Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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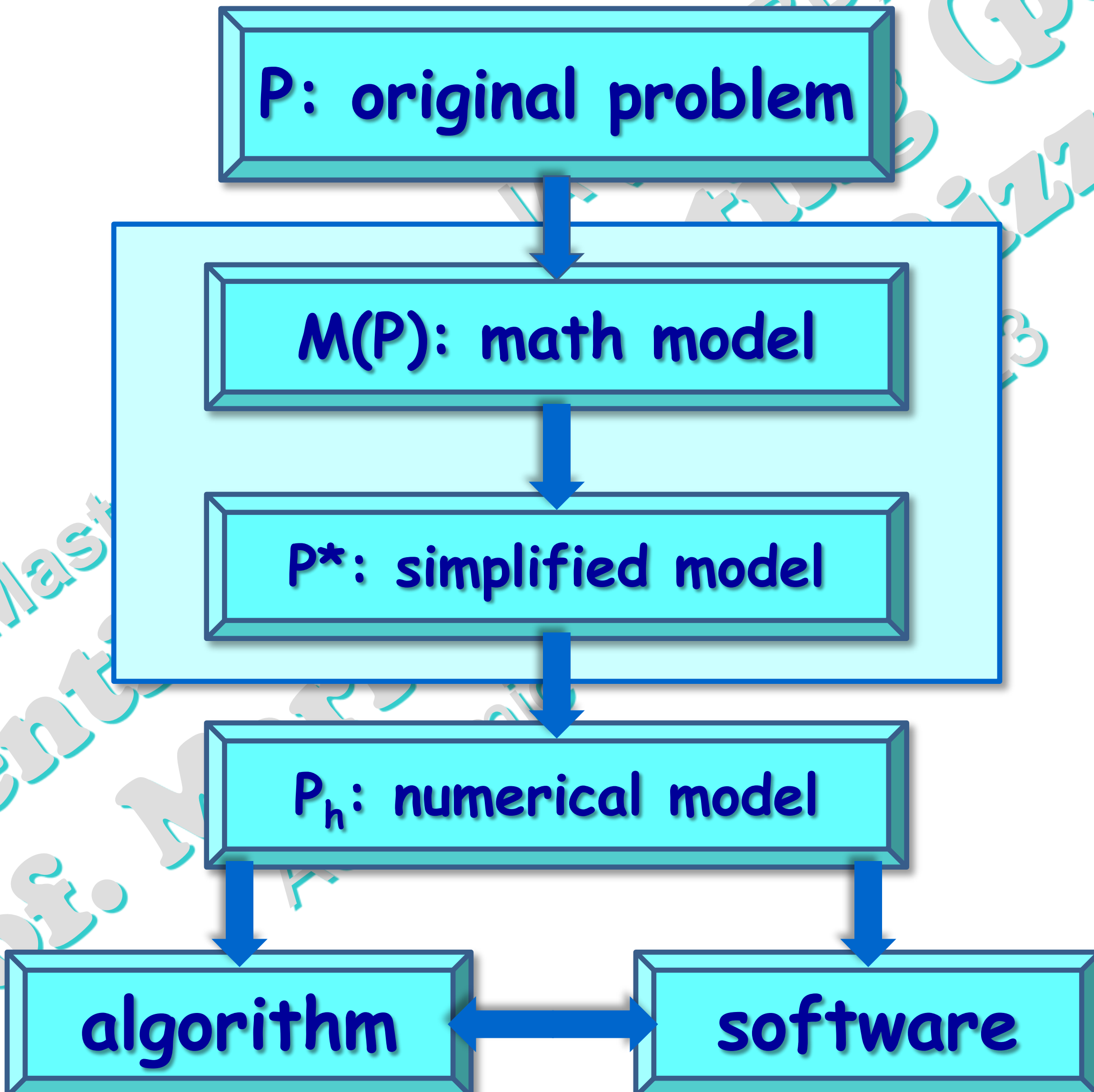
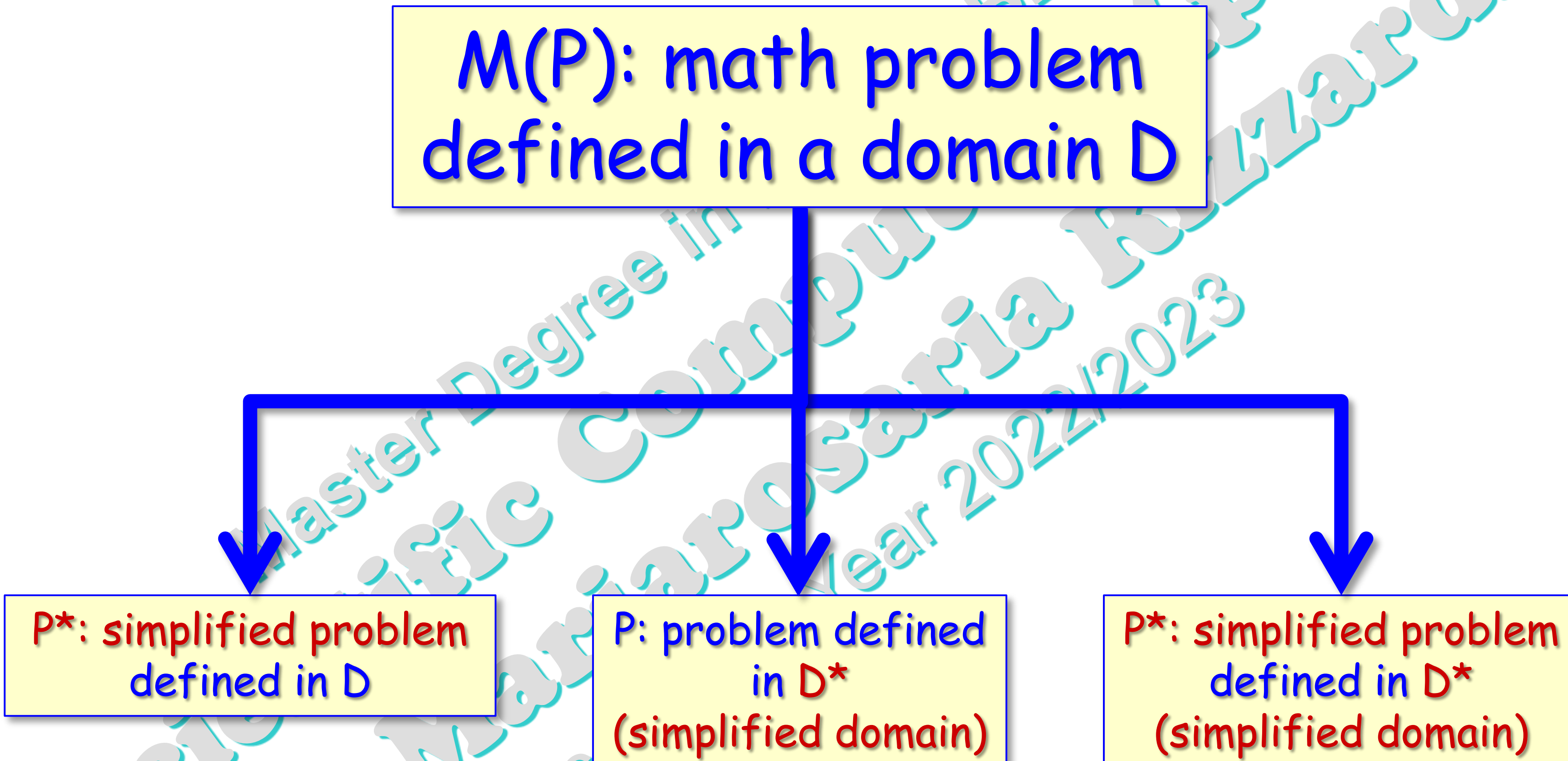
Contents

- Steps in solving a problem by means of a computer.
- Simplified problem and/or simplified problem domain. Examples: geometric and function transformations.
- Complex limit of a complex function and complex differentiability (holomorphism).
- Cauchy-Riemann Equations and equivalent statements.

Steps in the process to solve a problem by means of a computer



Model Transformation



Example 1: domain transformation

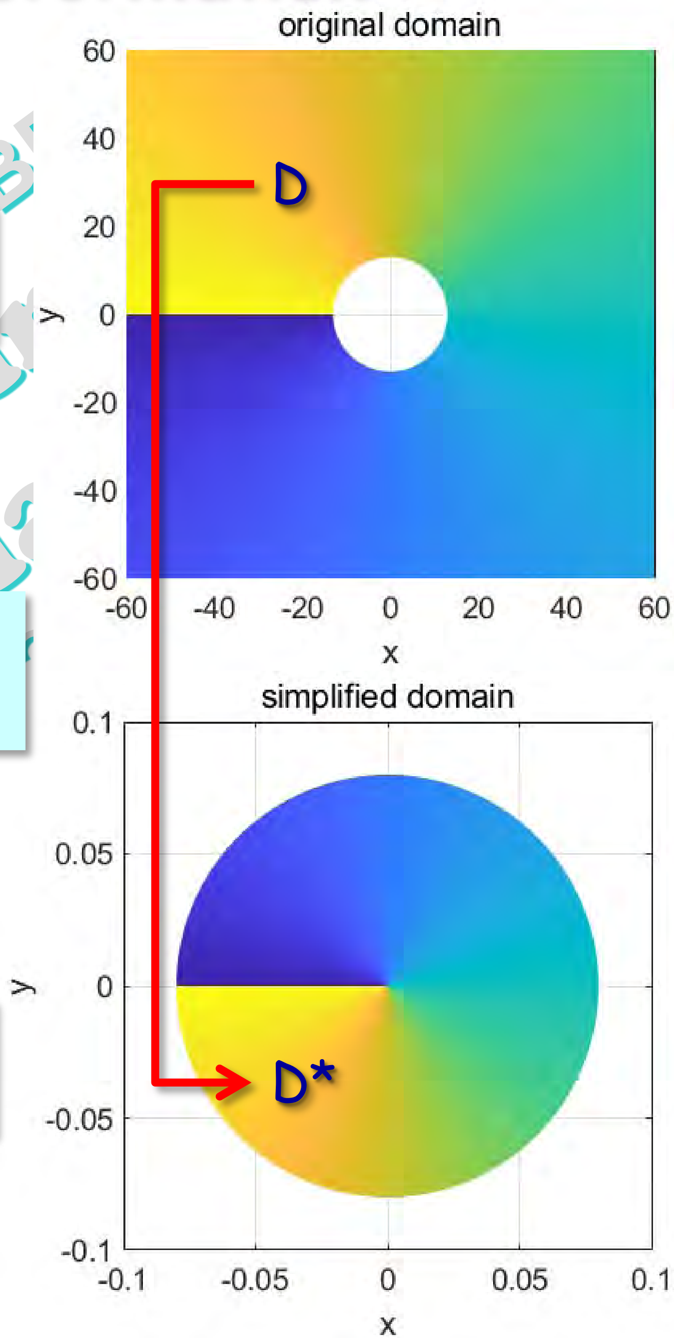
P: problem defined in domain D where
$D = \{z \in \mathbb{C} : |z| > R\} \subseteq \mathbb{R}^2$

D: unbounded domain

Transformation:
inversion → $w = 1/z, \quad z \in \mathbb{C}$

D*: bounded domain

P: problem defined in simplified domain D*
$D^* = \{w \in \mathbb{C} : |w| < 1/R\}$



Example 2: domain transformation

Fluid dynamics application of symmetric Joukowski Transform $T_J(z)\ :\ w=\frac{1}{2}\left(z+\frac{1}{z}\right),\ z\in\mathbb{C}$

Nikolai Yegorovich Zhukovskij, 1910

T_J is used to solve for the two-dimensional potential flow around a class of airfoils known as Joukowski airfoils. In particular flow around a disk.

Problem: display all the curves (streamlines) that are inverse images, by means of T_J, of horizontal lines in the w-plane: $\mathrm{Im}(w)=k$, with k constant.

$$z=x+iy \qquad w=u+iv \qquad \left(i=\sqrt{-1}\right)$$

$$w=u+iv=\frac{1}{2}\left(\frac{z^2+1}{z}\right)=\frac{1}{2}\frac{\left(z^2+1\right)\cdot\bar{z}}{z\cdot\bar{z}}=\frac{1}{2}\frac{|z|^2 z+\bar{z}}{|z|^2}=\frac{1}{2}\frac{\left(x^2+y^2\right)(x+iy)+x-iy}{x^2+y^2}=$$

$$=\frac{1}{2}\frac{\left(x^2+y^2\right)(x+iy)+x-iy}{x^2+y^2}=\frac{1}{2}\left\{x\left[1+\frac{1}{x^2+y^2}\right]+iy\left[1-\frac{1}{x^2+y^2}\right]\right\}$$

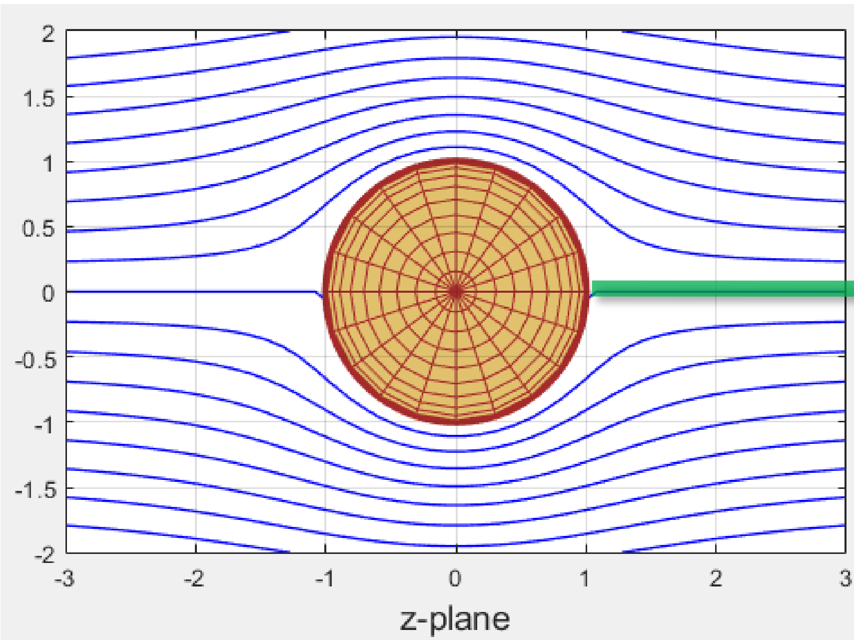
$\mathrm{Im}(w)$

$$w=u(x,y)+iv(x,y)=\frac{x}{2}\left[1+\frac{1}{x^2+y^2}\right]+i\frac{y}{2}\left[1-\frac{1}{x^2+y^2}\right]$$

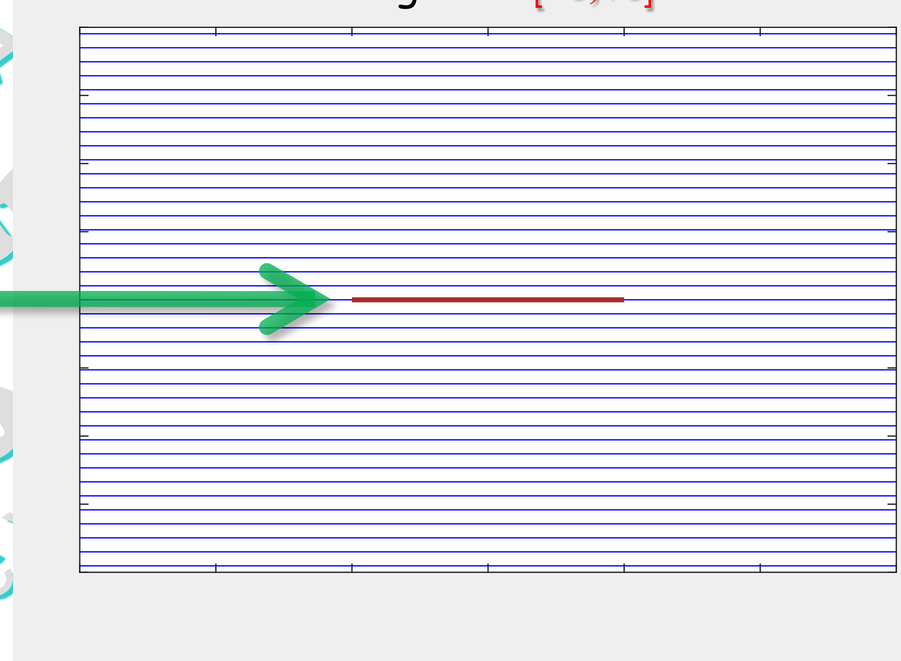
Example 2: domain transformation (cont.)

Fluid dynamic application of symmetric Joukowski transform $T_J(z)$

Streamlines for an incompressible potential flow around a circular cylinder in a uniform stream.



The unity circle has been transformed into the real segment $[-1,+1]$

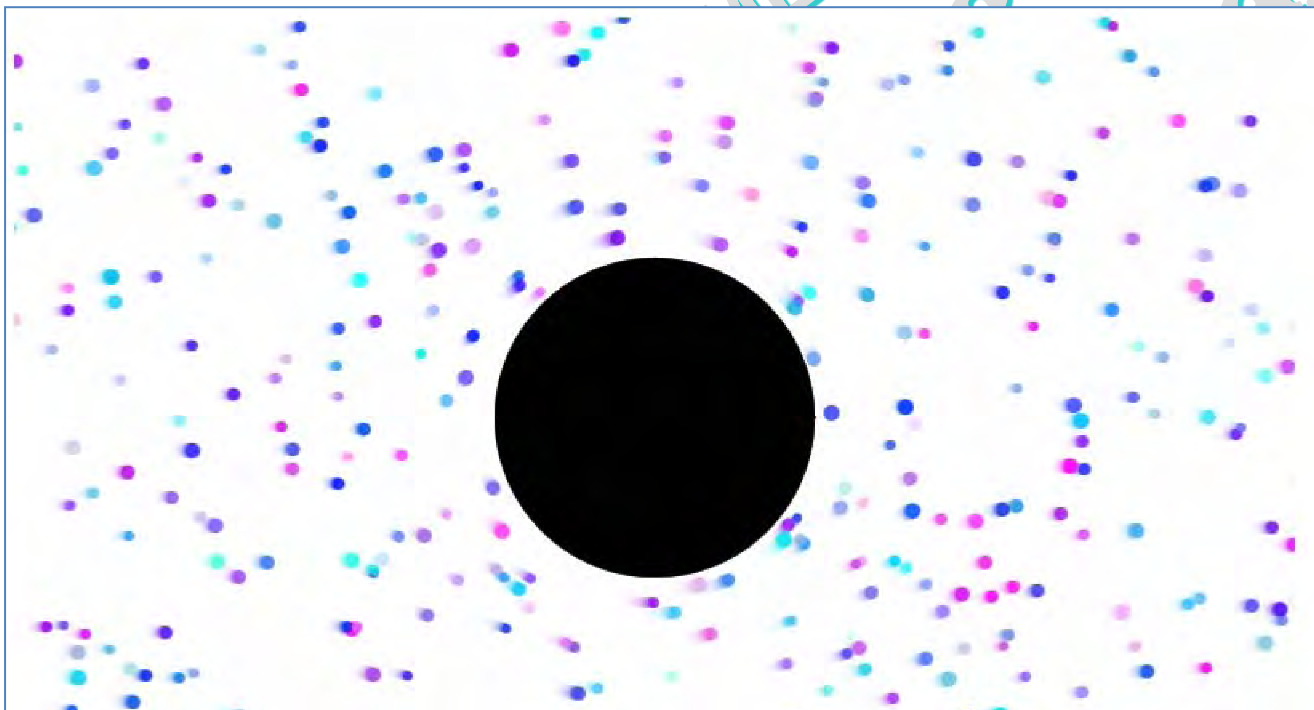


$$z = x + iy, \quad w = u(x,y) + iv(x,y) = \frac{x}{2}\left[1 + \frac{1}{x^2+y^2}\right] + i\frac{y}{2}\left[1 - \frac{1}{x^2+y^2}\right]$$

```
a=-4; b=4; N=64; [x,y]=meshgrid(linspace(a,b,N));
v = y/2.*(1-1./(x.^2+y.^2)); % v=cost
contour(x,y,v,50,'b'); hold on; sphere
axis([-3 3 -2 2]); axis equal; grid on
```

$v = \mathrm{Im}(w)$

Joukowski airfoils



https://teaching.smp.uq.edu.au/scims/Complex_analysis/JoukowskyAirfoil.html

symmetric Joukowski Transformation $z=\frac{1}{2}\left(z+\frac{1}{z}\right)$

download: **airfoil.m**

Γ = Γ(-0.1+0.2i,1.118)

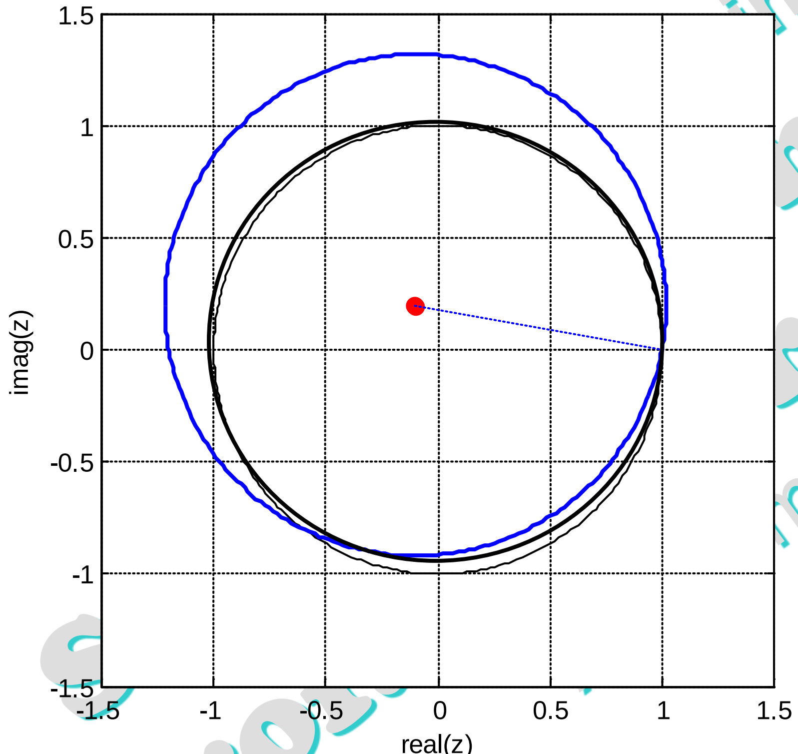
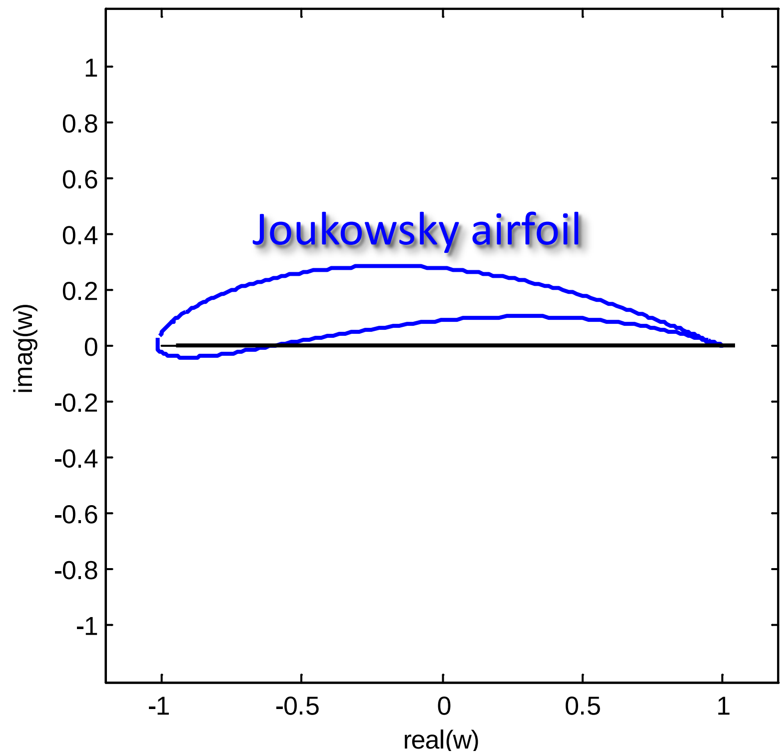


image of Γ by Joukowsky: w = (z+1/z)/2



The blue circle Γ has been transformed into Joukowsky airfoil.

The black unit circle has been transformed into the segment [-1,+1]

(prof. M. Rizzardi)

Example 3: problem transformation

$$u_t = c^2 u_{xx}$$

c^2 *thermal diffusivity*

problem P
defined in domain D

↓

simplified problem P*
defined in domain D

PDE

1D Heat Equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) & x>0,\ t>0 \\ u(x,0^+) = u_0(x) & \text{initial condition} \\ u(0,t) = \varphi_0(t) & \text{boundary condition} \end{cases}$$

Laplace Transform:

$$F(s) = \mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,\mathrm{d}t, \quad s \in \mathbb{C} \wedge \mathrm{Re}(s) > \sigma_0$$

Laplace $\mathscr{L}$-transformation
applied to PDE
(Laplace's method)

$$U(x,s) = \mathscr{L}_t[u(x,t)]$$

ODE

$$\begin{cases} U''(x,s) = sU(x,s) - u_0(x) & x>0,\ s\in\mathbb{C} \\ U(0,s) = \mathscr{L}[\varphi_0(t)] \end{cases}$$

x: differentiation var.
s: parameter

A complex function and its complex limit

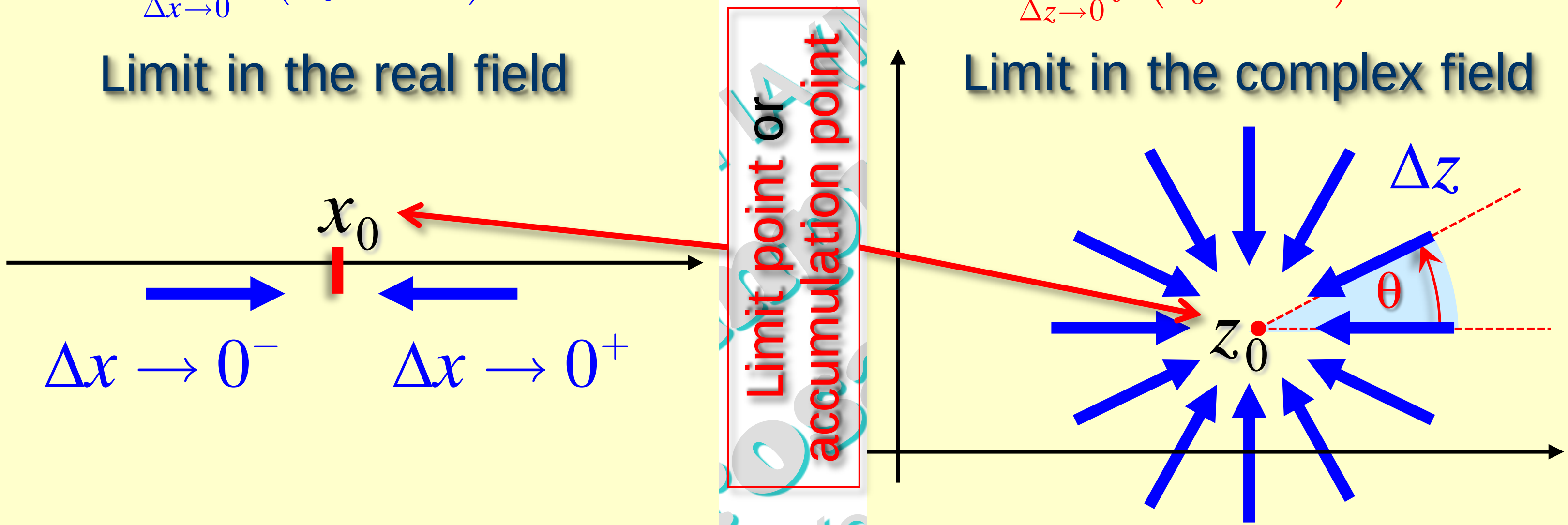
$$f : z = \underbrace{x}_{\mathrm{Re}(z)} + i\underbrace{y}_{\mathrm{Im}(z)} \in \mathbb{C} \longrightarrow f(z) = f(x,y) = \underbrace{u(x,y)}_{\mathrm{Re}(f)} + i\underbrace{v(x,y)}_{\mathrm{Im}(f)} \in \mathbb{C}$$

$$\lim_{\Delta x \to 0} f(x_0 + \Delta x) = \ell$$

Limit in the real field

$$\lim_{\Delta z \to 0} f(z_0 + \Delta z) = \lambda$$

Limit in the complex field



Uniform convergence of limit for $|\Delta x| \to 0$ from left and from right.

Uniform convergence of limit for $|\Delta z| \to 0$ with respect to any θ.

⇒ **real derivative**

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

⇒ **complex derivative**

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$$

The **complex differentiability (holomorphism)** is a stronger condition than the real differentiability.

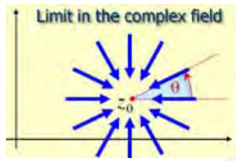
A complex function and its complex limit: example 1

$$\lim_{z\to 0} f(z) \quad ?$$

complex conjugate

$$f(z)=\frac{\bar{z}}{z}=\frac{x-iy}{x+iy}$$

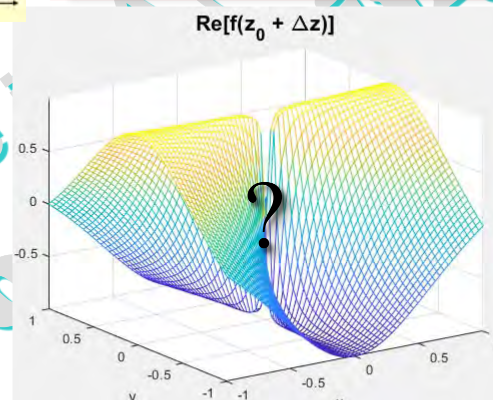
$$z_0=0$$

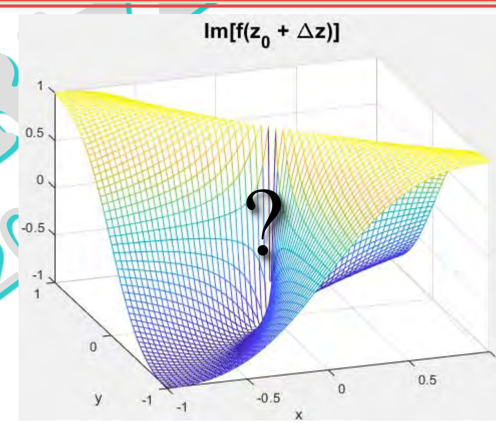


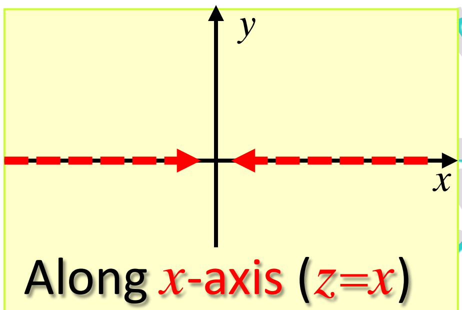
the limit as $\rho\to 0$ depends on θ

```
syms x y real; z=x+1i*y; f=conj(z)/z;
ff=subs(f,{x,y},{rho*cos(th),rho*sin(th)});
ff=simplify(ff)
ff = (cos(th)-sin(th)*1i)/(cos(th)+sin(th)*1i)
```

```
syms x y real; z=x+1i*y;
f=conj(z)/z;
```





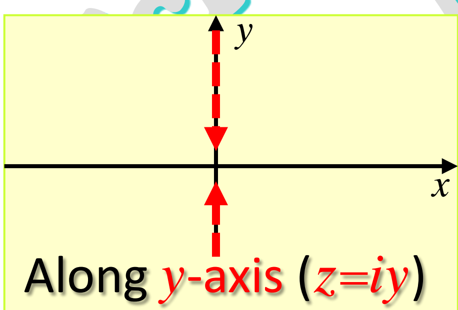


$$\lim_{\substack{y=0\\x\to 0}} f(z)=\lim_{\substack{y=0\\x\to 0}}\frac{x-iy}{x+iy}=\lim_{x\to 0}\frac{x}{x}=1$$

```
f1=subs(f,y,0); disp(limit(f1,x,0))
1
```

$\neq$

2 directions where the limits differ



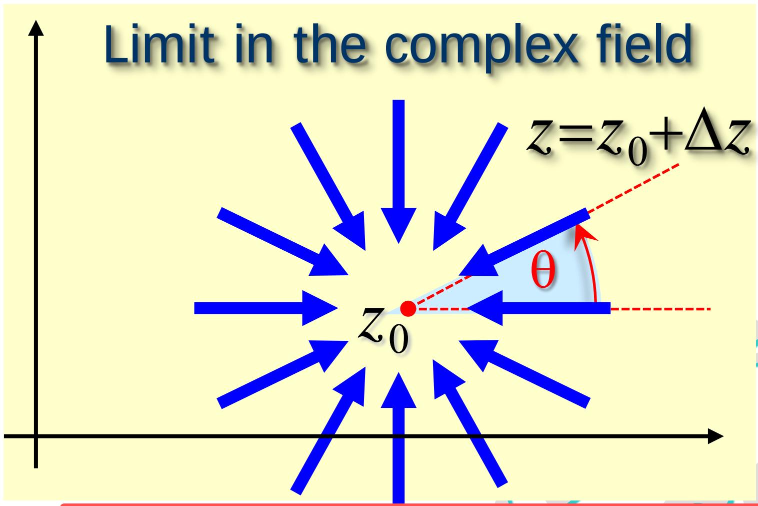
$$\lim_{\substack{x=0\\y\to 0}} f(z)=\lim_{\substack{x=0\\y\to 0}}\frac{-iy}{+iy}=\lim_{y\to 0}\frac{-y}{+y}=-1$$

```
f2=subs(f,x,0); disp(limit(f2,y,0))
-1
```

The limit does not exist

A complex function and its complex limit: example 2

Evaluate the limit of $f(z) = z^2 + z + 1$ as $z \to 1+2i$.



1) Write the function as $f(z_0+\Delta z)$.

```
syms Dx Dy real; Dz=Dx+1i*Dy; f=@(Z)Z^2+Z+1;
z0=1+2i;   z=z0+Dz;   fz=f(z);
```

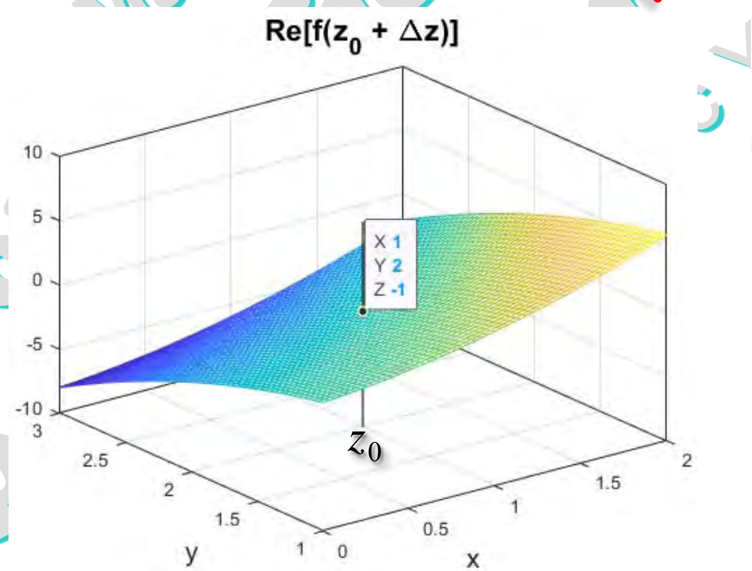
2) Insert polar coordinates of Δz.

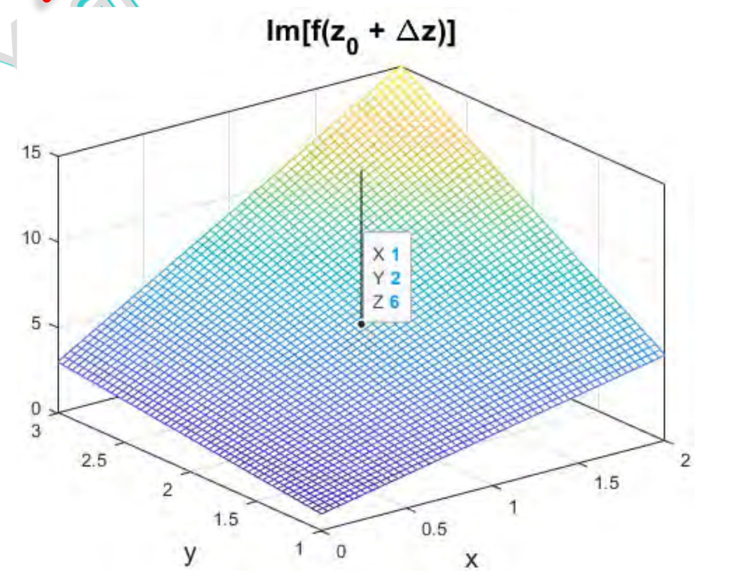
```
syms th real; syms rho positive
ff=subs(fz,{Dx,Dy},{rho*cos(th),rho*sin(th)})
ff=simplify(ff)
```

3) Evaluate the limit.

```
disp(z0^2+z0+1)  % the true value
- 1 + 6i
```

```
disp(limit(ff,rho,0))
- 1 + 6i
```





Holomorphic functions

$f(z)$ holomorphic at z_0 with respect to z

The following complex limit of the difference quotient of f_z exists

$$f_z'(z_0) = \frac{df}{dz}(z_0) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

(has a complex derivative f' at z_0 with respect to z)

$f(z)$ holomorphic at z_0 with respect to $\overline{z}$

The following complex limit of the difference quotient of $f_{\overline{z}}$ exists

$$f_{\overline{z}}'(z_0) = \frac{df}{d\overline{z}}(z_0) = \lim_{\Delta z \to 0} \frac{\Delta f}{\overline{\Delta z}} = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\overline{\Delta z}}$$

(has a complex derivative f' at z_0 with respect to $\overline{z}$)

f differentiable w.r.t. z — THEOR. → f not differentiable w.r.t. $\overline{z}$

f differentiable w.r.t. $\overline{z}$ — THEOR. → f not differentiable w.r.t. z

Basics of complex analysis

$$z = x + iy \in \mathbb{C} \to f(z) \in \mathbb{C}$$

$$f(z) = f(x,y) = u(x,y) + iv(x,y)$$

where

$$x = \mathrm{Re}[z]$$
$$y = \mathrm{Im}[z]$$
$$u(x,y) = \mathrm{Re}[f(x,y)]$$
$$v(x,y) = \mathrm{Im}[f(x,y)]$$

$f(z)$ holomorphic $\Leftrightarrow$ $f(z)$ satisfies Cauchy-Riemann Eqs

THEOR.: *Cauchy-Riemann equations*

A complex function $f(z)$ has a complex derivative $f'(z)$ at z_0 if, and only if, its real and imaginary parts are continuously differentiable and satisfy the Cauchy–Riemann equations at $z_0 = x_0 + iy_0$:

$$\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0) = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial u}{\partial x}(z_0) = \dfrac{\partial v}{\partial y}(z_0) \\ \dfrac{\partial u}{\partial y}(z_0) = -\dfrac{\partial v}{\partial x}(z_0) \end{cases}$$

complex form

real form

In this case the complex derivative is equal to any of the following expressions:

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$$

Cauchy-Riemann Equations at z_0

$$z = x + iy \in \mathbb{C} \rightarrow f(z) \in \mathbb{C}$$

$$f(z) = f(x,y) = u(x,y) + iv(x,y)$$

w.r.t. z

$$\begin{cases} \dfrac{\partial u}{\partial x}(z_0) = +\dfrac{\partial v}{\partial y}(z_0) \\ \dfrac{\partial u}{\partial y}(z_0) = -\dfrac{\partial v}{\partial x}(z_0) \end{cases}$$

$$\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0) = 0$$

real form

complex form

w.r.t. $\overline{z}$

$$\begin{cases} \dfrac{\partial u}{\partial x}(z_0) = -\dfrac{\partial v}{\partial y}(z_0) \\ \dfrac{\partial u}{\partial y}(z_0) = +\dfrac{\partial v}{\partial x}(z_0) \end{cases}$$

$$\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0) = 0$$

Basics of complex analysis

$$f : z = x + iy \in \mathbb{C} \longrightarrow f(z) = f(x,y) = u(x,y) + iv(x,y) \in \mathbb{C}$$

THEOR.: The following items are equivalent

⟺ 1) $f(z)$ is holomorphic (complex differetiability) at z_0.
(w.r.t. z)

⟺ 2) $f(z)$ is analytic (sum of a power series) at z_0.

⟺ 3) $f(z)$ satisfies the ***Cauchy-Riemann equations*** at z_0.

⟺ 4) $f(x,y)$, $u(x,y)$, $v(x,y)$ satisfy ***Laplace's equation***

$$\nabla^2 g = 0 \iff g_{xx} + g_{yy} = 0 \iff g(x,y) : \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0 \iff \Delta g = 0$$

A function that satisfies *Laplace's equation* is said *harmonic**.

* Harmonic functions are used in robotics applications for motion planning in a known environment

If f is a *holomorphic function* then u and v are said *harmonic conjugate*.

$f(x,y)$, $u(x,y)$, $v(x,y)$ satisfy the *Laplace Equation*

(u, v are said harmonic conjugate functions)

$$\varphi(x,y): \quad \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

Examples

```
syms x y real; z=x+i*y;
f = z^3;
u=simplify(real(f))
u =
x^3 - 3*x*y^2
v=simplify(imag(f))
v =
3*x^2*y - y^3
fcontour(u,[-5 5],'b')
axis equal; hold on
fcontour(v,[-5 5],'r')
disp(diff(f,x,2)+diff(f,y,2))
0
disp(diff(u,x,2)+diff(u,y,2))
0
disp(diff(v,x,2)+diff(v,y,2))
0
disp(diff(f,x)+1i*diff(f,y))
0
```

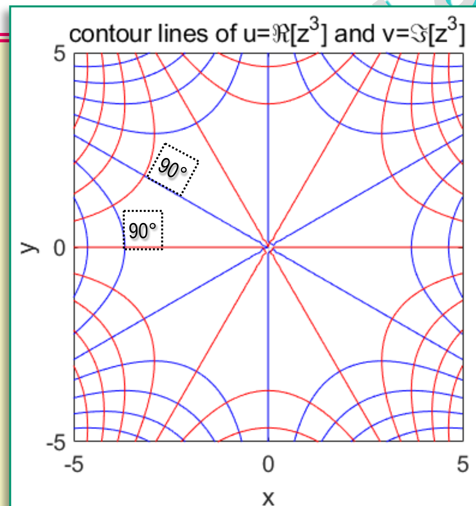
f satisfies the Laplace Eq.

u satisfies the Laplace Eq.

v satisfies the Laplace Eq.

f holomorphic w.r.t. z

$$\frac{\partial f}{\partial x}(z_0)+i\frac{\partial f}{\partial y}(z_0)=0$$



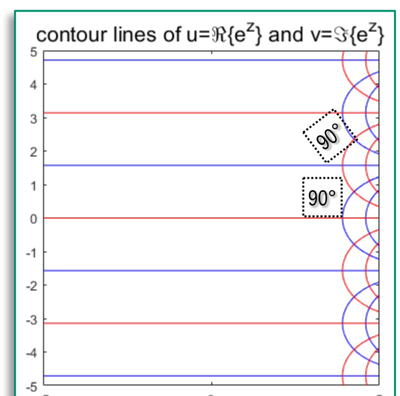

```
f = conj(z);
u=simplify(real(f))
u =
x
v=simplify(imag(f))
v =
-y
disp(diff(f,x)+1i*diff(f,y))
2
disp(diff(f,x)-1i*diff(f,y))
0
```

non-holomorphic w.r.t. z

holomorphic w.r.t. conj(z)

$$\frac{\partial f}{\partial x}(z_0)-i\frac{\partial f}{\partial y}(z_0)=0$$

```
f = exp(z);
u=simplify(real(f))
u =
exp(x)*cos(y)
v=simplify(imag(f))
v =
exp(x)*sin(y)
disp(diff(f,x,2)+diff(f,y,2))
0
disp(diff(u,x,2)+diff(u,y,2))
0
disp(diff(v,x,2)+diff(v,y,2))
0
disp(diff(f,x)+1i*diff(f,y))
0
```



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