('equal')
```

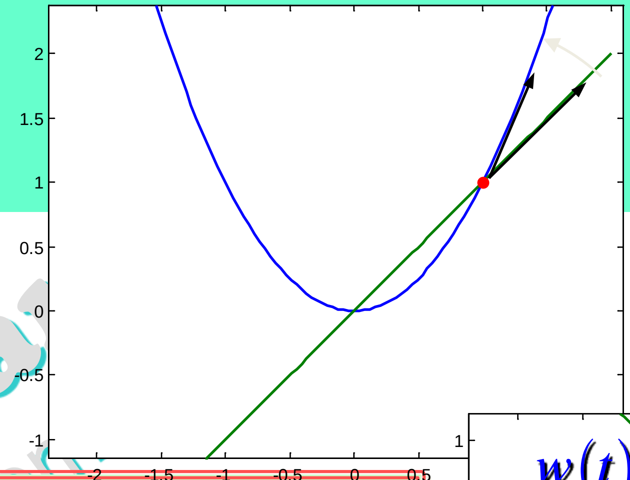
$$\tau : z=z(\rho)=z_0+\rho z'(t_0),\quad \rho\in\mathbb{R}$$

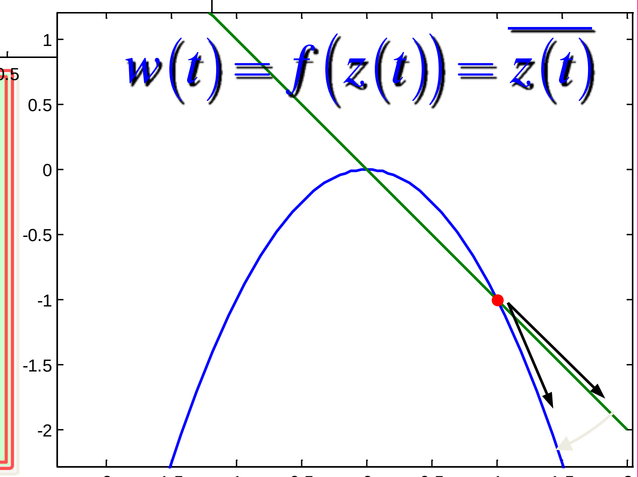
# Exercises

Change the previous symbolic code into numerical code, by approximating derivatives by means of difference quotients.

Compute the angle between the line, of eq.: $z_1=t+it$, and the parabola, of eq.: $z_2=t+it^2$, at their intersection point $P(1,1)$, and the angle between their image curves by means of the following mappings:

1. $w=f(z)=2z-1$
2. $w=f(z)=\overline{z}$



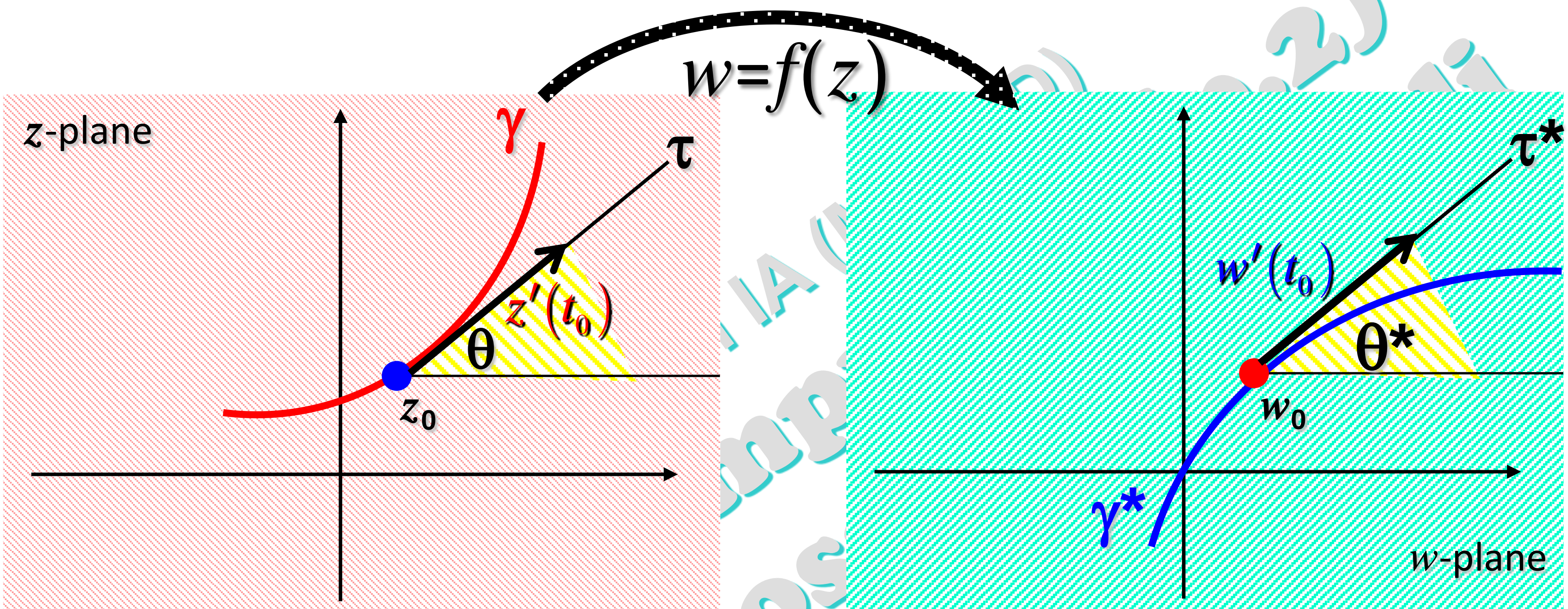



```
z1=@(t) t+i*t;   z2=@(t) t+i*t.^2;
N=99;  tt=linspace(-2,2,N);
z1t=z1(tt);   z2t=z2(tt); z0=1+i;
plot(real(z1t),imag(z1t),real(z2t),imag(z2t), ...
       real(z0),imag(z0),'.r'); axis('equal')
...
fz=@conj; w0=fz(z0); w1t=fz(z1t); w2t=fz(z2t);
plot(…
```

# Law on transformation of local arcs (1/5)








γ is a smooth arc through $z_0$

γ* is the image of γ by $w=f(z)$

$$\gamma : z = z(t), \quad t \in [a,b]$$

$$z_0 = z(t_0) \quad a < t_0 < b$$

$$\gamma^* : w = w(t) = f(z(t)), \quad t \in [a,b]$$

$$w_0 = f(z_0) = f(z(t_0))$$

The tangent τ to γ at $z_0$ has equation

$$\tau : z = z(\rho) = z_0 + \rho \cdot z'(t_0), \quad \rho \in \mathbb{R}$$

and the angle θ of γ at $z_0$ is given by

$$\theta = \arg z'(t_0)$$

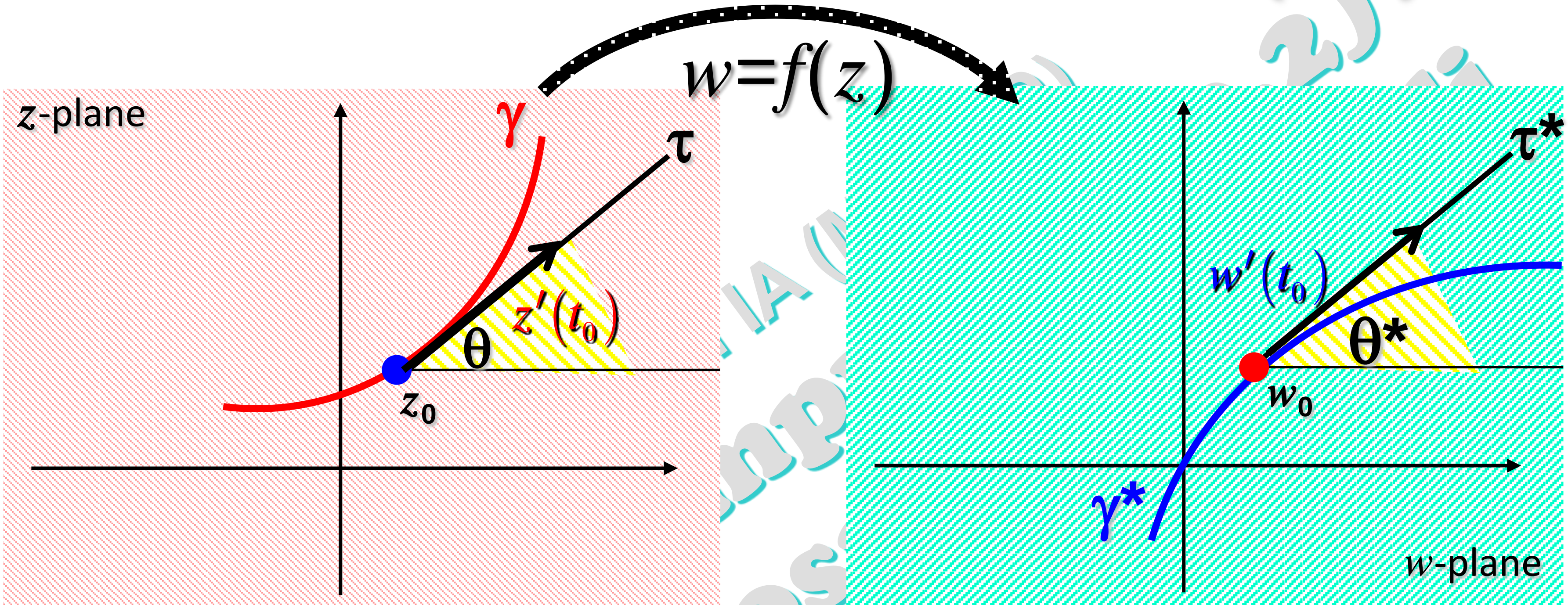
The tangent τ* to γ* at $w_0$ has equation

$$\tau^* : w = w(\rho) = w_0 + \rho \cdot w'(t_0), \ \rho \in \mathbb{R}$$

and the angle θ* of γ* at $w_0$ is given by

$$\theta^* = \arg w'(t_0)$$

# Law on local arc transformation (2/5)




$\gamma$ is a smooth arc through $z_0$ $\qquad$ $\gamma^*$ is the image of $\gamma$ by $w=f(z)$

$$\gamma : z = z(t), \quad t \in [a,b] \qquad \gamma^* : w = w(t) = f\big(z(t)\big), \quad t \in [a,b]$$

$$z_0 = z(t_0) \quad a < t_0 < b \qquad w_0 = f(z_0) = f\big(z(t_0)\big)$$

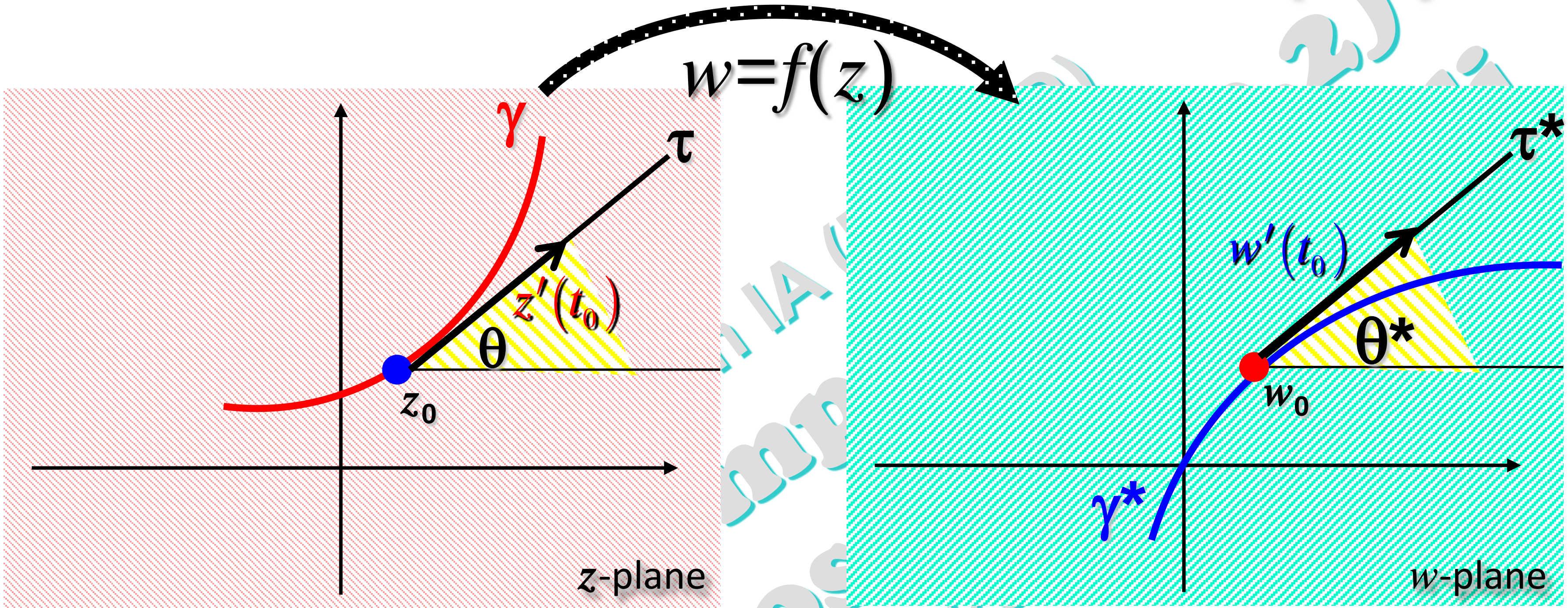
**If $\gamma$ is a regular curve and $f(z)$ is holomorphic, then $\gamma^*$ is regular.**

The tangent $\tau^*$ to $\gamma^*$ at $w_0$ has equation $\tau^* : w = w(\rho) = w_0 + \rho \cdot w'(t_0),\ \rho \in \mathbb{R}$

and $\theta^* = \arg w'(t_0)$

# Law on local arc transformation (3/5)








## In the $z$-plane, around $z_0$:

$\gamma$ = smooth arc at $z_0$

$\gamma : z = z(t), \quad t \in [a,b]$

$\tau : z = z(\rho) = z_0 + \rho z'(t_0), \quad \rho \in \mathbb{R}$

$\tau$ = tangent at $z_0$

## In the $w$-plane, around $w_0$:

$\gamma^*$ = image arc

$\gamma^* : w = w(t) = f(z(t)), \quad t \in [a,b]$

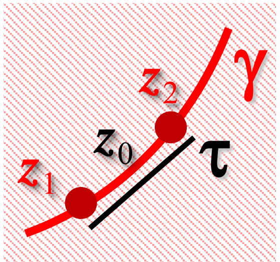
$\tau^* : w = w(\rho) = w_0 + \rho w'(t_0), \quad \rho \in \mathbb{R}$

$\tau^*$ = tangent at $w_0$

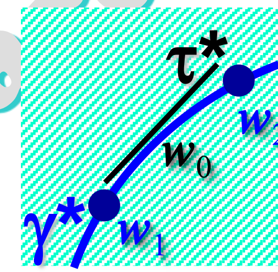
chain rule

$$w'(t_0) = f'(z_0) \cdot z'(t_0), \quad f'(z_0) \neq 0$$

# Law on local arc transformation (4/5)




In the $z$-plane, let $z_1$ and $z_2$ be two points on $\gamma$ very close to $z_0$ so that the chord between them can be considered equal to a tangent segment on $\tau$.




$$w'(t_0) = f'(z_0) \cdot z'(t_0), \quad f'(z_0) \neq 0$$

$z'(t_0)$: related to the curve at that point

$f'(z_0)$: related to the map at that point

$$z_2 - z_1 = z(\rho_2) - z(\rho_1) = [z_0 + \rho_2 z'(t_0)] - [z_0 + \rho_1 z'(t_0)] = [\rho_2 - \rho_1] z'(t_0)$$

$$w_2 - w_1 = w(\rho_2) - w(\rho_1) = [\rho_2 - \rho_1] w'(t_0) = [\rho_2 - \rho_1] z'(t_0) f'(t_0) = [z_2 - z_1] f'(z_0)$$

$$w_2 - w_1 = [z_2 - z_1] f'(z_0)$$

by the equality between complex numbers, we get

**1** Law on infinitesimal angles:

$$\theta^* = \arg[w_2 - w_1] = \theta + \arg f'(z_0)$$

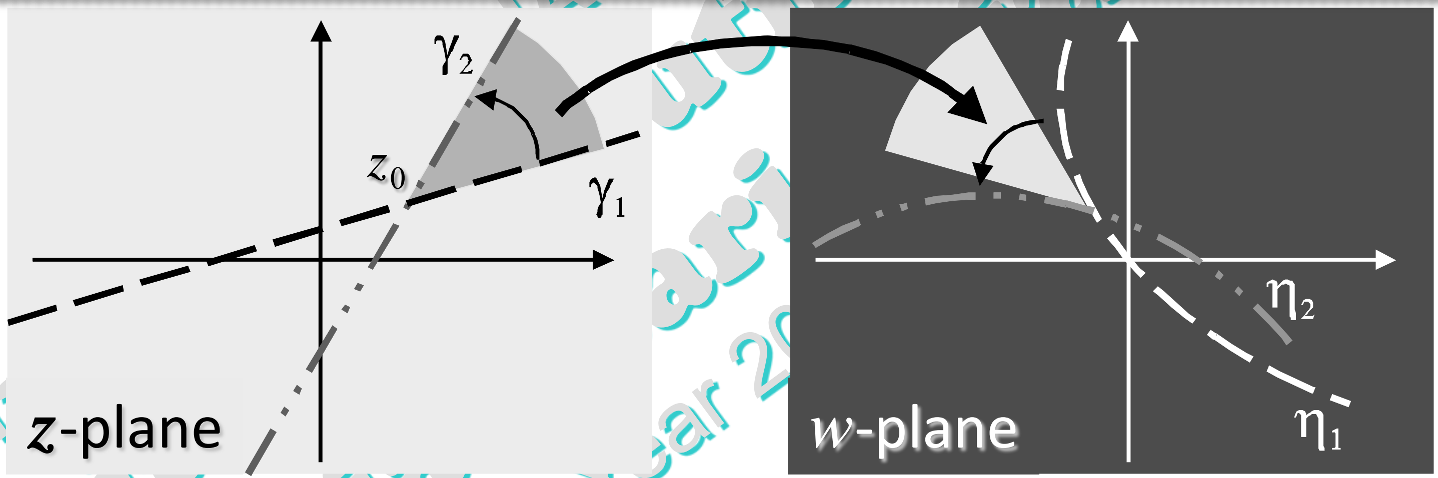
**2** Law on infinitesimal magnitudes:

$$\ell^* = |w_2 - w_1| = \ell \cdot |f'(z_0)|$$

# Consequences of the Law on infinitesimal angles:

$$\theta^* = \arg[w_2 - w_1] = \theta + \arg f'(z_0)$$

**Theor.** If $f(z)$ is holomorphic at $z_0$ and $f'(z_0)\neq 0$ then the map $w=f(z)$ is conformal at $z_0$




**Proof:**

Let $\alpha$ be the angle between two curves ($\gamma_1$ and $\gamma_2$) which are intersecting at $z_0$:

$$\alpha = \theta_2 - \theta_1$$

Let $\alpha^*$ (image of $\alpha$) be the angle between the curves ($\eta_1$ and $\eta_2$) which are the images of $\gamma_1$ and $\gamma_2$ by means of $w=f(z)$:

$$\alpha^* = \theta_2^* - \theta_1^*$$

Then

$$\alpha^* = \theta_2^* - \theta_1^* = [\theta_2 + \arg f'(z_0)] - [\theta_1 + \arg f'(z_0)] = \alpha$$

# Law on local arc transformation (5/5)

The map $w=f(z)$ locally behaves as:

Law on infinitesimal angles:

$$\theta^* = \theta + \arg f'(z_0)$$

← rotation by angle $\arg f'(z_0)$

Law on infinitesimal magnitudes:

$$\ell^* = \ell \cdot |f'(z_0)|$$

← homothety with factor $|f'(z_0)|$

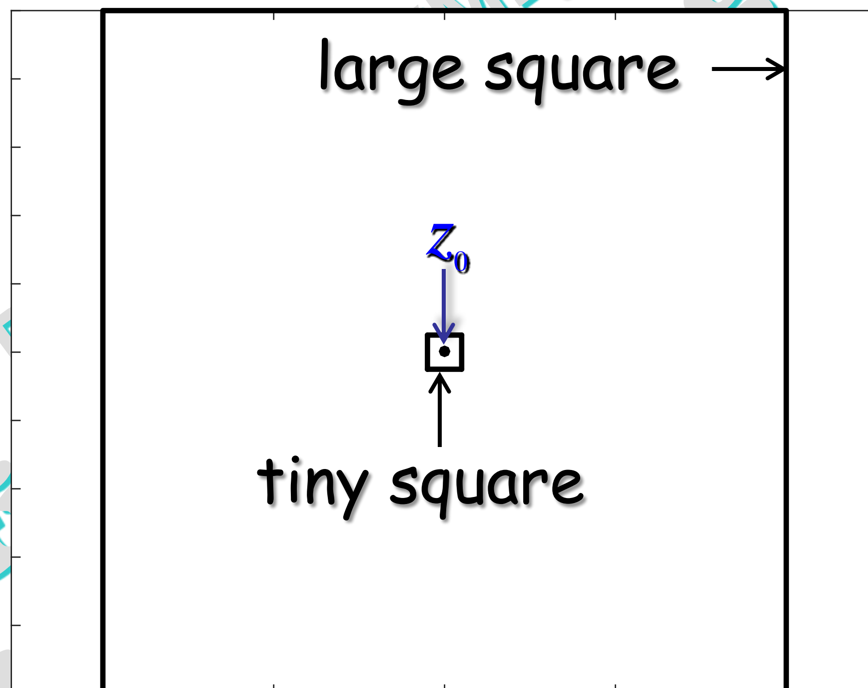
## Geometric interpretation of derivative of a complex fun

If $f(z)$ is holomorphic, then the map $w=f(z)$, at each point such that $f'(z)\neq 0$, can be locally considered as the composition of a rotation (by an angle $\arg f'(z)$) and a homothety (with a factor $|f'(z)|$).

# **Example**: what does "local behavior" mean? (1/3)

map $\quad w = f(z) = z^2 \qquad f'(z) = 2z \neq 0 \Leftrightarrow z \neq 0$

We consider two squares around $z_0$:




Then we transform both of them:

1. by means of: $w = z^2$
2. by means of: $W = w_0 + f'(z_0)[z - z_0]$

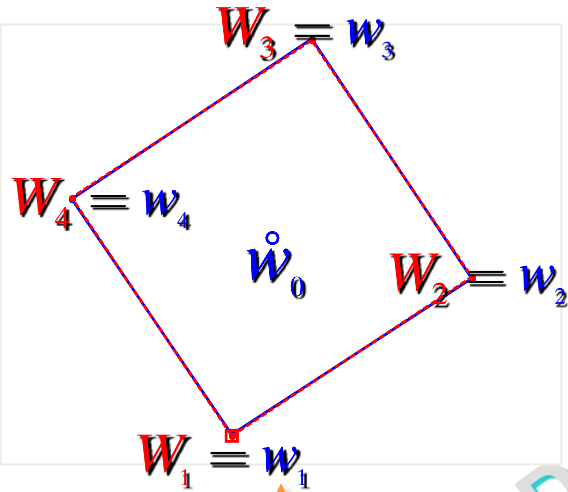
   and compare their images.

$$f'(z_0) = |f'(z_0)|\, e^{i \arg f'(z_0)}$$

homothety $\quad$ rotation

# Example: what does "local behavior" mean? (2/3)

$$w = f(z) = z^2 \qquad f'(z) = 2z$$

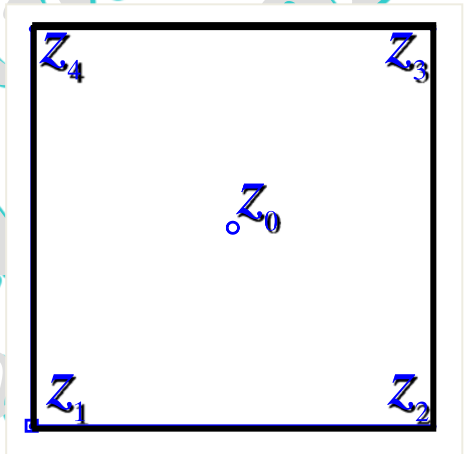



2 maps:

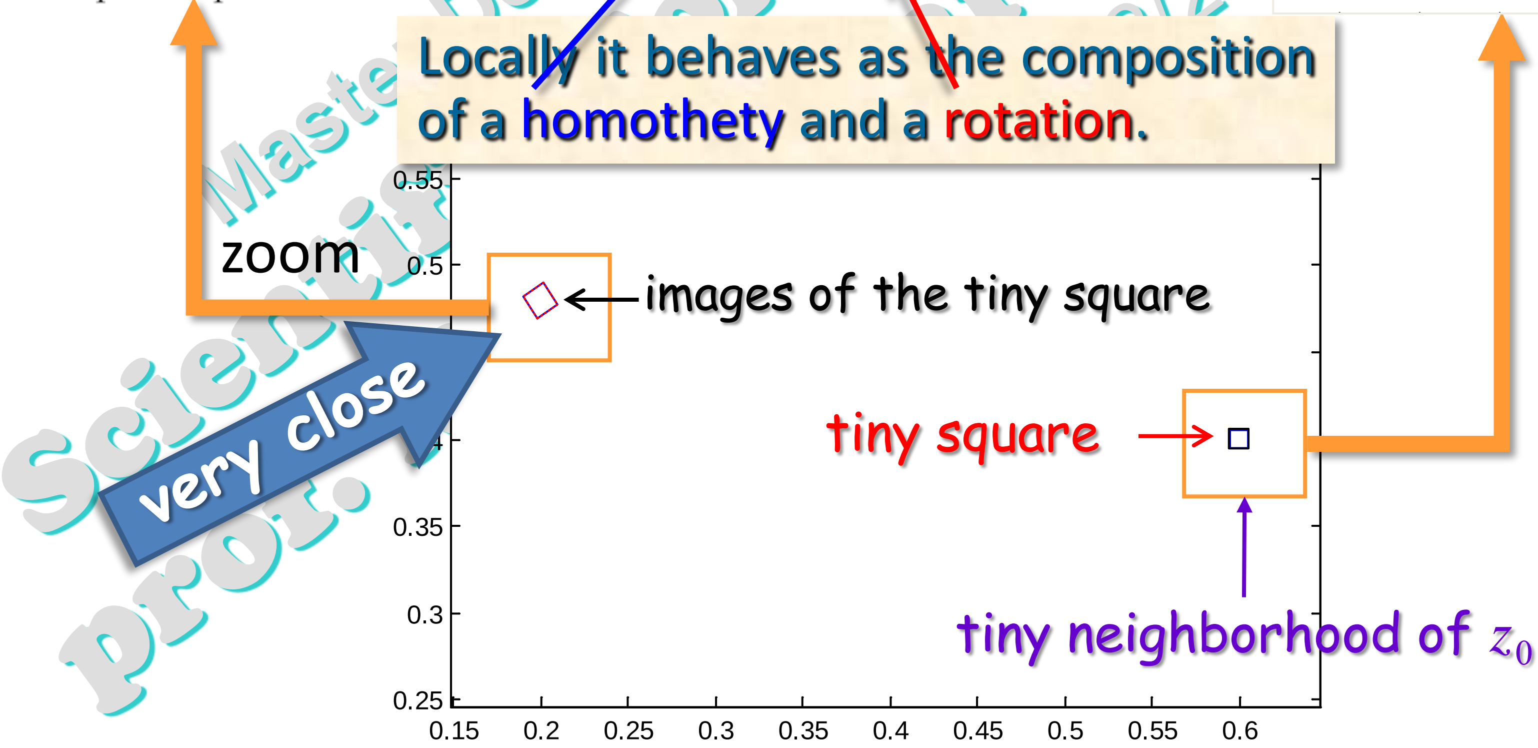
◇ $w_k = z_k^{\;2}$

◇ $W_k = w_0 + f'(z_0)[z_k - z_0]$

$$f'(z_0) = |f'(z_0)|\, e^{i \arg f'(z_0)}$$

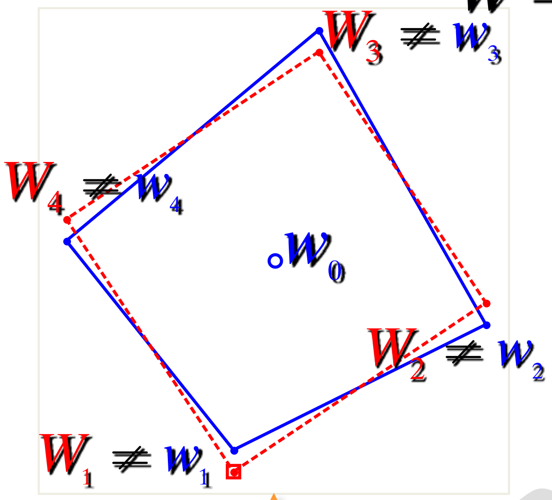



Locally it behaves as the composition of a homothety and a rotation.



# Example: what does "local behavior" mean? (3/3)

$$w = f(z) = z^2 \qquad f'(z) = 2z$$

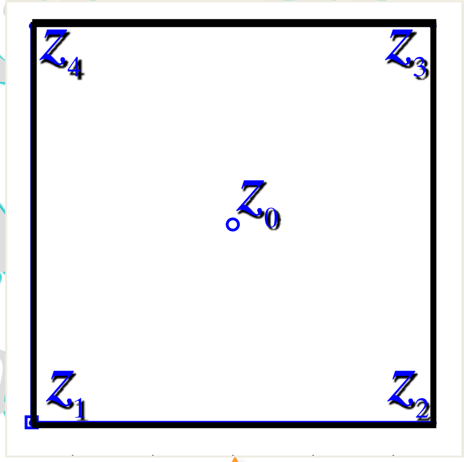


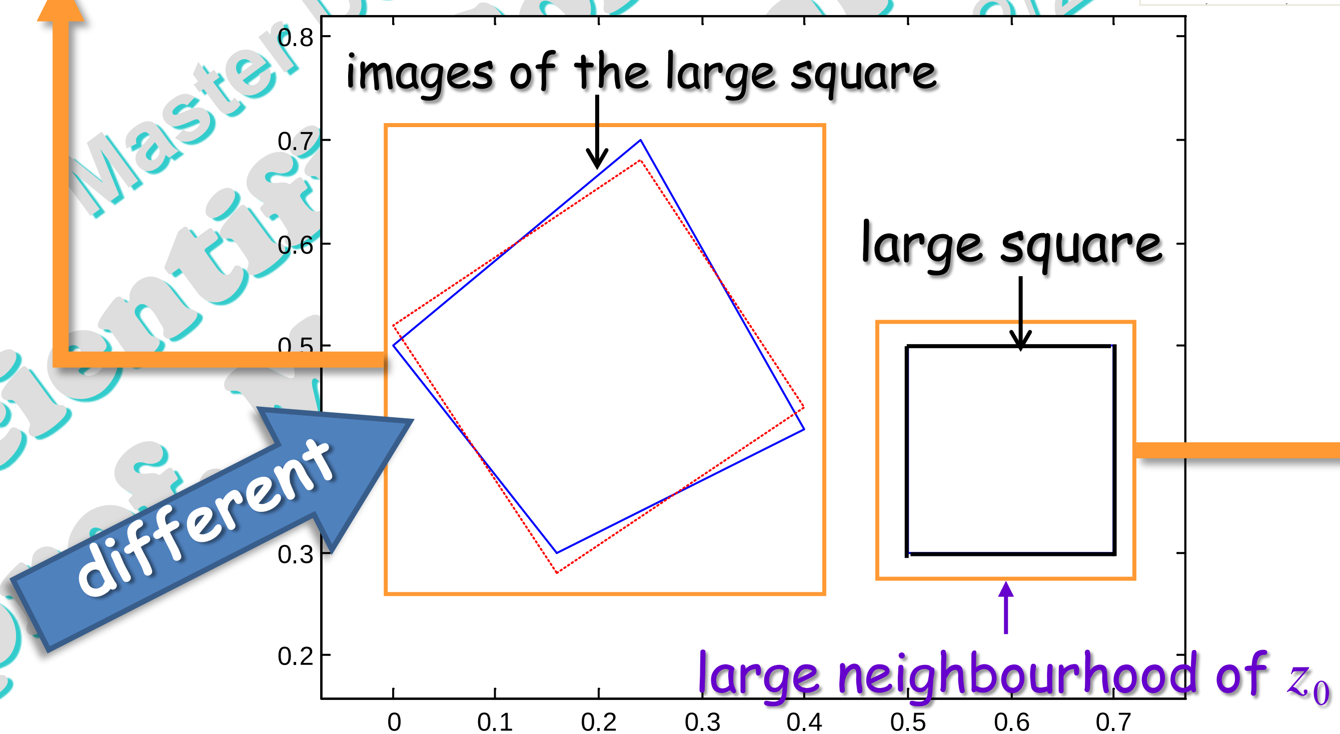

2 maps:

$w_k = z_k^{\,2}$

$W_k = w_0 + f'(z_0)[z_k - z_0]$

$$f'(z_0) = |f'(z_0)|\, e^{i \arg f'(z_0)}$$



# Critical points of a transformation

Given a transformation $w = f(z)$, all the points such that $f'(z) = 0$ are said critical points.

At each critical point:

- the transformation is non-invertible;
- the transformation is non-conformal.