Exercises

SC2_06 – Linear transformations.

1. Given the following transformation

$$F\ :\ A=\begin{pmatrix}\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\end{pmatrix}\in M_{3\times3}(\mathbb{R})\longrightarrow F(A)=A-A^{\mathsf{T}}\in M_{3\times3}(\mathbb{R})$$

check if it is linear, and compute, by means of the MATLAB Symbolic Math Toolbox, the subspaces $\mathcal{N}(F)$, Kernel of F, and $\mathcal{R}(F)$, Range of F.

2. Find the Kernel of the linear transformation that maps every differentiable function $f(x)$ to its derivative $f'(x)$.

3. Established a non-zero vector $\mathbf{v}\in\mathbf{R}^3$, say $\mathbf{v}=(3\ 2\ 1)^{\mathsf{T}}$, verify that the following linear transformation is surjective but not injective

$$F:\boldsymbol{u}\in\mathbf{R}^3\longrightarrow\alpha=\langle\mathbf{v},\boldsymbol{u}\rangle\in\mathbf{R}$$

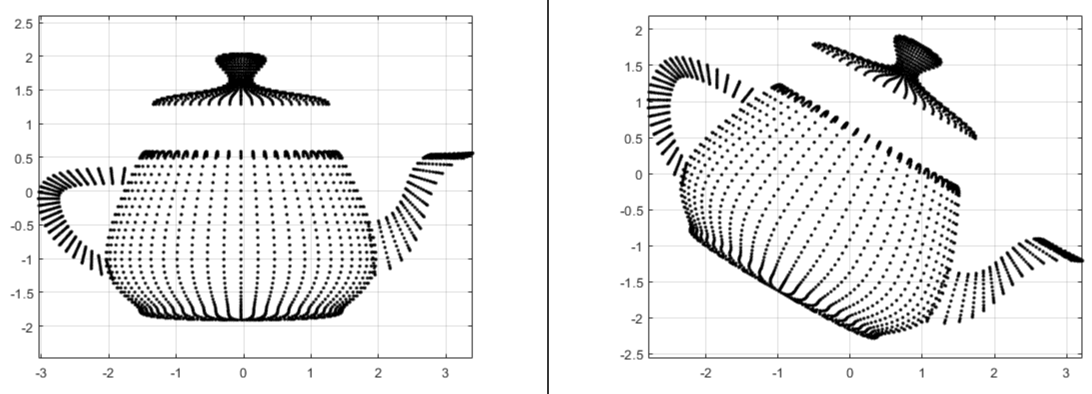
where $\langle,\rangle$ denotes the standard inner product in $\mathbf{R}^3$.
If $A=(3\ 2\ 1)$, verify that F corresponds to the mapping t_A, induced by the matrix A. Compute and draw the subspaces $\mathcal{N}(A)$ and $\mathcal{R}(A^{\mathsf{T}})$.

4. Determine the isomorphism between $\mathcal{R}(A^{\mathsf{T}})$ and $\mathcal{R}(A)$ where

$$A=\begin{pmatrix}1&2&0\\0&1&1\end{pmatrix},\qquad A=\begin{pmatrix}1&0\\2&1\\0&1\end{pmatrix},\qquad A=\begin{pmatrix}1&3\\2&6\\0&0\end{pmatrix}$$

5. Verify that a plane rotation by an angle θ is made up of two suitable shears (one horizontal and one vertical) and a non-uniform scaling. If R is the rotation matrix, S_o is the horizontal shear matrix, S_v is the vertical shear matrix and D is the non-uniform scaling matrix, how is the R matrix factored? Check this graphically, for $\theta=-\pi/6$, on the points in the file `my_teapot_2D_O.mat`, which you can download from the eLearning platform. Also display how the individual transformations act on the input points. The following code draws these points before and after the rotation.

```
load my_teapot_2D_O
th=-pi/6; R=[cos(th) -sin(th); sin(th)  cos(th)];
P=[x;y]; figure, plot(P(1,:),P(2,:),'.k'); axis equal; grid on
Q=R*P;   figure, plot(Q(1,:),Q(2,:),'.k'); axis equal; grid on
```



6. Let A be the matrix of the 2D orthogonal reflection $S_{\boldsymbol{a}} = t_A$ over a line r=span$\{\boldsymbol{a}\}$, where $\|\boldsymbol{a}\|_2=1$; prove, by means of Symbolic Math Toolbox, that
$$A = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

7. Let A be the matrix of the 2D orthogonal reflection $S_{\boldsymbol{a}} = t_A$ over a line r=span$\{\boldsymbol{a}\}$; find the four fundamental subspaces of A. Is the transformation t_A an automorphism? Verify that the *Householder reflector* $H_{\boldsymbol{a}}(\mathbf{v}) = -S_{\boldsymbol{a}}(\mathbf{v})$ describes the orthogonal reflection over $r^{\perp}$.

8. Compute the subspaces $\mathcal{N}(A)$ and $\mathcal{N}(\mathrm{I}-A)$ where A is the matrix of the 2D orthogonal projection onto a generic line r=span$\{\boldsymbol{a}\}$. Verify your answer in the particular case of $\boldsymbol{a}=(2,1)^{\mathsf{T}}$, and draw also the two subspaces. What is the mapping induced by the matrix $\mathrm{I}-A$?

9. Compute the distance between two parallel lines as an application of the 2D orthogonal projection onto a line. Draw results for some particular lines.

10. Given a square matrix `A` of size (2×2) and computed by `Nmax=6; A=randi(Nmax,2)-Nmax/2;`, factorize it into elementary linear transformations.

11. Given a square matrix, computed by `A=rand(2)` or by `A=randn(2)`, find the elementary linear maps obtained by the following factorizations of `A`:
    - `[L,U,P]=lu(A);`
    - `[U,S,V]=svd(A);`

12. Prove that, for all orthogonal matrices A, $\det(A) = \pm 1$ holds.

13. What is the linear transformation induced by the permutation matrix `P` defined as:
    `P=[1 0 0;0 0 1;0 1 0]`
    And by $\mathtt{P}^{-1}$? Display their effects.

14. Let A be the matrix of the 3D orthogonal reflection $S_{\boldsymbol{a}} = t_A$ over a line r=span$\{\boldsymbol{a}\}$, where $\boldsymbol{a}\in\mathbf{R}^3$; what do the four fundamental subspaces of A represent? And $\mathcal{N}(A-\mathrm{I})$? And $\mathcal{N}(\mathrm{I}-A)$? Is t_A an automorphism?

15. Draw with MATLAB the shadow parallel to the XZ plane (or to the YZ plane) of a solid of revolution [see `SC2_07e.pdf` pg. 17]. This shadow can be computed as the orthogonal projection of the points of the solid on that Cartesian plane, then translated onto a side of the graphic figure.

16. Given a vector $\boldsymbol{a}$ in $\mathbf{R}^3$, find the matrix form of the endomorphism for the orthogonal projection on the straight line r=span$\{\boldsymbol{a}\}$.

17. Compute in $\mathbf{R}^3$ the distance between a point P and a line r, and between P and a plane π. This can be considered an application of the orthogonal projection. Since $\mathbf{R}^3$ is assumed to be a vector space, the line and the plane pass through the origin, and the point has to be understood as the endpoint of OP vector.

18. Compute a generalized inverse of the matrix: `A=[4 4 -2;4 4 -2;-2 -2 10];` and, then, of the matrix: `diag(diag(A))`. Compute also the pseudoinverse matrix of `A` and compare it to the `pinv(A)` output.

19. Check that the matrix X_p, obtained as a particular right inverse of the following matrix [see `SC2_06f.pdf` pag. 11]:

$$A=\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 1\end{pmatrix}$$

is not the Moore-Penrose pseudoinverse of $\boldsymbol{A}$.

20. Apply the algorithm on pag. 10-11 of `SC2_06f.pdf`, to compute a particular left inverse $\boldsymbol{X}_p$ and the general form of the left inverses of the following matrix $\boldsymbol{A}$ (3×2):

$$A=\begin{pmatrix}1 & 1\\ 2 & 0\\ 0 & 1\end{pmatrix}$$

Compute also the left pseudoinverse of $\boldsymbol{A}$ as the orthogonal projection of $\boldsymbol{X}_p$ onto $\mathscr{R}(\boldsymbol{A}^{\top})$, and compare it with $B=\left(A^{\top}A\right)^{-1}A^{\top}$.

21. Solve with MATLAB the following underdetermined linear systems, and compute both the general solution and the least-norm solution. Display their results graphically.
    - `A=[1 2 3;4 6 6;7 8 9]; b=[2;2;2];`
    - `A=[1 4 7;2 3 9;2 2 8]; b=[6;7;6];`
    - `A=[1 3 8;1 2 6;0 1 2]; b=[12;9;3];`