Course of
"Automatic Control Systems"
2022/23

Robust stability: Phase and Gain stability margins

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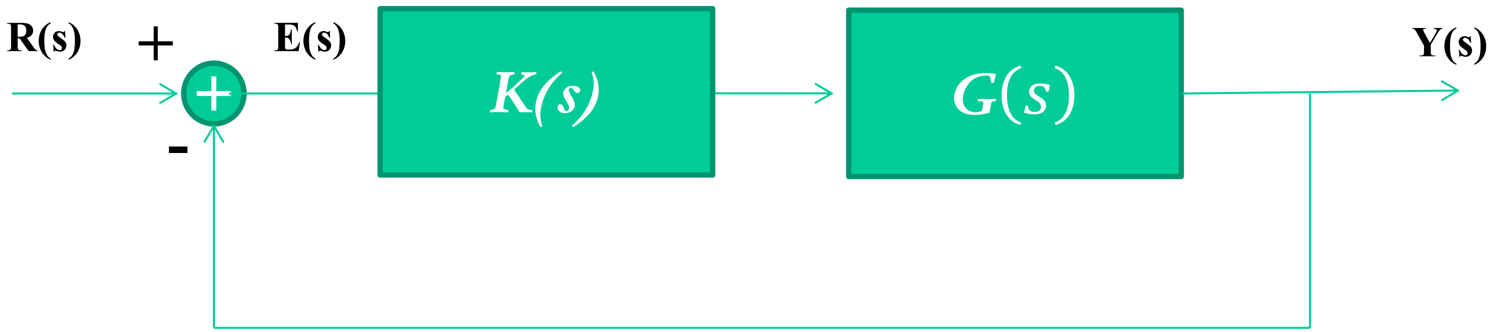
Stability of the closed loop system

- Let us consider a closed loop system in the form



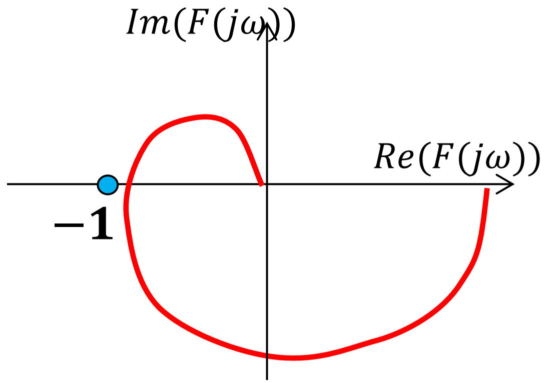
- In this lesson we present two parameters, ***the phase and gain margins***, able to ***quantify the robustness of the closed loop stability*** with respect to system uncertainties.

- From now on ***we assume that the closed loop system is regularly stable***, that is there exists a critical gain $\bar{k}$ after which the closed loop system becomes unstable

Ideal parameter to quantify the robust stability of closed loop system

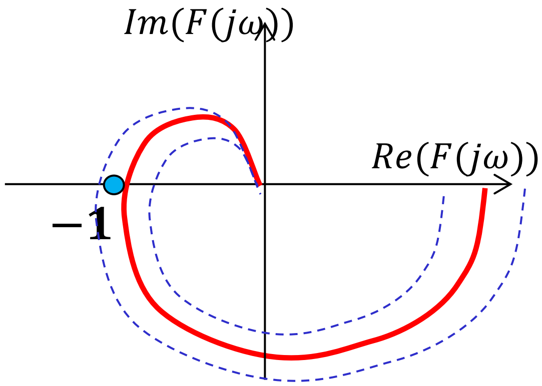
- Let us consider a regular stable closed loop system with $n_p^+(F(s)) = 0$.

- Let us assume that, for a given controller $K(s)$, the Nyquist plot of the transfer function $F(s)$ is



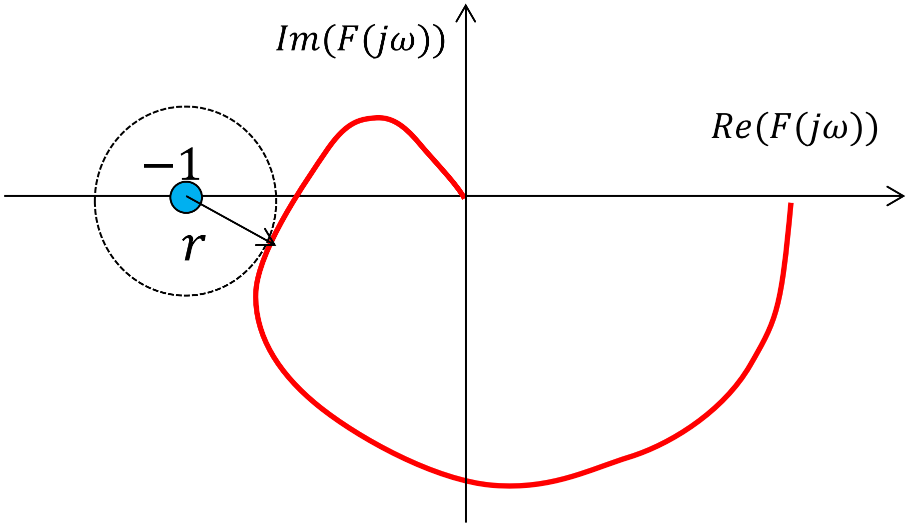
For the Nyquist stability criterion, the closed loop system is asymptotically stable because $\overleftarrow{\mathcal{N}} = 0$.

- However, if the plant model is uncertain, the Nyquist plot of $F(s)$ risks to encircle the critical point $-1 + j0$ and hence stability property of the closed loop system is lost.



Ideal parameter to quantify the robust stability of closed loop system

- An ideal parameter able to quantify the robustness of the closed loop system stability is the minimum distance between the critical point and the Nyquist plot of $F(s)$.

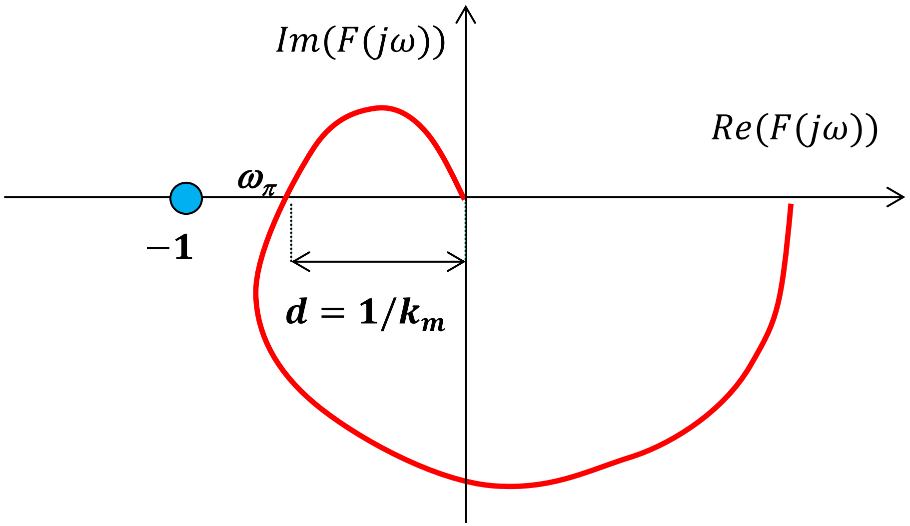


- In order to compute r, we need to evaluate the Nyquist plot of $F(s)$ precisely in each point of the diagram.

- For this reason, we will define two simpler parameters, ***the phase and gain margins,*** that can be easily computed on the Nyquist plot and are able to ***quantify the robustness of the closed loop stability.***

Gain stability margin

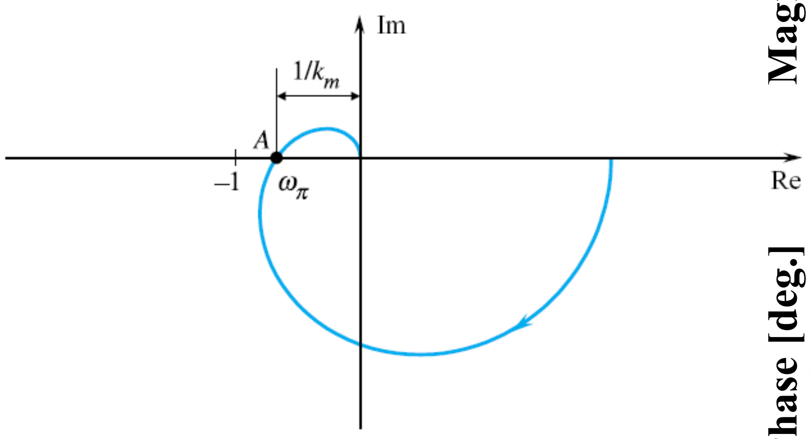
- The ***gain stability margin*** $\boldsymbol{k_m}$ is represented in the following figure.

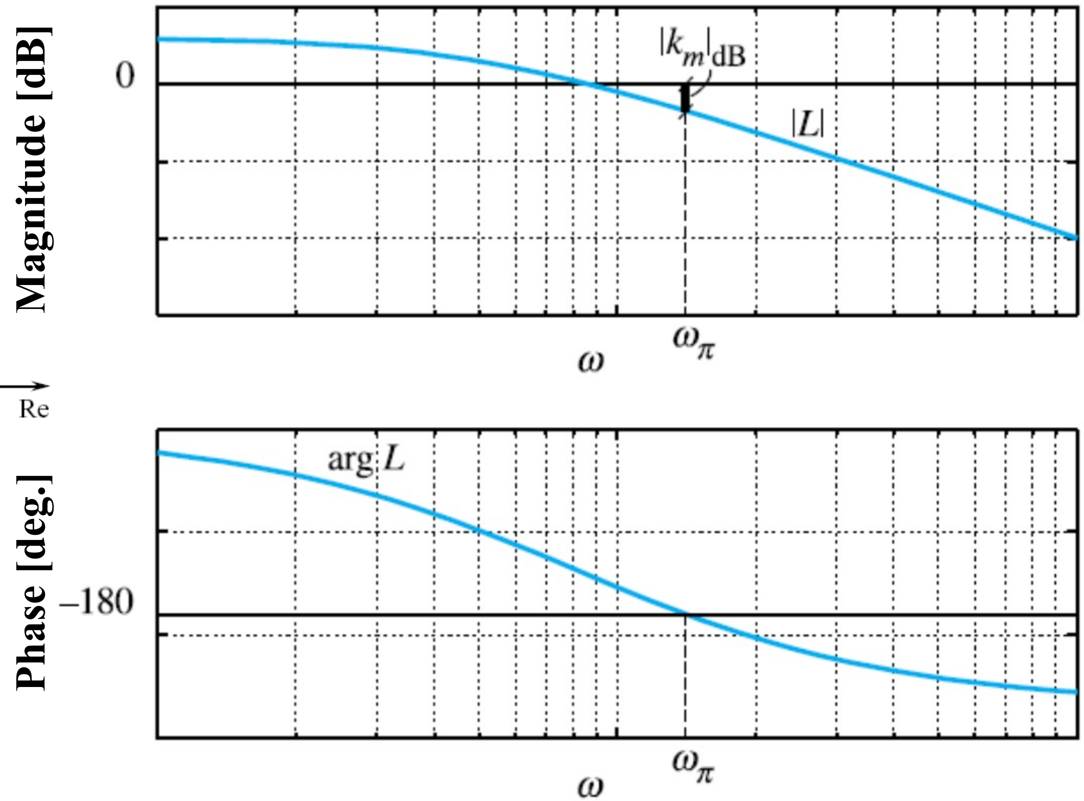


- Said
  - ω_π the frequency where the Nyquist plot of $F(j\omega)$ intersects the negative real axis, that is the phase of $F(j\omega)$ is equal to $-\pi$.
  - $\boldsymbol{d}$ the module of $F(j\omega)$ in ω_π $(d = |F(j\omega_\pi)|)$

the gain stability margin is defined as the inverse of d.

$F(s)$ or $L(s)$



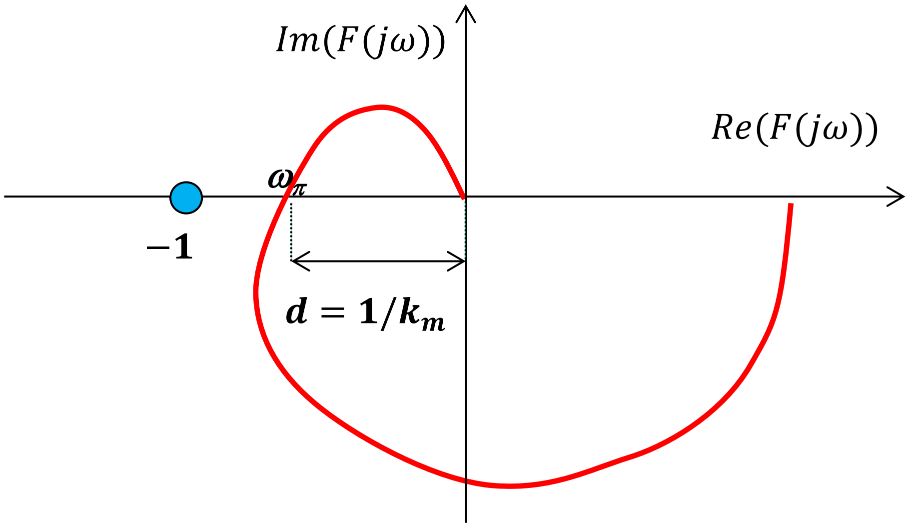


$$k_m = \frac{1}{|L(j\omega_\pi)|}, \arg L(j\omega_\pi) = -180°. \qquad |\boldsymbol{k_m}|_{\boldsymbol{dB}} = 20\log_{10}\frac{1}{|L(j\omega_\pi)|} = -|\boldsymbol{L(j\omega_\pi)}|_{\mathbf{dB}}.$$

The closed loop system is A.S. if k_m>1.

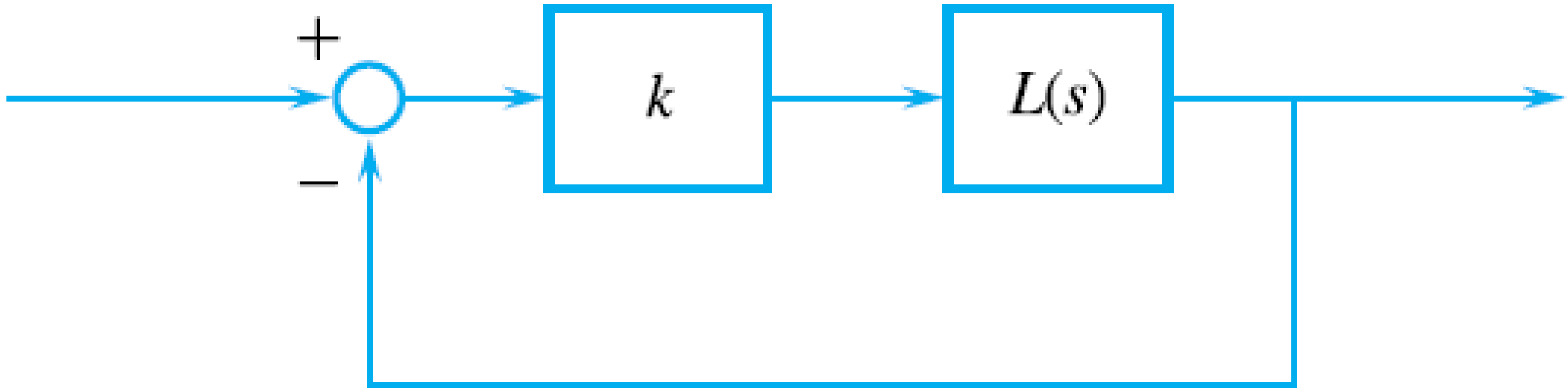
Gain stability margin

- The ***gain stability margin*** $\boldsymbol{k_m}$ indicates the maximum gain uncertainty of $F(j\omega)$ before the critical point $-1 + j0$ is encircled, and hence closed loop stability is lost.



- If $F(j\omega)$ doesn't intersects the negative real axis, the gain stability margin is not defined (the closed loop system is asymptotically stable for all gain uncertainty of $F(j\omega)$).

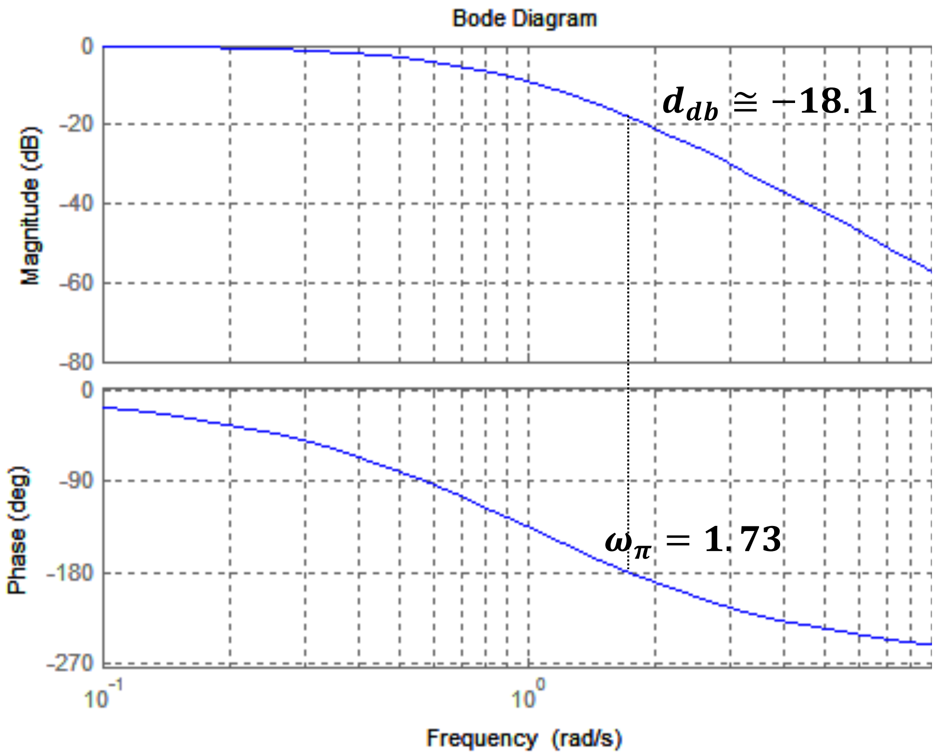
$F(s)$ or $L(s)$

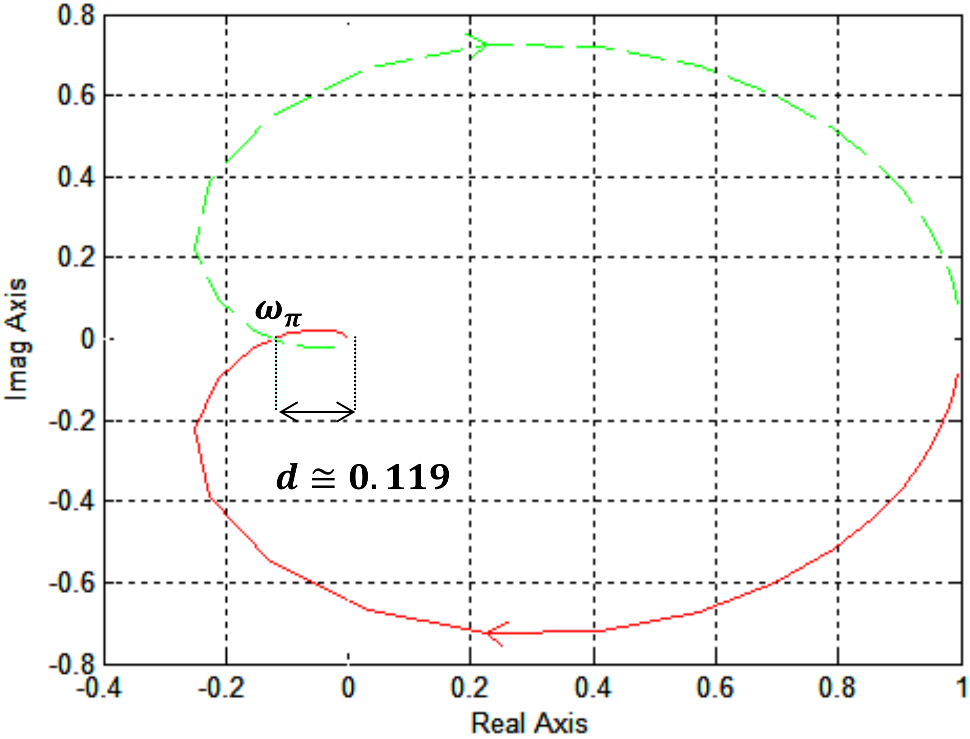


- The closed loop system is A.S. for $k < k_m$, i.e. ***gain stability margin***
- k_m indicates the maximum gain for which $F(s)$ or $L(s)$ can be multiplied without leading to an unstable closed loop system.

Gain stability margin: example

- Let us consider the transfer function $F(s) = \frac{1}{(1+s)^3}$

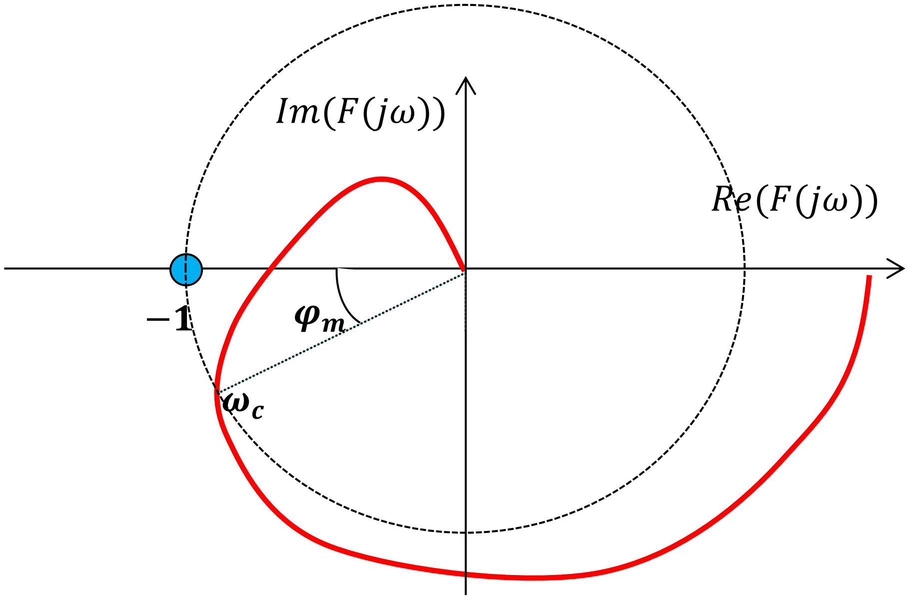




Gain stability margin $k_m \cong 8.04 \rightarrow k_{mdb} \cong 18.1$

Phase stability margin

- The ***phase stability margin*** $\boldsymbol{\varphi_m}$ is represented in the following figure.

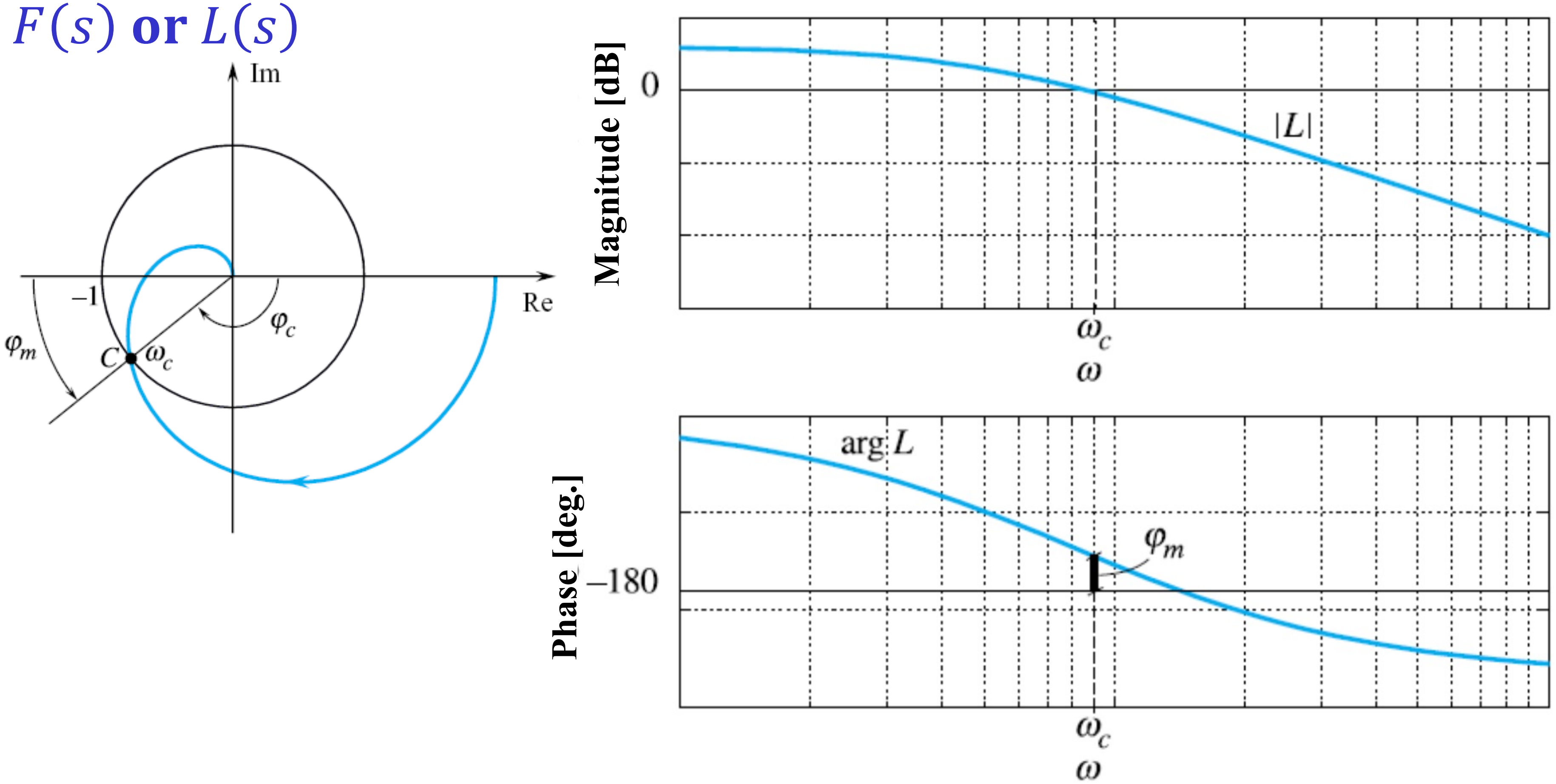


- Said
  - $\boldsymbol{\omega_c}$ the frequency where the Nyquist plot of $F(j\omega)$ intersects the unit circle, that is the module of $F(j\omega)$ is equal to 1.
  - $\boldsymbol{\varphi_c}$ the phase of $F(j\omega)$ in ω_c $\left(\varphi_c = \arg F(j\omega_c)\right)$

The phase stability margin is defined as $\boldsymbol{\varphi_m = 180 - |\varphi_c|}$

Phase stability margin

$F(s)$ **or** $L(s)$

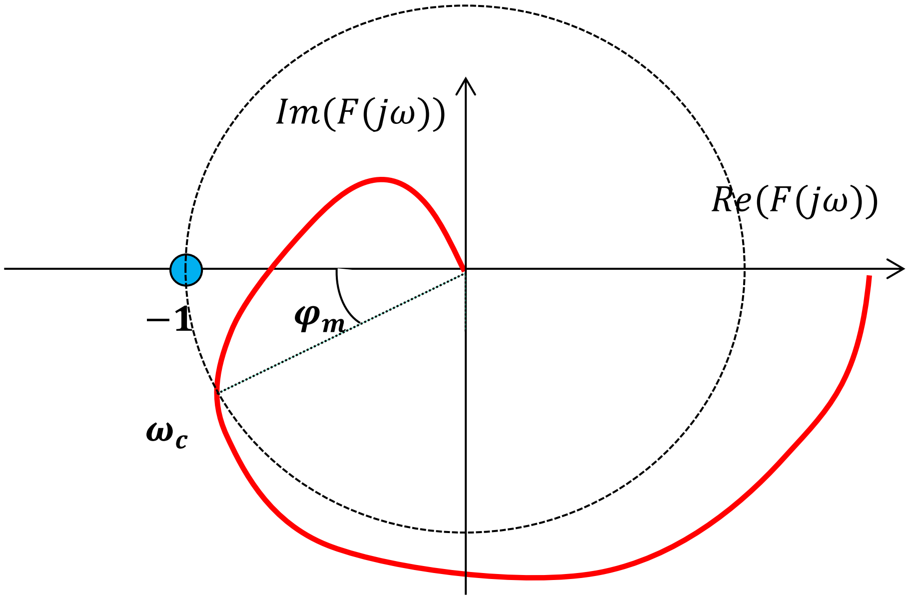


$\varphi_m = 180° - |\varphi_c|, \varphi_c = \arg L(j\omega_c), |L(j\omega_c)| = 1, \quad |L(j\omega_c)|_{dB} = 0.$

The closed loop system is A.S if φ_m>0.

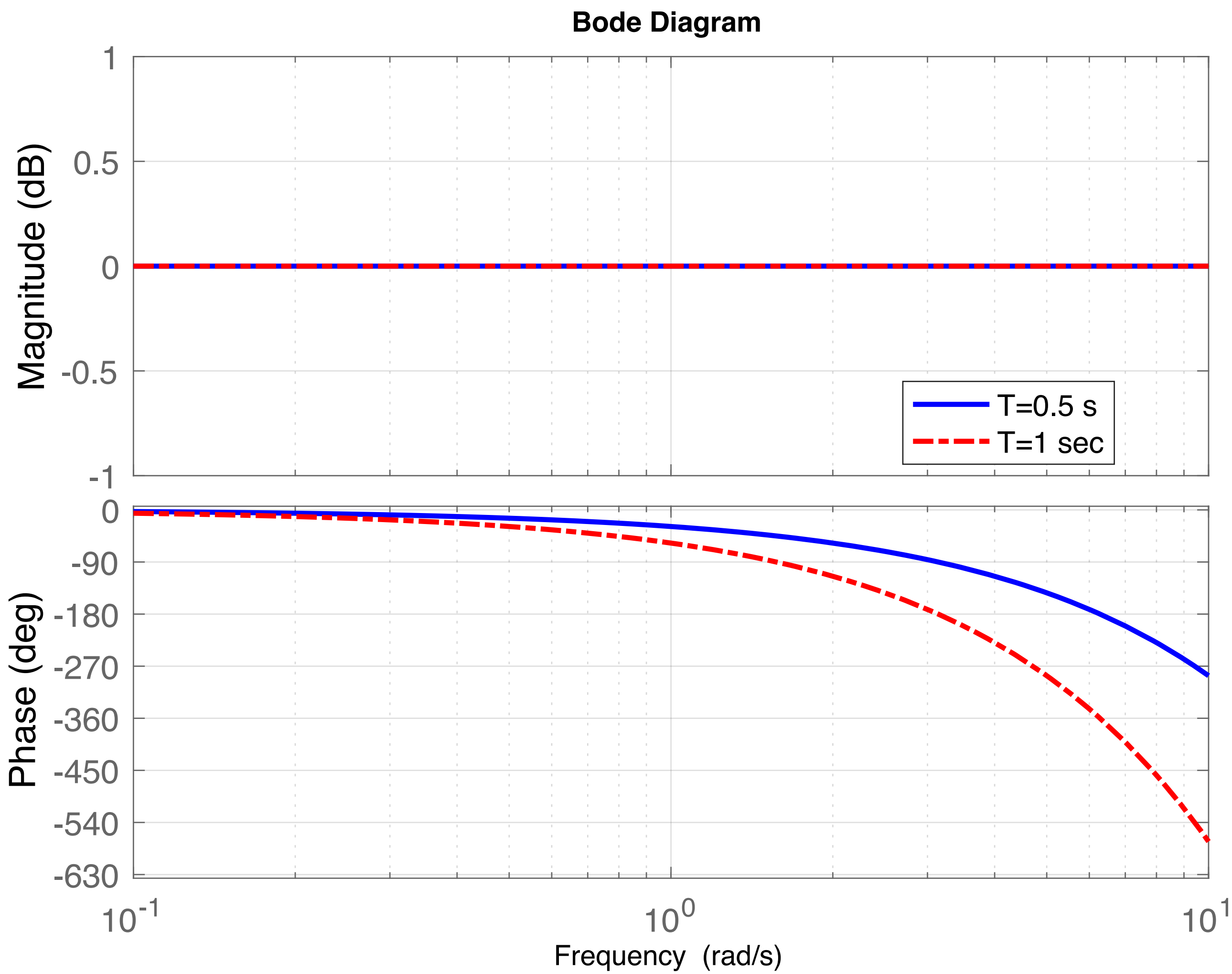
Phase stability margin

- The ***phase stability margin*** $\boldsymbol{\varphi_m}$ indicates the maximum phase uncertainty of $F(j\omega)$ before the critical point $-1 + j0$ is encircled, and hence closed loop stability is lost.



- If $F(j\omega)$ doesn't intersects the unit circle because the amplitude of $F(j\omega)$ is less than 1 for all ω, then the phase stability margin is not defined (the closed loop system is asymptotically stable for all phase uncertainty of $F(j\omega)$).

Time delay system



T.f. of time delay system:

$$\boldsymbol{G}(s) = e^{-Ts}.$$

Indeed, for a system with time delay T,

$$y(t) = u(t - T).$$

By making Laplace,

$Y(s) = U(s)e^{-sT}$,

then

$$W(s) = \frac{Y(s)}{U(s)} = e^{-sT}.$$

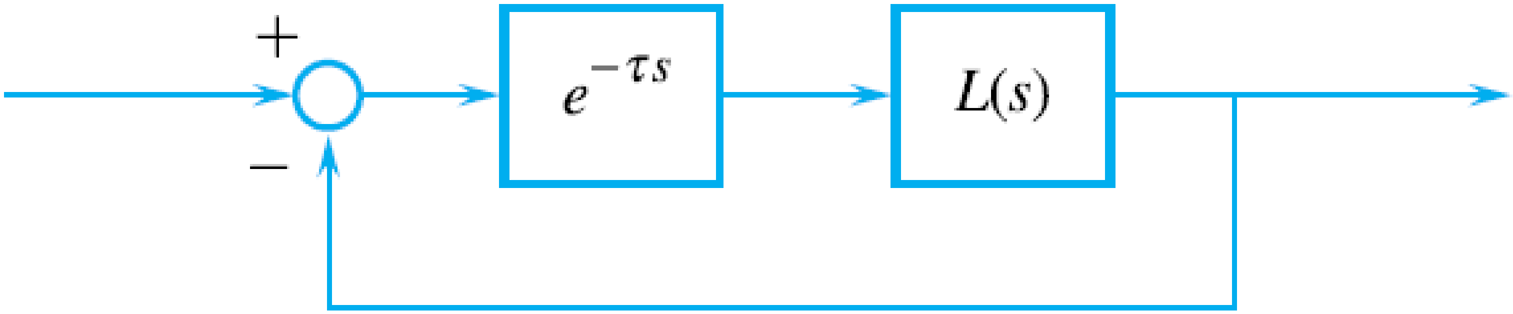
By substituting s=$j\omega$,

$$W(j\omega) = e^{-j\omega T}.$$

Then the magnitude is one for each ω and the phase is $-\omega T$.

Phase stability margin

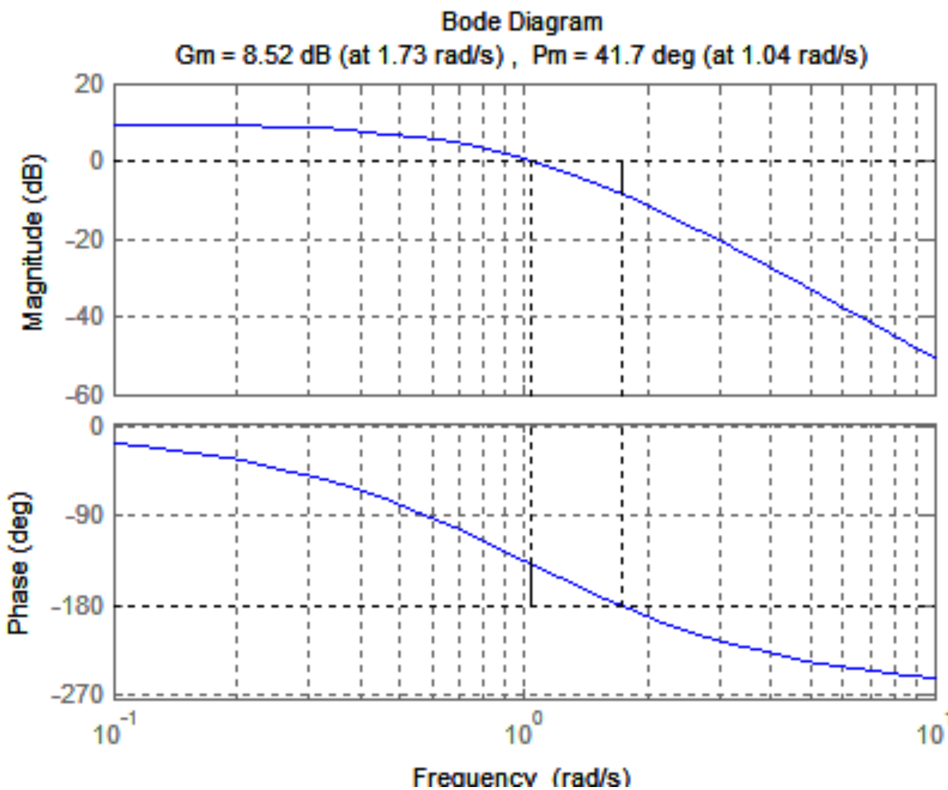
$F(s)$ **or** $L(s)$

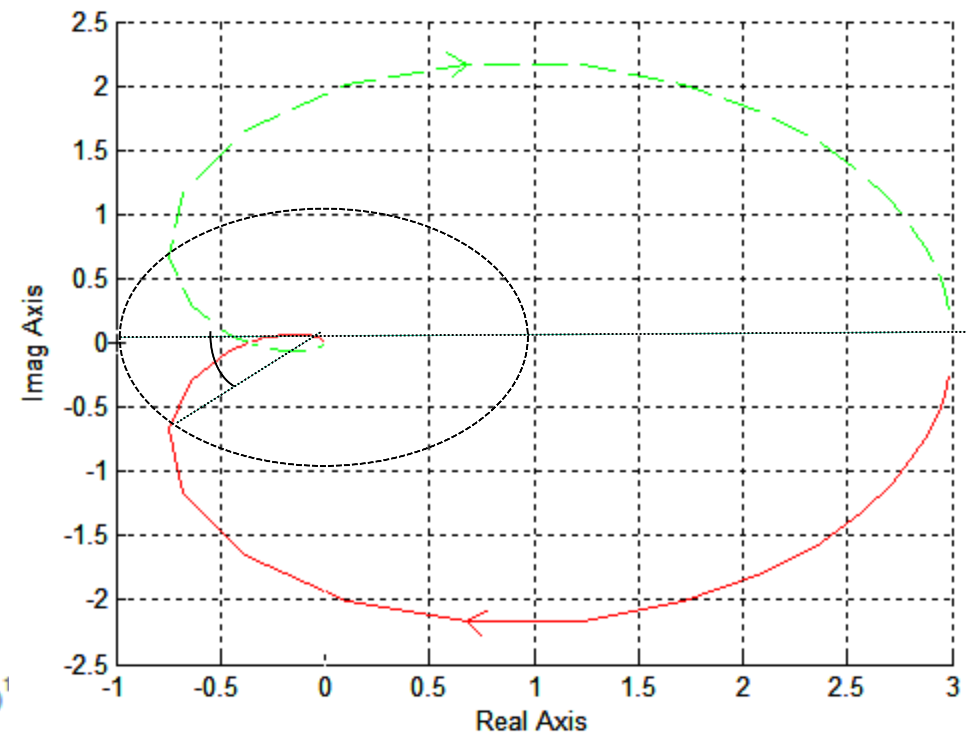


The closed loop system is A.S., if $\omega_c \tau < \varphi_m \pi / 180°$.

Then $\tau < \dfrac{\varphi_m \pi / 180°}{\omega_c}$

- Let us consider the transfer function $F(s) = \frac{3}{(1+s)^3}$





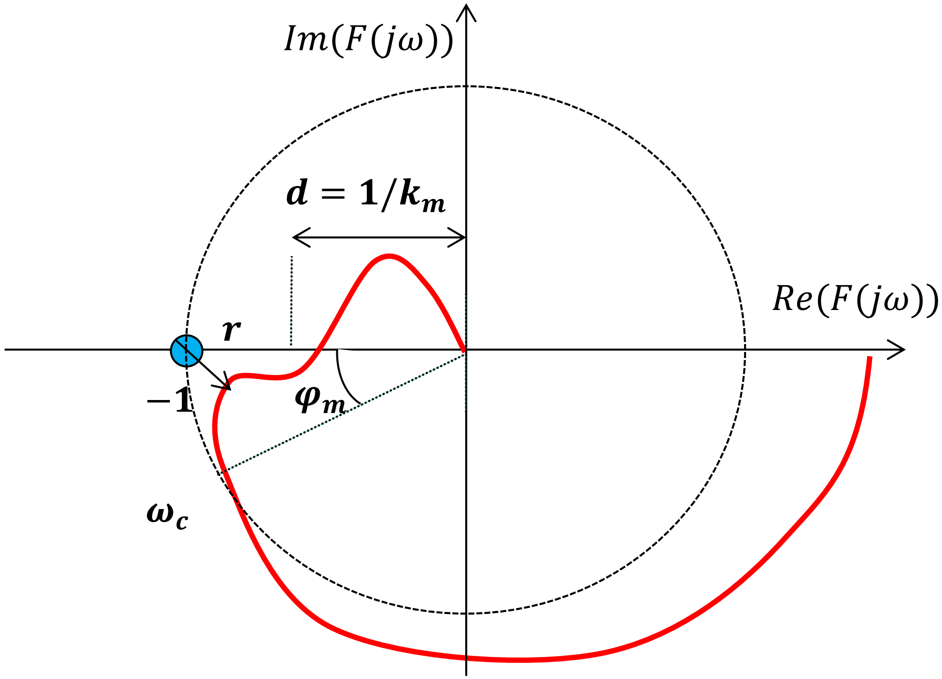
Phase stability margin $\varphi_m \cong 41.7° \;\; at \quad \omega_c = 1.04 rad/s$

$$\tau_{max} \cong \frac{\varphi_m \pi / 180}{\omega_c} = 0.7\ s$$

Phase and gain stability margins

- The ***phase and gain stability margins*** quantify the robustness of the closed loop stability with respect to phase and gain uncertainties on the transfer function $F(s)|_{s=j\omega}$

- However, ***they consider the phase and gain uncertainties separately***. Hence there are $F(j\omega)$ having high phase and gain margins but a low ideal parameter r quantify the robustness of the closed loop stability.



$k_m \gg 0$
$\varphi_m \gg 0$

but very small r