Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST
Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

prof. Mariarosaria Rizzardi

Centro Direzionale di Napoli – Bldg. C4
room: n. 423 – North Side, 4th floor
phone: 081 547 6545
email: mariarosaria.rizzardi@uniparthenope.it

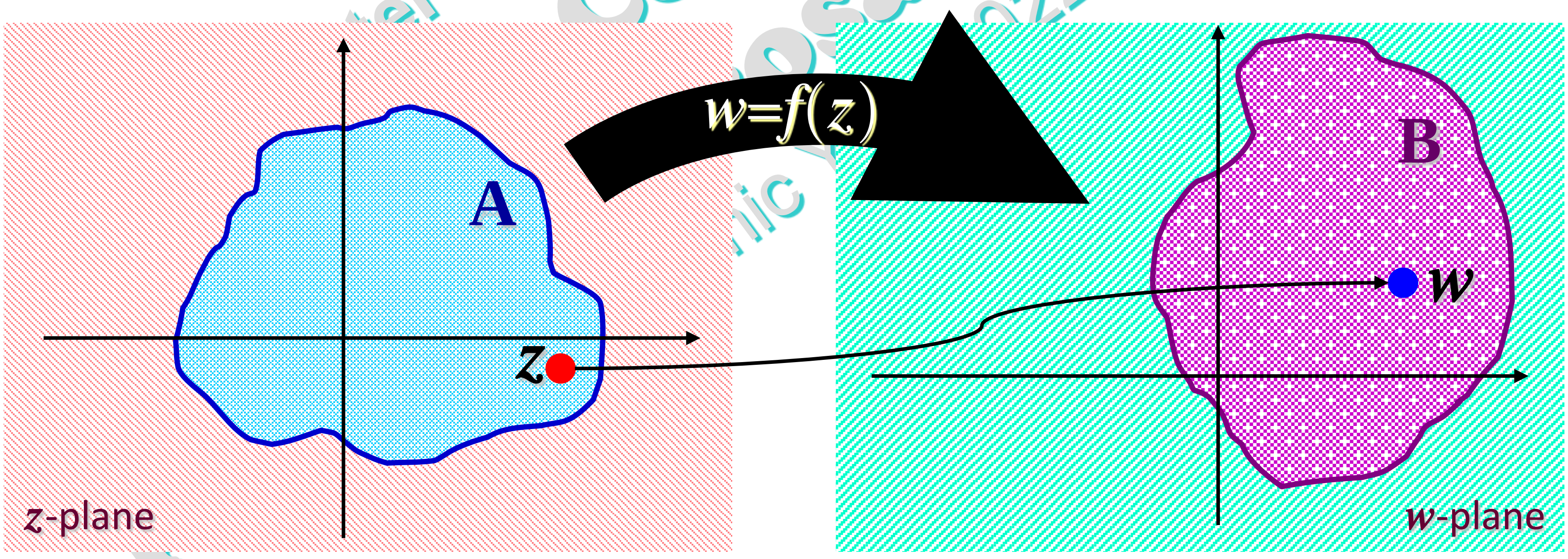
Contents

- Complex form of 2D mappings.
- Conformal and anticonformal mappings.
- Holomorphic functions and conformal mappings.
- Origin of conformal mappings.
- Local invertibility of plane mappings.
- Jacobian of 2D transformations.
- Critical points.

2D mappings in complex form

Complex-valued functions of a complex variable $f(z)$ can be considered as **mappings between two complex planes**: the origin domain is the z-plane, where $z=x+iy$, and the image domain is the w-plane where $w=f(z)$

$$z = x + iy, \quad w = f(z) = u + iv \quad \Rightarrow \quad \mathrm{T}: \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$



Examples

***z*-plane = (*x*,*y*)-plane**

$z = x + iy$

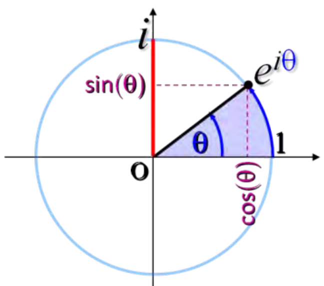
***w*-plane = (*u*,*v*)-plane**

$w = u + iv$

$w = f(z) = iz$

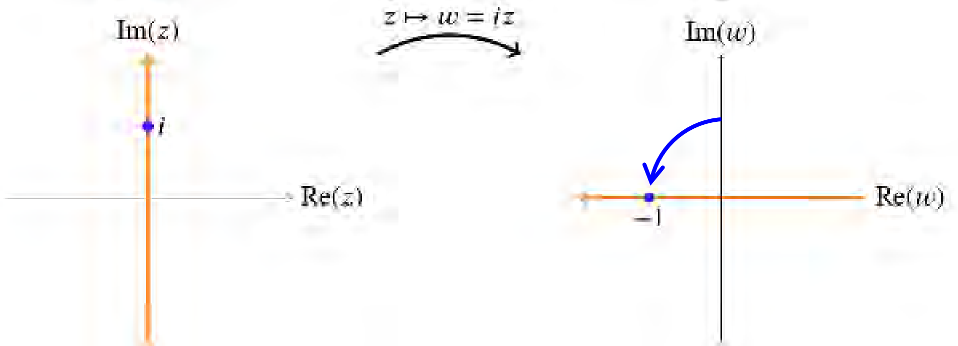
$$w = \mathbf{T}(x,y): \begin{cases} u = -y \\ v = x \end{cases}$$

Since $i = e^{+i\pi/2}$, the transformation corresponds to a 90 degree rotation ($w = ze^{+i\pi/2}$).



Euler's formula

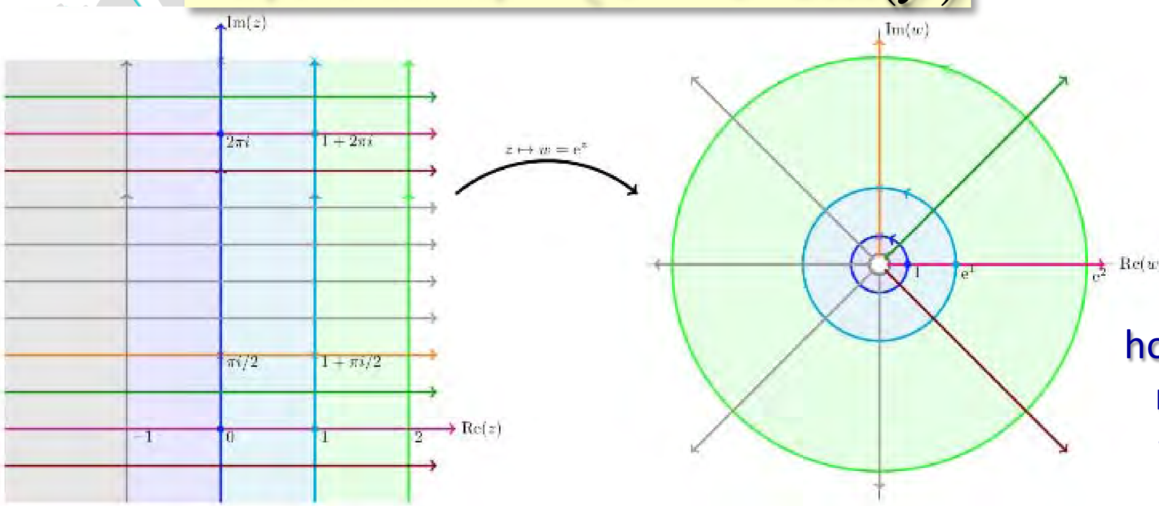
$e^{i\theta} = \cos\theta + i\sin\theta$



$w = f(z) = e^z$

$$w = \mathbf{T}(x,y): \begin{cases} u = e^x \cos(y) \\ v = e^x \sin(y) \end{cases}$$

exponential map



vertical lines are mapped to circles

horizontal lines are mapped to rays from the origin

Example

z-plane = (x,y)-plane

$$z = x + iy$$

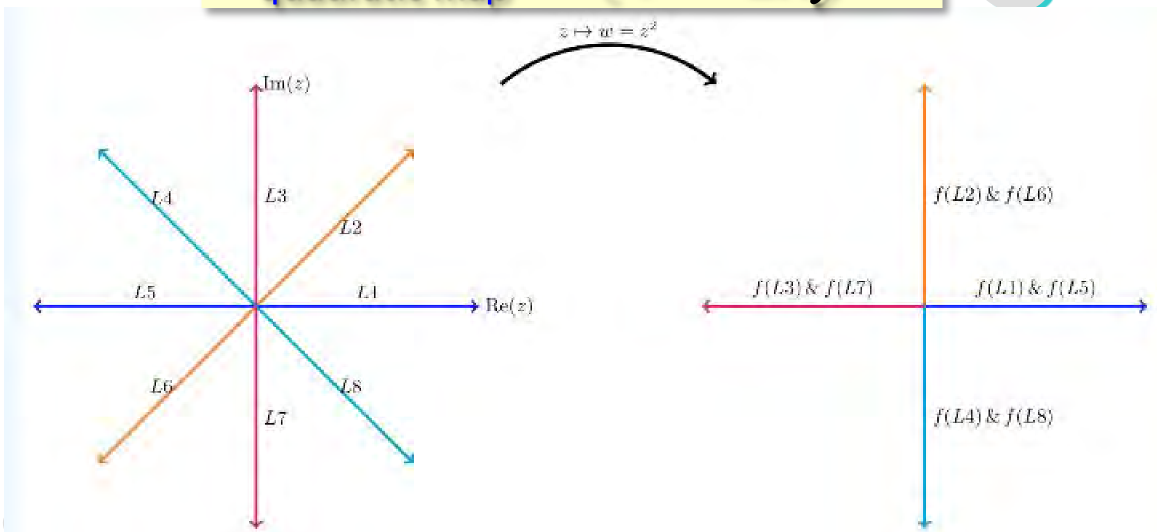
$$w = f(z) = z^2$$

$$w = \mathbf{T}(x,y): \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

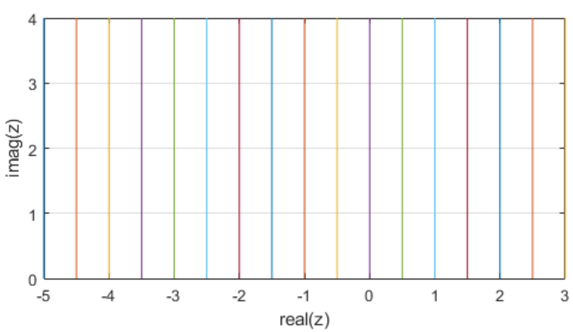
quadratic map

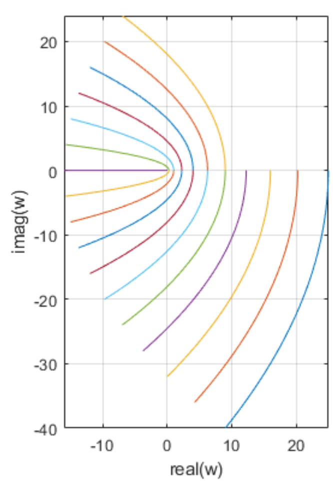
w-plane = (u,v)-plane

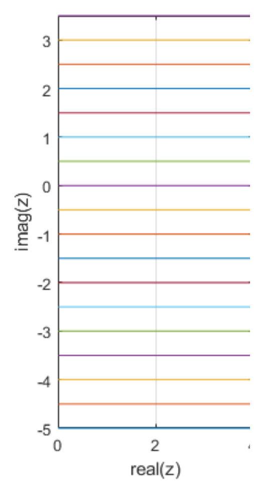
$$w = u + iv$$

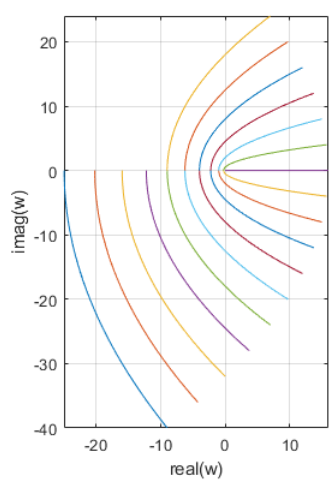


vertical lines are mapped to left facing parabolas





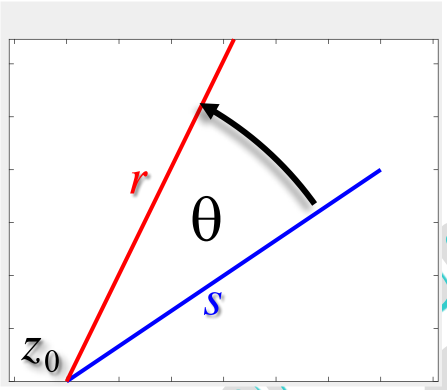




horizontal lines are mapped to right facing parabolas

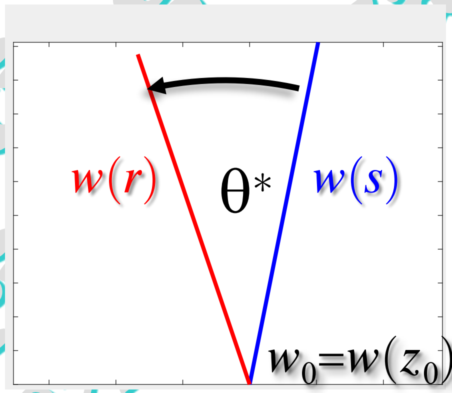
Conformal mappings: definitions

DEF: **Conformal Mapping:** the magnitude of local angles and their orientation are preserved (the angle remains the same).

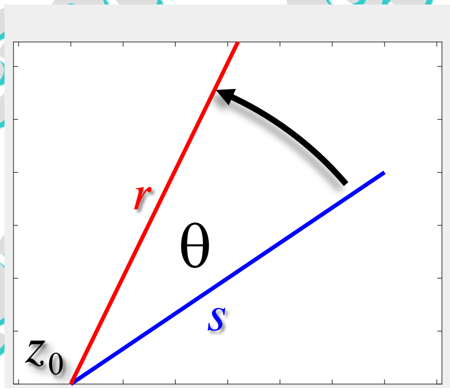


$$w = f(z)$$

conformal

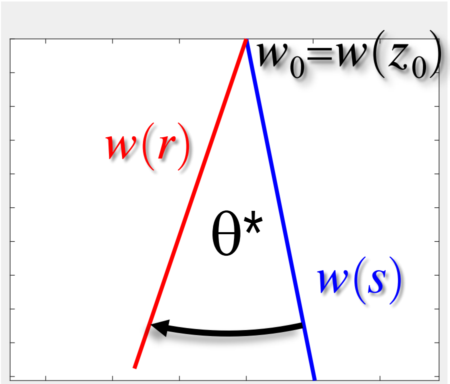


DEF: **Anticonformal Mapping:** the magnitude of local angles is preserved, but their orientation is inverted (the angle changes its sign).



$$w = f(z)$$

anticonformal



Holomorphic functions and conformal mappings

THEOREM

Every holomorphic function (w.r.t. z) describes a plane transformation that is conformal in all the points where its derivative is not zero.

[… we will prove it later]

The opposite is also true.

Every conformal plane transformation originates from a holomorphic function (w.r.t. z) whose derivative is not zero.

A similar theorem holds for holomorphic functions w.r.t. $\overline{z}$ and anti-conformal mappings.

(prof. M. Rizzardi)

Where do conformal mappings originate from?

What conditions must be satisfied by a transformation of the complex plane into itself, $w=f(z)$, in order to leave the Laplace eq. unchanged? That is, what are the transformations able to preserve the harmonicity?

$$f \; : \; z=x+iy\in\mathbb{C} \longrightarrow w=f(z)=u+iv\in\mathbb{C} \qquad w=f(z) \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

Property

If $\Psi(u,v)$ is a harmonic function of u,v and the function $w=f(z)$ is a holomorphic function, then the composite function

$$\varphi(x,y) = \Psi(u(x,y),v(x,y))$$

is a harmonic function of x, y.

The proof applies the "chain rule" to differentiate a composite function.

Where do conformal mappings originate from?

Theor.: In order to mantain the Laplace Equation unchanged, after applying the plane transformation

$$T: \quad z = x + iy \longrightarrow w = w(z) = w(x,y) \quad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \Rightarrow \quad \Psi_{uu} + \Psi_{vv} = 0$$

this map must satisfy the following equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} = +\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy-Riemann eqs w.r.t. z

or alternatively

$$\begin{cases} \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = +\dfrac{\partial v}{\partial x} \end{cases}$$

Cauchy-Riemann eqs w.r.t. $\bar{z}$

T conformal map*

T anticonformal map

*...we will prove it later

The mapping must be conformal or anticonformal

(prof. M. Rizzardi)

Example 1 $w=f(z)=z^2$

$f(z)$ holomorphic at z_0 w.r.t. z

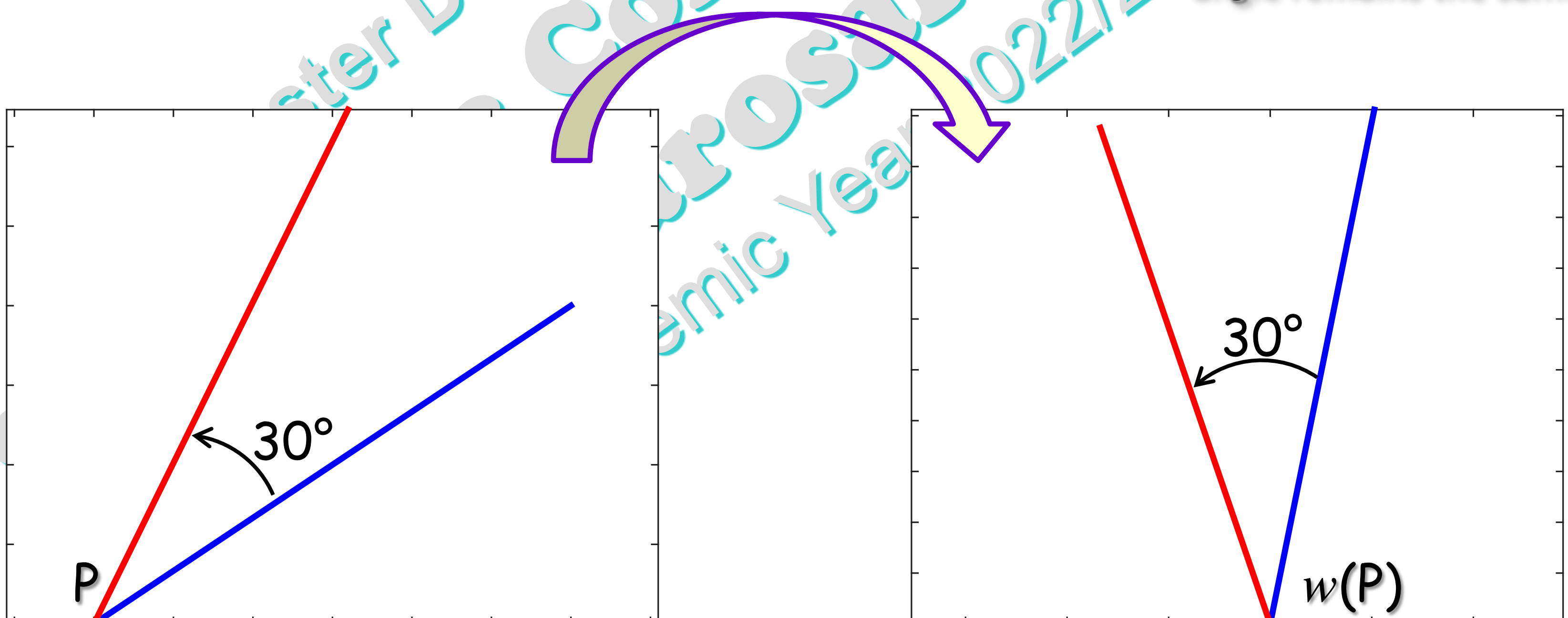
$$\frac{df}{dz}(z_0)=\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z}=\lim_{z\to z_0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$\begin{cases}\dfrac{\partial u}{\partial x}=+\dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}$$

conformal

$$w=u+iv:\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}\qquad w=z^2:\begin{cases}u(x,y)=x^2-y^2\\ v(x,y)=2xy\end{cases}$$

DEF.: Conformal Mapping: the magnitude of local angles and their orientation are preserved (the angle remains the same).



Example 2 $w=f(z)=\bar{z}^2$

$f(z)$ holomorphic at z_0 w.r.t. $\bar{z}$

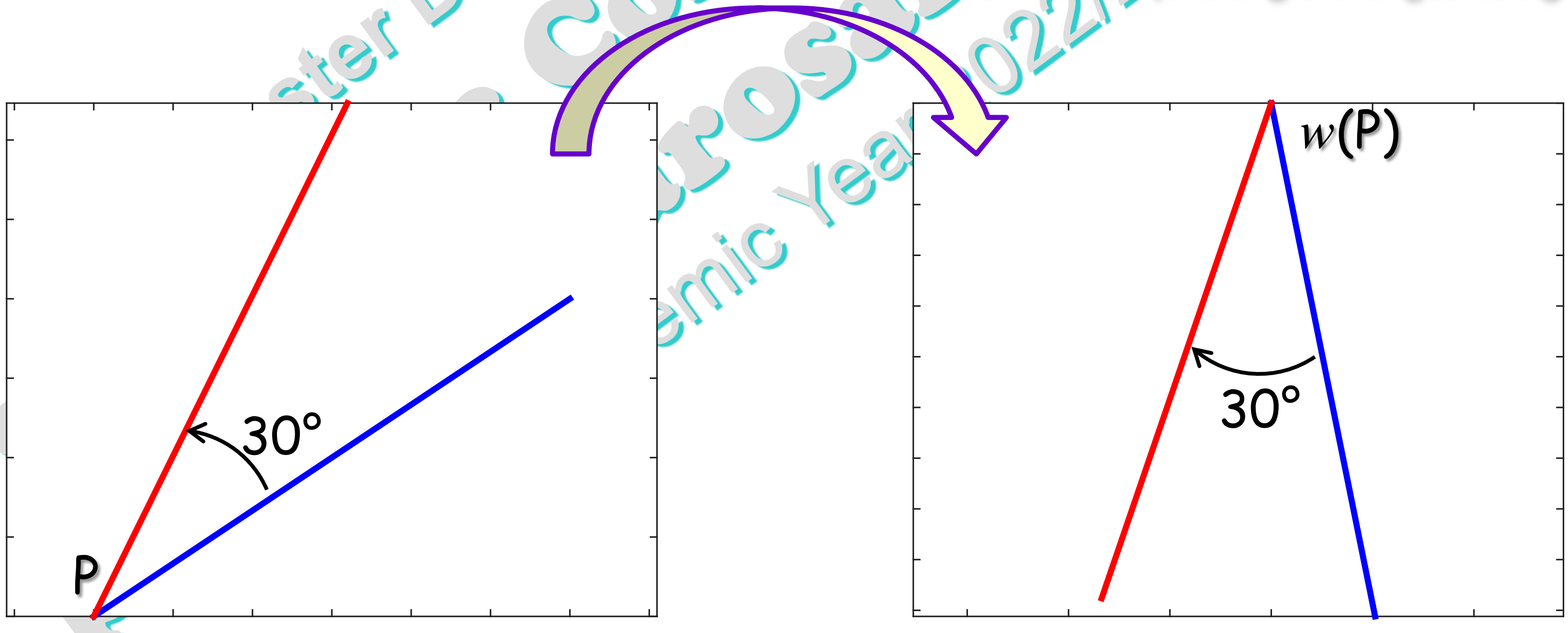
$$\frac{df}{d\bar{z}}(z_0)=\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta \bar{z}}=\lim_{z\to z_0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta \bar{z}}$$

$$\begin{cases}\dfrac{\partial u}{\partial x}=-\dfrac{\partial v}{\partial y}\\[2mm]\dfrac{\partial u}{\partial y}=+\dfrac{\partial v}{\partial x}\end{cases}$$

anticonformal

$$w=u+iv:\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}\qquad w=\bar{z}^2:\begin{cases}u(x,y)=x^2-y^2\\v(x,y)=-2xy\end{cases}$$

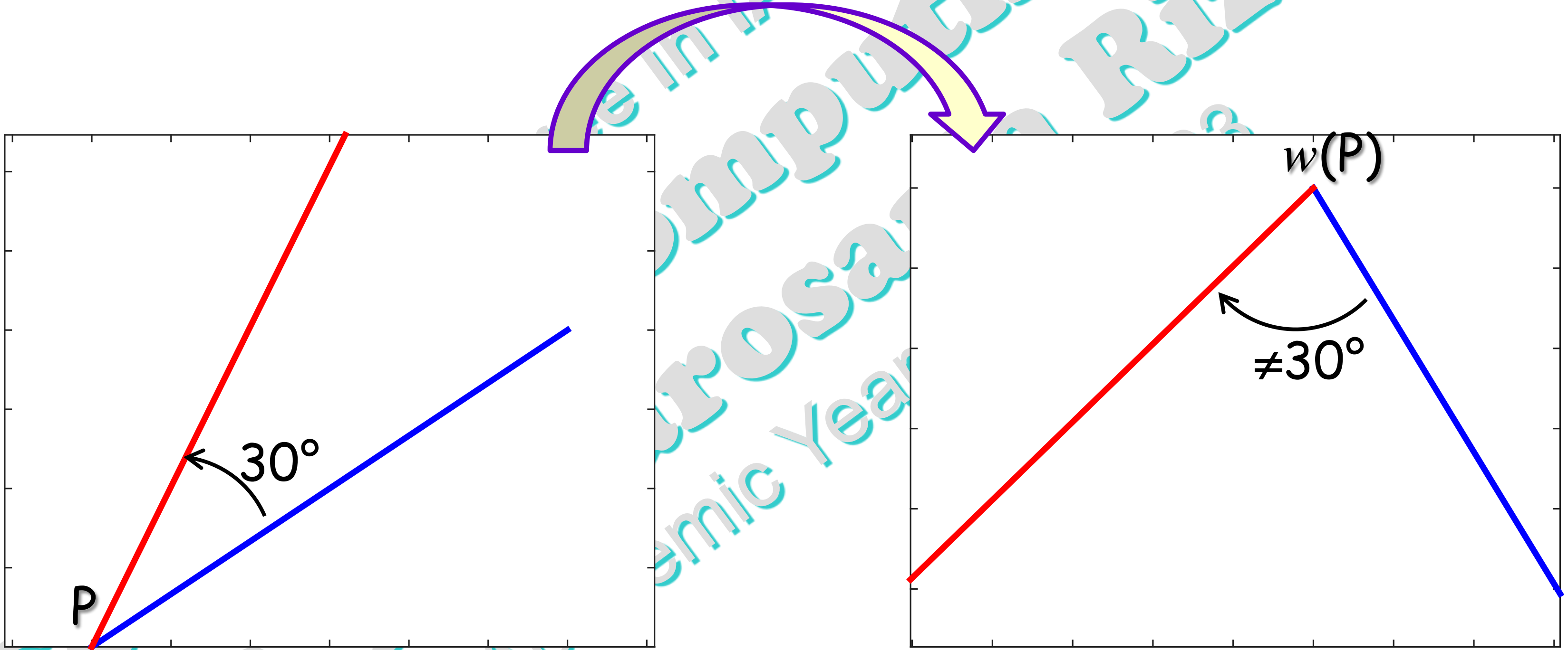
DEF: Anticonformal Mapping: the magnitude of local angles is preserved, but their orientation is inverted (the angle changes its sign).



Example 3: $w=f(z)=z^2+2\bar{z}^2$

neither conformal nor anticonformal

$$w=u+iv:\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases} \qquad w=z^2+2\bar{z}^2:\begin{cases}u(x,y)=3x^2-3y^2\\ v(x,y)=-2xy\end{cases}$$



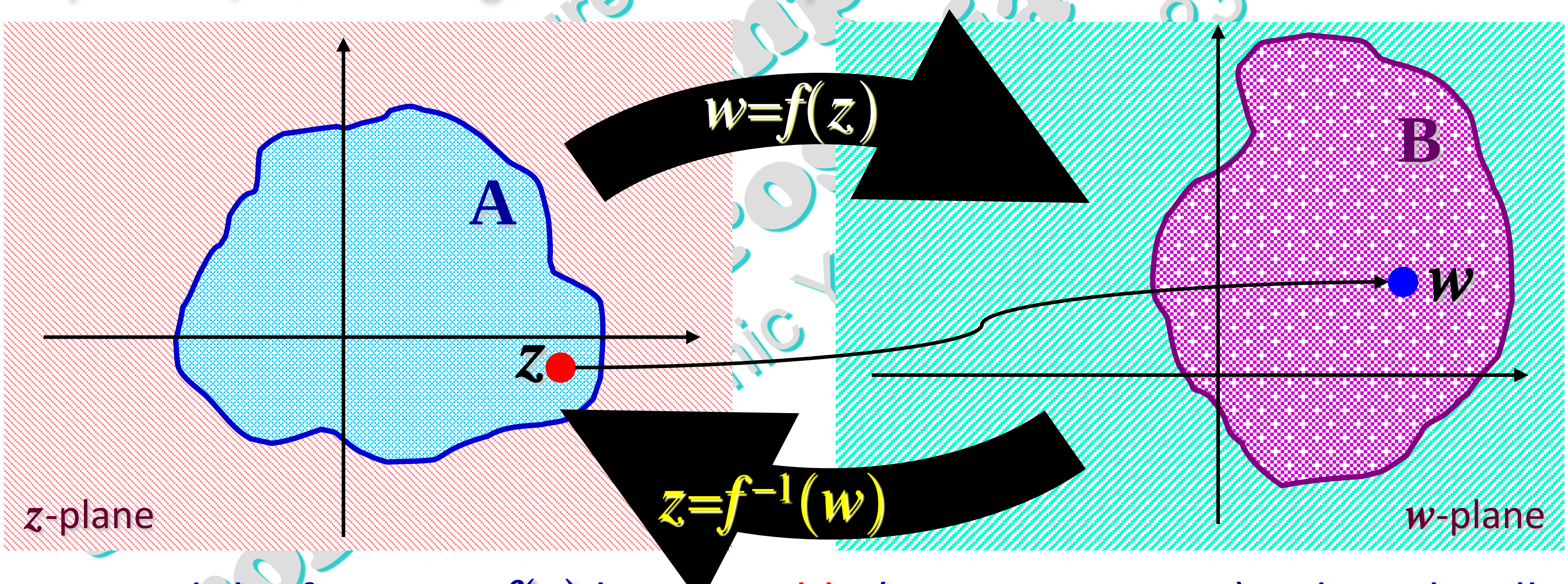
If a complex function is defined by a formula containing both z and $\bar{z}$, then it is differentiable neither w.r.t. z nor w.r.t. $\bar{z}$.

2D mappings in complex form

Complex-valued functions of a complex variable $f(z)$ can be considered as **mappings between two complex planes**: the origin domain is the z-plane, where $z=x+iy$, and the image domain is the w-plane where $w=f(z)$

$$z=x+iy,\quad w=f(z)=u+iv \quad\Rightarrow\quad \mathrm{T}:\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$$

Once we have transformed the origin domain into the image domain, and solved the problem, we want to go back to the z-plane.



We need the function $f(z)$ be invertible (one-to-one map) at least locally.

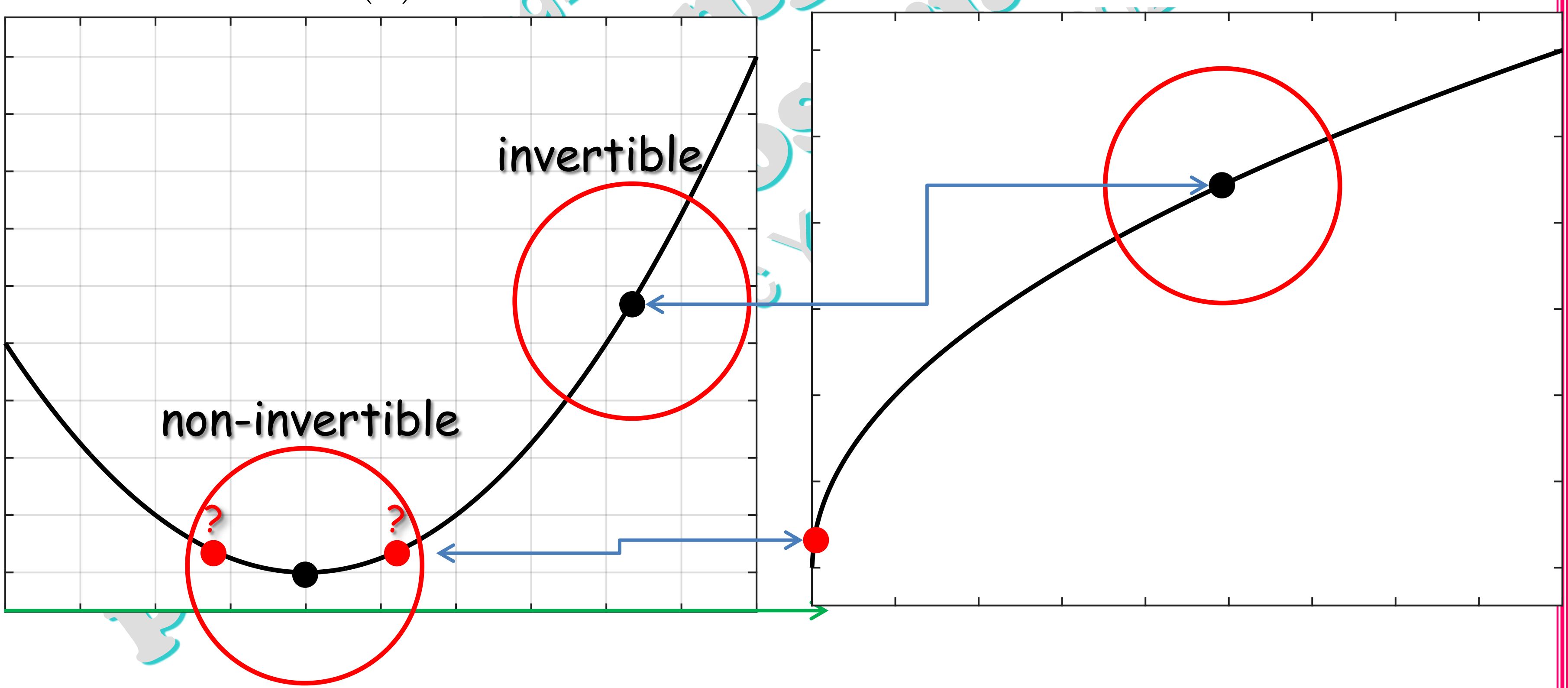
2D mapping f: $w=f(z)$ ⟺ inverse 2D mapping f^{-1}: $z=f^{-1}(w)$

Basics: Invertibility of a real function

A real-valued function of a single real argument, $y=f(x)$, is **locally invertible** at x_0 if, and only if, $f'(x_0)\neq 0$ (locally monotonic). Moreover, the derivative of the inverse function is

$$y_0 = f(x_0) \quad \Rightarrow \left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

$$y = f(x) = x^2 \qquad\qquad x = f^{-1}(y) = \sqrt{y}$$



Jacobian matrix of a 2D transformation

$$w=\mathrm{T}(z)\ :\ \mathrm{T}(x,y)=\begin{pmatrix}u\\v\end{pmatrix}:\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$$

z-plane $z=x+iy=\begin{pmatrix}x\\y\end{pmatrix}$ —transformation→ w-plane $w=u+iv=\begin{pmatrix}u\\v\end{pmatrix}$

Definition A coordinate transformation $\mathrm{T}(x,y)$ is differentiable at a point (x_0,y_0) if there exists a matrix $J(x_0,y_0)$ such that

$$\lim_{\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x_0\\y_0\end{pmatrix}}\frac{\left\|\mathrm{T}(x,y)-\mathrm{T}(x_0,y_0)-J(x_0,y_0)\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right]\right\|}{\left\|\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|}=0$$

$$\mathrm{T}(x,y)=\mathrm{T}(x_0,y_0)+J(x_0,y_0)\left[\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right]+o\left(\left\|\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|^2\right)$$

infinitesimal of order 2

When it exists, $J(x_0,y_0)$ is the total derivative of $\mathrm{T}(x,y)$ at (x_0,y_0). It can be shown that this matrix is given by the Jacobian Matrix of the transformation: $J(x_0,y_0)=\dfrac{\partial(u,v)}{\partial(x,y)}(x_0,y_0)$

If $z=z(\tau)$ is a curve in the (x,y)-plane and $w=w(\tau)$ its image in the (u,v)-plane:

$$z(\tau)=\begin{pmatrix}x(\tau)\\y(\tau)\end{pmatrix},\quad \tau\in[a,b]\qquad w(\tau)=\mathrm{T}(x(\tau),y(\tau))=\begin{pmatrix}u(x(\tau),y(\tau))\\v(x(\tau),y(\tau))\end{pmatrix},\quad \tau\in[a,b]$$

and if $z(\tau)$ is smooth, then the Jacobian matrix maps any tangent vector to a curve at a given point, in the z-plane, to a tangent vector to the image of the curve at the image of that point, in the w-plane:

$$w'(\tau)\longrightarrow\ \frac{dw}{d\tau}=\begin{pmatrix}u_x x'(\tau)+u_y y'(\tau)\\v_x x'(\tau)+v_y y'(\tau)\end{pmatrix}=\begin{bmatrix}u_x & u_y\\v_x & v_y\end{bmatrix}\begin{pmatrix}x'(\tau)\\y'(\tau)\end{pmatrix}=J(x(\tau),y(\tau))\frac{dz}{d\tau}\ \longleftarrow z'(\tau)$$

Example

z-plane = (x,y)-plane

$$z(\tau)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\tau\\\tau^2\end{pmatrix}$$

$$w=\mathrm{T}(x,y):\begin{cases}u=x^2-y^2\\v=2xy\end{cases}$$

quadratic map

w-plane = (u,v)-plane

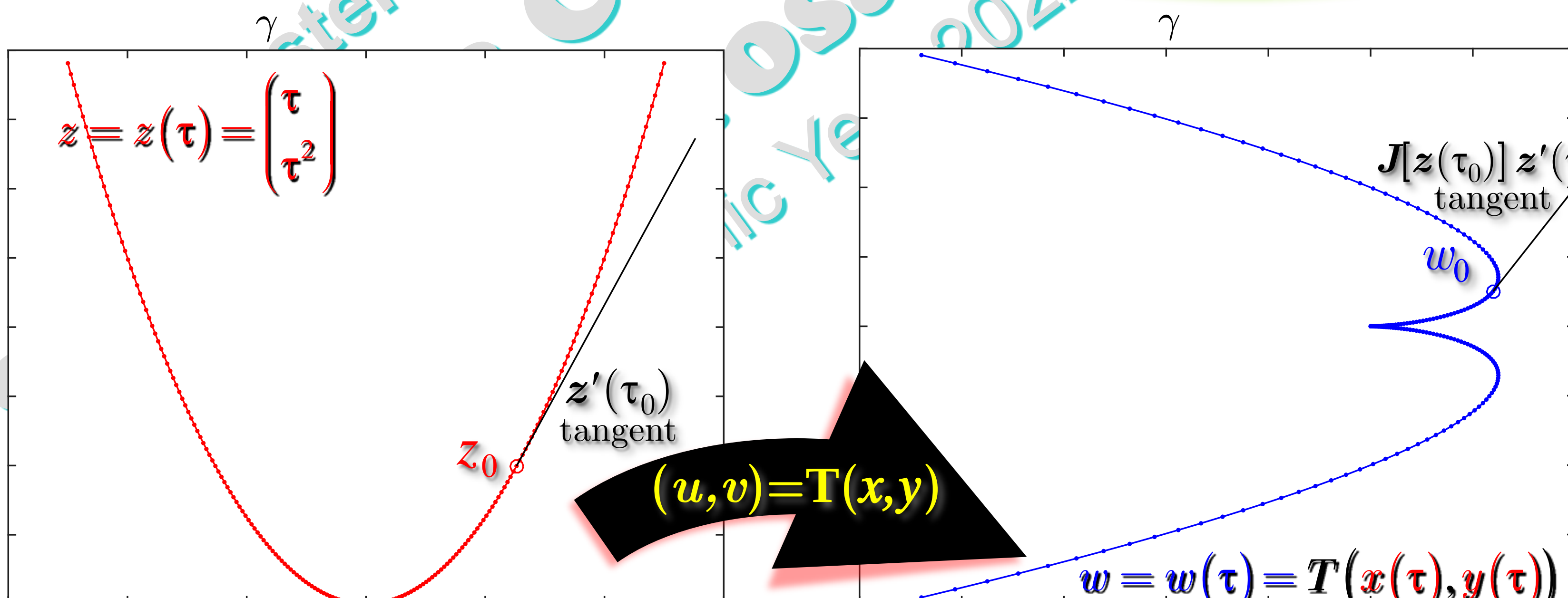
$$w(\tau)=T(x(\tau),y(\tau))=\begin{pmatrix}u(\tau)\\v(\tau)\end{pmatrix}$$

tangent line at $z_0=z(\tau_0)$ in z-plane: $\tau: z=z(\tau_0)+\lambda z'(\tau_0),\quad \lambda\in\mathbb{R}$

tangent line at $w_0=w(\tau_0)$ in w-plane:

in place of $\tau^*: w=w(\tau_0)+\lambda w'(\tau_0),\quad \lambda\in\mathbb{R}$

we can use $\tau^*: w=w(\tau_0)+\lambda J\big(x(\tau_0),y(\tau_0)\big)z'(\tau_0)$



Theorem on local invertibility of mappings

A 2D tranformation $f\ :\ z\in\mathbb{C}^* \longrightarrow w=f(z)=u(x,y)+iv(x,y)\in\mathbb{C}^*$ $\quad \mathrm{T}\begin{cases} u=u(x,y)\\ v=v(x,y)\end{cases}$
is one-to-one at a point (locally invertible) if the Jacobian $|J|$ (i.e. the determinant of the Jacobian matrix J) does not vanish at that point.

Theor.: If $f(z)$ is holomorphic in A, then the mapping $w=f(z)$ is regular invertible at every point $z_0\in A$ such that $f'(z_0)\neq 0$.

Proof: If $f(z)$ is holomorphic at z_0, then the Cauchy-Riemann Equations hold and the Jacobian of the mapping is such that:

$$|J(x_0,y_0)| = \begin{vmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{vmatrix} = \left[\frac{\partial u}{\partial x}(x_0,y_0)\right]^2 + \left[\frac{\partial v}{\partial x}(x_0,y_0)\right]^2 = |f'(z_0)|^2$$

Then the mapping $w=f(z)$ is regular invertible at every point z_0 such that $f'(z_0)\neq 0$.

The points z^* such that $f'(z^*)=0$ are said **critical points** of the mapping $w=f(z)$.

The mapping is not invertible at each critical point.

Example: $z=0$ is a critical point for the quadratic map