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delle **Scienze**, dell'**Ingegneria**
e della **Salute**

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E TECNOLOGIE

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Scientific Computing (part 2 – 6 credits)

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- Affine Spaces and subspaces.
- Related properties.
- Parallelism between affine subspaces.
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- Affinely independent points and affine reference systems of subspaces.

Affine Spaces

From a geometrical point of view, curves and surfaces are usually considered as sets of “points” with some special properties. Typically, one is interested in geometric properties, invariant under certain transformations: for example, translations, rotations, projections, etc.

Modeling the space of points as a Linear Space is not very satisfactory, mainly because the point corresponding to the zero vector, called the **origin**, plays a special role, when there is really no reason to have a privileged origin.

An Affine Space is a geometric structure that makes possible to deal with points, curves, surfaces, etc., independently of any specific choice of a coordinate system (no preferential origin).

Definition of Affine Spaces

A structure $\langle \Sigma, \mathrm{V}, \varphi \rangle$ is said an **Affine Space** Σ, if:

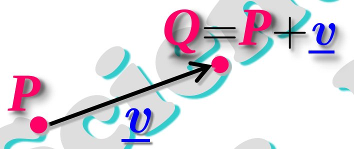
- Σ (geometric space) is a non-empty set of **points** P;
- V (direction space) is a Linear Space on the field K ($\mathbb{R}$ or $\mathbb{C}$);
- φ (difference function) $\varphi : (P,Q) \in \Sigma \times \Sigma \longrightarrow \varphi(P,Q) = \vec{v} \in \mathrm{V}$

  is a mapping usually denoted as

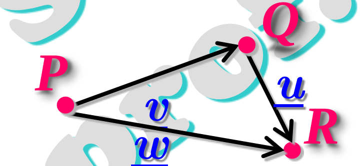
$$\varphi(P,Q) = \boxed{Q - P} = \overrightarrow{PQ} = \vec{v} \Longrightarrow Q = P + \underline{v}$$

($\underline{v}$: displacement vector or translation vector)

and such that:



(1) $\forall P \in \Sigma, \forall v \in \mathrm{V} \quad \exists! Q \in \Sigma : \varphi(P,Q) = v$



(2) $\forall P,Q,R \in \Sigma \quad \varphi(P,Q) + \varphi(Q,R) = \varphi(P,R)$

(Head-to-Tail Axiom)

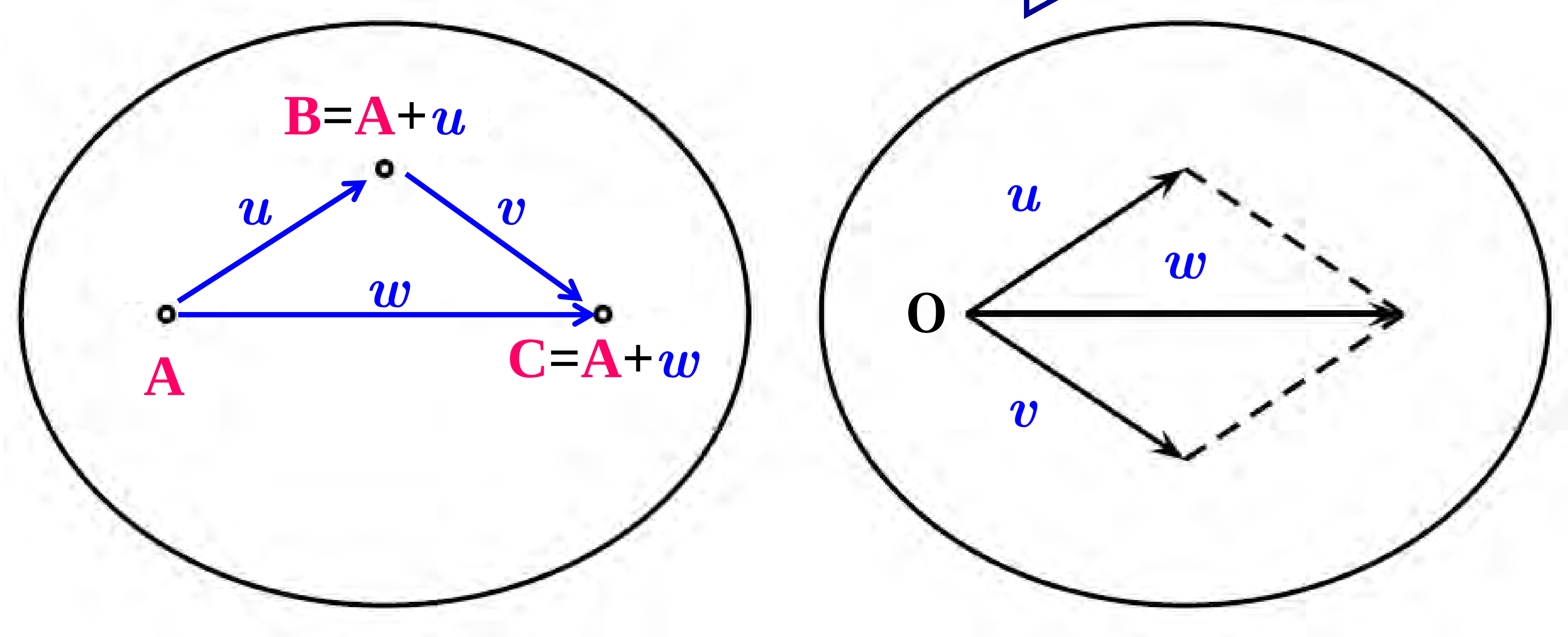
Intuitive picture of an **Affine Space**

$$\langle \Sigma, \mathbf{V}, \varphi \rangle$$

A, **B**, **C**: points

u, *v*, *w*: vectors

$$\Sigma \xrightarrow{\varphi(P,Q)} \mathbf{V}$$



Examples of Affine Spaces

- The set of points on a line of $\mathbb{R}^n$, even if it does not pass through the origin.

- The set of points on a plane of $\mathbb{R}^n$, even if it does not pass through the origin.

- The set of solutions of a non-homogeneous and compatible linear system $Ax=b$.

Linear Spaces contain the so called free vectors.

Affine Spaces introduce the “sum between a point and a vector”:

$$\varphi(P,Q)=Q-P=\overrightarrow{PQ}=\vec{v} \quad \Longleftrightarrow \quad Q=P+\varphi(P,Q)=P+\vec{v}$$

They contain the so called Euclidean (or geometric) vectors, that connect an initial point P to an end point Q.

Properties

Chosen a point $O\in\Sigma$, and defined $\forall P\in\Sigma,\quad \varphi(O,P)=\overrightarrow{OP}$, then

- ➢ $\forall P\in\Sigma,\quad \varphi(P,P)=\vec{0}$
- ➢ $\forall P,Q\in\Sigma,\ \varphi(P,Q)+\varphi(Q,P)=\vec{0}$ so that we set $\varphi(Q,P)=-\varphi(P,Q)$

- ➢ Each Linear Space V can be equipped with an Affine Space structure $\langle V,V,\varphi\rangle$ [by picking an origin $\mathbf{O}$, and by defining $\boldsymbol{a}=\varphi(\mathbf{O},\mathbf{A})$ and $\varphi(\mathbf{A},\mathbf{B})=\boldsymbol{b}-\boldsymbol{a},\ \forall \boldsymbol{a},\boldsymbol{b}\in V$, so that $V=\{\mathbf{A}:\mathbf{A}=\mathbf{O}+\varphi(\mathbf{O},\mathbf{A})=\mathbf{O}+\boldsymbol{a},\ \forall \boldsymbol{a}\in V\}$]
- ➢ Each Affine Space Σ can be equipped with a Linear Space structure [by defining the vectors of the Linear Space as $\boldsymbol{a}=\varphi(\mathbf{O},\mathbf{A})\ \ \forall \mathbf{A}\in\Sigma$, where $\mathbf{O}$ is the origin]

An **Affine Space** Σ, whose direction space V is a Normed Linear Space is said an Euclidean Space.

- The **dimension** of an Affine Space $\langle \Sigma, V, \varphi \rangle$ is defined as

$$\dim \Sigma = \dim V$$

In order to give a reference system to an Affine Space [affine reference system $\mathscr{R}(\mathbf{O}, \mathrm{B})$], you need:

- to establish a point $\mathbf{O} \in \Sigma$ (the origin of the reference $\mathscr{R}$).
- to choose a basis B = $\{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ for V.

In the $\mathscr{R}(\mathbf{O},\mathrm{B})$ reference system, the affine coordinates $(p_1, p_2, \dots, p_n)$ of a point $\mathbf{P}$ are defined by the components of the vector $\varphi(\mathbf{O},\mathbf{P})$ w.r.t. the selected basis B and origin $\mathbf{O}$.

Components of the vector between two points are given by the difference between point coordinates.

Proof: if $V = \text{span}\{B\} = \text{span}\{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ then $\forall\ \mathbf{P}, \mathbf{Q} \in \Sigma$

$\boxed{\varphi(\mathbf{P},\mathbf{O}) = -\varphi(\mathbf{O},\mathbf{P})} \Longrightarrow$

(2)

$$\mathbf{Q} - \mathbf{P} = \boxed{\varphi(\mathbf{P},\mathbf{Q}) = \varphi(\mathbf{P},\mathbf{O}) + \varphi(\mathbf{O},\mathbf{Q})} = \varphi(\mathbf{O},\mathbf{Q}) + \varphi(\mathbf{P},\mathbf{O}) =$$

$$= \varphi(\mathbf{O},\mathbf{Q}) - \varphi(\mathbf{O},\mathbf{P}) = (q_1 - p_1)\underline{b}_1 + (q_2 - p_2)\underline{b}_2 + \dots + (q_n - p_n)\underline{b}_n$$

A non-empty subset $\Sigma' \subseteq \Sigma$ is said to be an **Affine Subspace** of $\langle \Sigma, V, \varphi \rangle$ if there exists V', a linear subspace of V, such that the restriction of φ to Σ' admits V' as its direction space.

Particular affine subspaces of $\mathbb{R}^n$ ($\Sigma = \mathbb{R}^n$, $V = \mathbb{R}^n$)

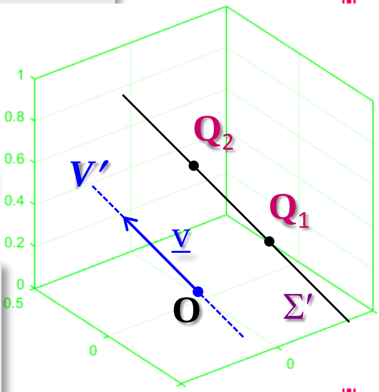
The only linear subspace of $V = \mathbb{R}^n$ with dim=0 is $\{\underline{0}\}$: therefore all the points of $\Sigma = \mathbb{R}^n$ are the only affine subspaces with dim=0.

The lines in $\Sigma = \mathbb{R}^n$ are the only affine subspaces with dim=1.

Indeed, given $\underline{v} \in V$: $V' = \text{span}\{\underline{v}\} = \lambda\,\underline{v}$ (dim $V'=1$), a line is described by

parametric eq. ⟶ $\Sigma' = \{\mathbf{P} \in \Sigma : \mathbf{P} = \mathbf{P}_0 + \lambda\,\underline{v} : \lambda \in \mathbb{R}\}$

(Σ' is said as an affine subspace passing through P_0 and parallel to V')



For example, $\Sigma' \subseteq \mathbb{R}^3$ is a line : $\Sigma' \equiv Q_1 + V' \equiv Q_2 + V'$, parallel to $V' = \text{span}\{\underline{v}\}$

The planes in $\Sigma = \mathbb{R}^n$ are the only affine subspaces with dim=2.

parametric eq. ⟶ $\Sigma'' = \{\mathbf{P} \in \Sigma : \mathbf{P} = \mathbf{P}_0 + \lambda\,\underline{v} + \mu\,\underline{w} : \lambda, \mu \in \mathbb{R}\}$

For example, $\Sigma'' \subseteq \mathbb{R}^3$ is a plane : $\Sigma'' \equiv \mathbf{P}_0 + V''$, parallel to $V'' = \text{span}\{\underline{v}, \underline{w}\}$

Examples of real Affine Subspaces

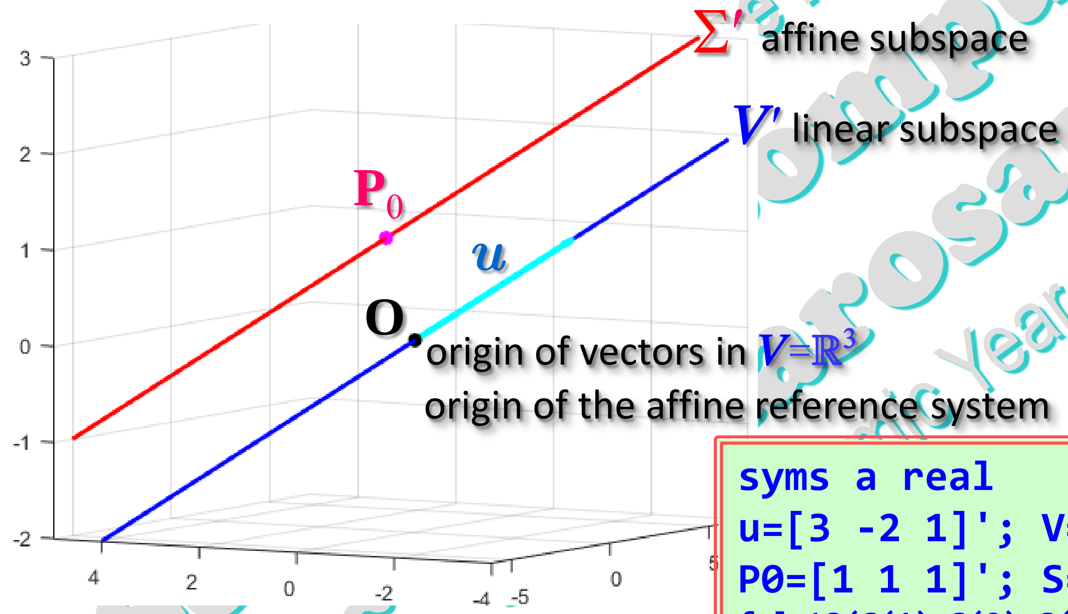
- ☐ The set of points of $\mathbb{R}^3$ on a line Σ', even if it does not pass through the origin:

$$\Sigma' = \{\mathbf{P} \in \Sigma = \mathbb{R}^3 \ : \ \mathbf{P} = \mathbf{P}_0 + \lambda \underline{\boldsymbol{u}}, \ \lambda \in \mathbb{R}, \ \underline{\boldsymbol{u}} \in V = \mathbb{R}^3\}$$

(parametric eq.)

$V' = \text{span}\{\underline{\boldsymbol{u}}\}$

V': direction space of Σ'



Σ' affine subspace

V' linear subspace

origin of vectors in $V = \mathbb{R}^3$
origin of the affine reference system

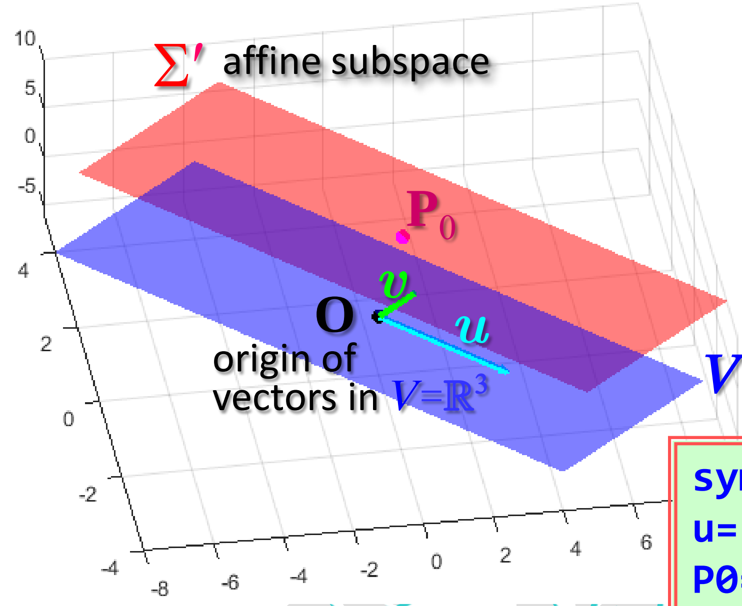
```
syms a real
u=[3 -2 1]'; V=a*u;
P0=[1 1 1]'; S=P0+V;
fplot3(S(1),S(2),S(3),[-2 2],'Color','r','LineWidth',2)
hold on
fplot3(V(1),V(2),V(3),[-2 2],'Color','b','LineWidth',2)
plot3(0,0,0,'ok','MarkerFaceColor','k')
plot3(P0(1),P0(2),P0(3),'om','MarkerFaceColor','m')
quiver3(0,0,0,u(1),u(2),u(3),1,'Color','c','LineWidth',3)
```

Examples of real Affine Subspaces

- ☐ The set of points of $\mathbb{R}^3$ on a plane Σ', even if it does not pass through the origin:

$$\Sigma' = \{\mathbf{P} \in \Sigma = \mathbb{R}^3 : \mathbf{P} = \mathbf{P}_0 + \lambda \underline{u} + \mu \underline{v},\ \ \lambda,\mu \in \mathbb{R},\ \underline{u},\underline{v} \in V = \mathbb{R}^3\}$$

(parametric eq.)



$V' = \text{span}\{\underline{u},\underline{v}\}$

V': direction space of Σ'

```
syms a b real
u=[3 -2 1]'; v=[1 0 2]'; V=[u v]*[a;b];
P0=[1 1 4]'; S=P0+V;
fmesh(V(1),V(2),V(3),[-2 2],'EdgeColor','none', ...
                                'FaceColor','b','FaceAlpha',0.5)
hold on
quiver3(0,0,0,u(1),u(2),u(3),1,'Color','c','LineWidth',3)
quiver3(0,0,0,v(1),v(2),v(3),1,'Color','g','LineWidth',3)
fmesh(S(1),S(2),S(3),[-2 2],'EdgeColor','none', ...
                                'FaceColor','r','FaceAlpha',0.5)
plot3(P0(1),P0(2),P0(3),'om','MarkerFaceColor','m')
plot3(0,0,0,'ok','MarkerFaceColor','k')
```

Parallelism and intersection in Affine Spaces

Two affine subspaces $\Sigma_1,\Sigma_2\subseteq\Sigma$, with **equal dimensions**, are said to be parallel if they have the same direction space.

Example: parallel lines; parallel planes.

Two affine subspaces $\Sigma_1,\Sigma_2\subseteq\Sigma$ with **different dimensions**, are said to be parallel if the direction space of the smaller one is contained in the direction space of the other.

Example: a line parallel to a plane.

The intersection $\Sigma_1\cap\Sigma_2$ between two affine subspaces $\Sigma_1,\Sigma_2\subseteq\Sigma$ is the set of points in Σ that belong to both of them. These points must satisfy both the parametric equations of Σ_1 and of Σ_2.

Example: intersection between lines, between planes, between a line and a plane.

Exercise

Verify which among the following affine subspaces are mutually parallel. Display with MATLAB the subspaces and their direction space.

In $\mathbb{R}^2$

Σ_1 : x - 2*y + 1 = 0

Σ_2 : x - 2*y + 3 = 0

Σ_3 : 2*x + y + 1 = 0

Σ_4 : x - y + 1 = 0

In $\mathbb{R}^3$

Σ_1 : P = [2;1;2] + ρ[0;0;1]

Σ_2 : x – y = 0

Σ_3 : P = [1;0;0] + ρ[1;-1;0]

Σ_4 : x - y + z + 1 = 0

Laboratory

Find the intersection between the two affine subspaces of $\mathbb{R}^3$:

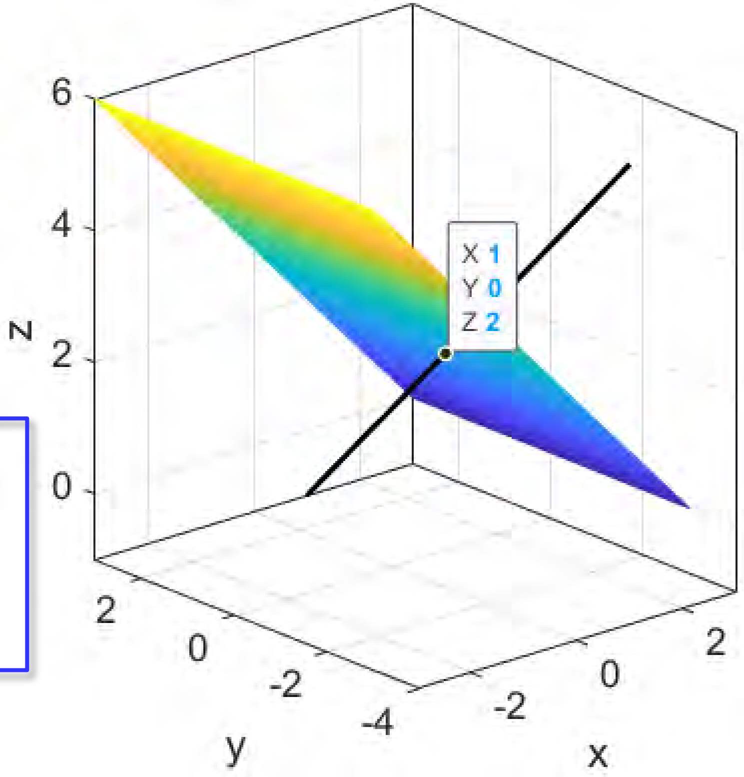
$$\pi \ : \ x+z=3, \qquad r \ : \ (x,y,z)^T = (1,1,1)^T + \lambda(0,1,-1)^T$$

We draw the subspaces:

numerically

```
[x,y]=meshgrid(linspace(-3,3,55));
z=3-x; h=mesh(x,y,z); % draw the plane
axis equal; box on; hold on
syms a real
r=[1,1,1]' + a*[0,1,-1]';
h=fplot3(r(1),r(2),r(3),[-5 2]);
set(h,'Color','k','LineWidth',2)
xlabel('x'); ylabel('y'); zlabel('z')
```

symbolically



Two solution algorithms:

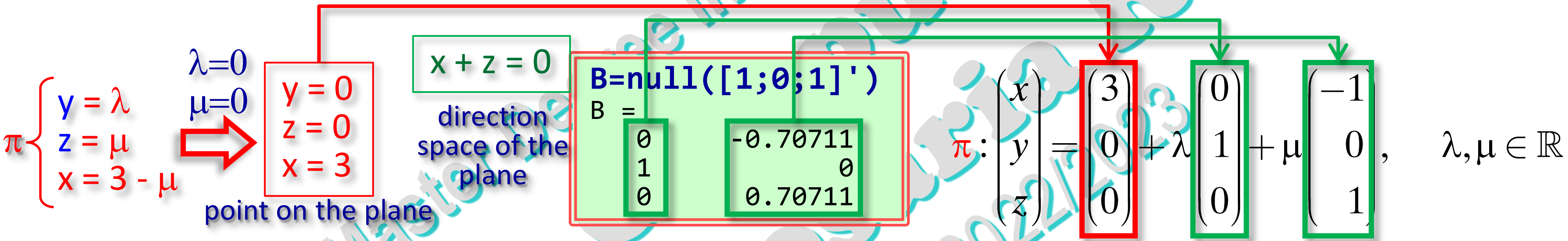
1. from cartesian eq. to parametric eq.
2. from parametric eq. to cartesian eq.

1. from cartesian to parametric eq.

The *r* line is already given by its parametric equation:

$$r:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\-1\end{pmatrix},\quad \alpha\in\mathbb{R}$$

The plane is given by its cartesian eq.: $\pi : x+z = 3$. Choosing y and z as free parameters, we are able to obtain its parametric eq.:



$$\pi\begin{cases}y=\lambda\\ z=\mu\\ x=3-\mu\end{cases}\ \xRightarrow[\mu=0]{\lambda=0}\ \boxed{\begin{matrix}y=0\\ z=0\\ x=3\end{matrix}}$$

point on the plane

$\boxed{x + z = 0}$

direction space of the plane

```
B=null([1;0;1]')
B =
  0   -0.70711
  1    0
  0    0.70711
```

$$\pi:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix},\qquad \lambda,\mu\in\mathbb{R}$$

To find $\pi\cap r$, we have to solve the linear system obtained by forcing the points, of coordinates (x, y, z), to lie both on the plane and on the line:

$$\begin{pmatrix}3\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}0\\1\\-1\end{pmatrix}\ \overset{\text{reordering}}{\Longleftrightarrow}\ \alpha\begin{pmatrix}0\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}$$

linear system

$$\begin{pmatrix}0&0&-1\\-1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}\alpha\\\lambda\\\mu\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

```
A=[0 0 -1;-1 1 0; 1 0 1];   b=[-2;1;1];
param=A\b;
sol1=[1,1,1]' + param(1)*[0,1,-1]'; % ∈r
sol1'
        1       0       2
```

intersection point coordinates

2. from parametric to cartesian eq.

The plane $\pi : x+z = 3$ is already expressed by its cartesian equation.

About the r line, let us write its scalar parametric equations and remove the parameter α from the linear system:

$$r:\begin{cases} x=1 \\ y=1+\alpha \\ z=1-\alpha \end{cases} \qquad r:\begin{cases} x=1 \\ \boxed{\alpha=y-1} \\ z=1-\alpha \end{cases} \qquad r:\begin{cases} x=1 \\ y+z=2 \end{cases}$$

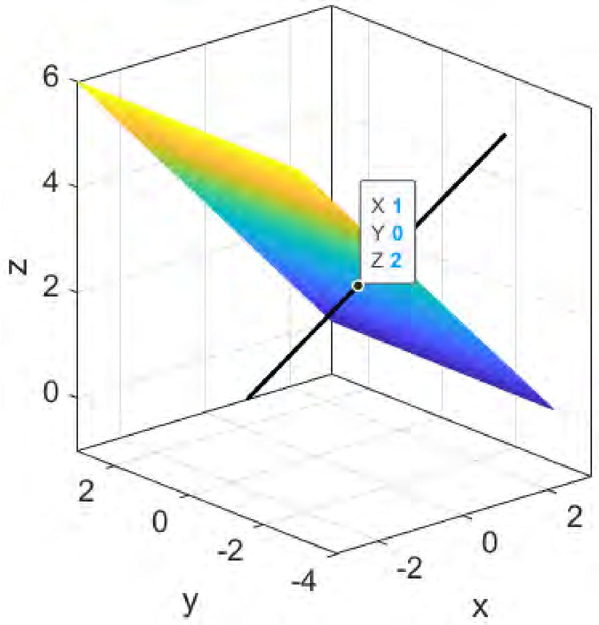
r as intersection of two planes

To find $\pi \cap r$, we solve the linear system:

$$\pi \cap r:\begin{cases} x=1 \\ y+z=2 \\ x+z=3 \end{cases} \quad \begin{matrix} \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} r \\ \pi \end{matrix}$$

```
A=[1 0 0; 0 1 1; 1 0 1];    b=[1;2;3];
sol2=A\b
           1      0      2
```

intersection point coordinates



Affinely independent points

In an Affine Space, $N+1$ points $P_0, P_1, \dots, P_N \in \Sigma$ are said to be affinely independent if, and only if, the N vectors

$$\vec{v}_1 = \overrightarrow{P_0P_1},\ \vec{v}_2 = \overrightarrow{P_0P_2},\ \dots,\ \vec{v}_N = \overrightarrow{P_0P_N} \in V$$

are linearly independent.

V: direction space of Σ

Example

```
P=[0 1 -1; 2 3 2]
P =
     0     1    -1
     2     3     2
v=P(:,2:end) - repmat(P(:,1),1,size(P,2)-1)
P =
     1    -1
     1     0
rank(v)
ans =
         2
```

N+1=3 points

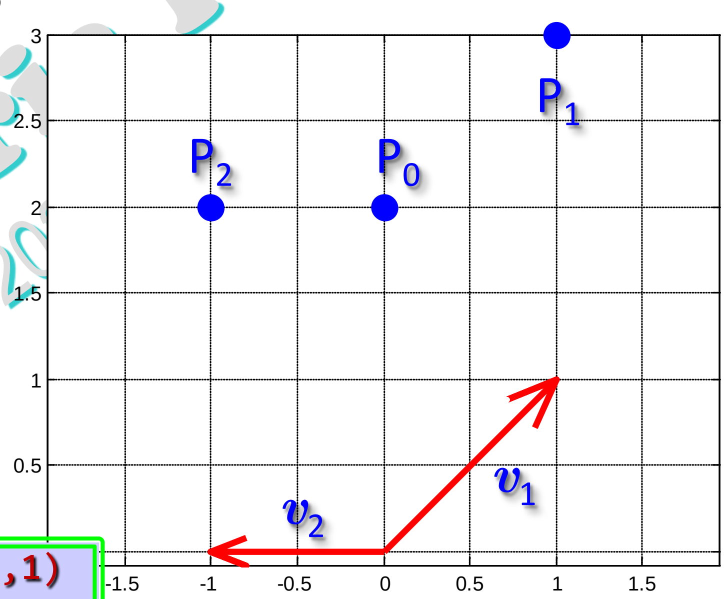
N=2 vectors

linearly independent

```
repmat(P(:,1),1,size(P,2)-1)
ans =
     0     0
     2     2
```

```
v=P(:,2:end) - P(:,1)
v =
     1    -1
     1     0
```

the repmat() function is no longer required



The points P_0, P_1, P_2 are affinely independent

Affinely independent points

If the $N+1$ points $P_0, P_1, \ldots, P_N \in \Sigma$ are affinely independent in the Affine Space Σ, then we can choose the following N vectors

$$\vec{v}_1 = \overrightarrow{P_0P_1},\ \vec{v}_2 = \overrightarrow{P_0P_2},\ \ldots,\ \vec{v}_N = \overrightarrow{P_0P_N} \in \boldsymbol{V}$$

$\boldsymbol{V}$: direction space of Σ

as a basis B for $\boldsymbol{V}$: in this way we introduce a new affine reference system $\mathscr{R}$(O, B), where P_0 is the origin of the new reference $\mathscr{R}$.

Of course we can choose any of the $N+1$ points as the origin (changing the basis accordingly).

Example

The standard affine reference system of $\Sigma=\mathbb{R}^n$ consists of points:

$$P_0=(\underbrace{0,0,\ldots,0}_{N}),\ P_1=(1,0,\ldots,0),\ P_2=(0,1,\ldots,0),\ \ldots,\ P_N=(0,0,\ldots,1).$$

any point coordinates univocally identified

⇒ $$Q=(\alpha_1, \alpha_2, \ldots, \alpha_N) = \cancel{P_0} + \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_N P_N$$

Affine reference system

Example 1

A new affine reference system of $\mathbb{R}^2$ could consist of the following points:

$$P_0=(0,1),\ P_1=(2,3),\ P_2=(2,1).$$

What are the affine coordinates of a point Q w.r.t. the new affine reference system, if we know that its coordinates w.r.t. the standard affine reference system were $(5,-1)$?

$$(5,-1)=Q=O+\varphi(O,Q)=P_0+\alpha_1(P_1-P_0)+\alpha_2(P_2-P_0)$$

$$(5,-1)=Q=(0,1)+\alpha_1[(2,3)-(0,1)]+\alpha_2[(2,1)-(0,1)]$$

$$(5,-1)^T-(0,1)^T=\alpha_1[(2,3)-(0,1)]^T+\alpha_2[(2,1)-(0,1)]^T$$

$$P_0=O \implies \left[(P_1-P_0)^T \quad (P_2-P_0)^T\right]\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=[Q-P_0]^T$$

This is a determined linear system because the points are affinely independent

Example 1 (contd.)

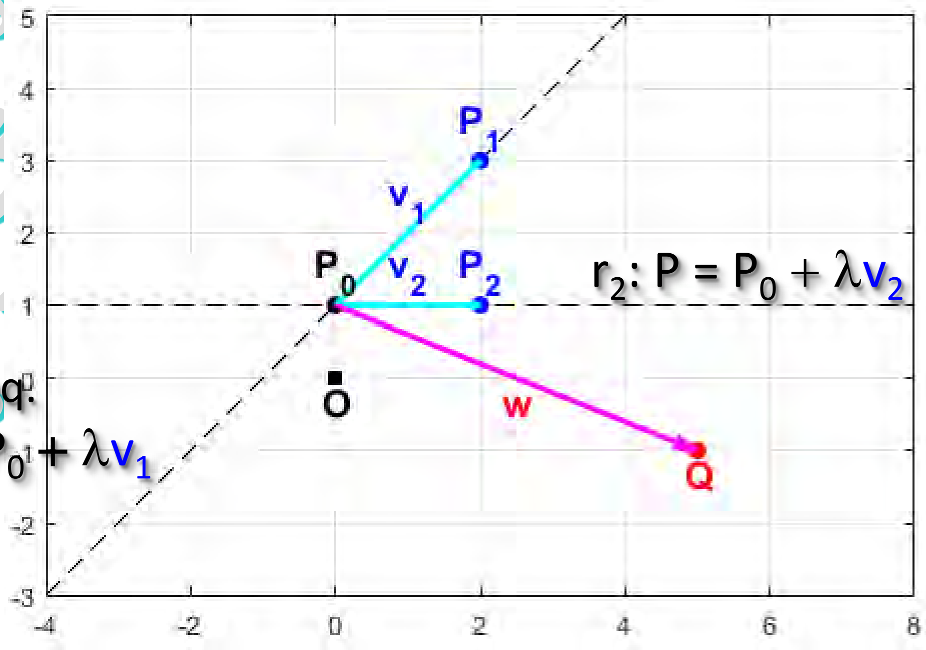
```
P0=[0 1]; P1=[2 3]; P2=[2 1];
v1=(P1-P0)'; v2=(P2-P0)'; A=[v1 v2];
rank(A)
ans =
     2
Q=[5 -1]; w=(Q-P0)'; alpha=A\w
alpha =
             -1
            3.5
```

basis of the direction space

3 affinely independent points

check

```
A*alpha
ans =
     5
    -2
P0' + A*alpha
ans =
     5
    -1
```



r_2: $P = P_0 + \lambda v_2$

parametric eq.

r_1: $P = P_0 + \lambda v_1$

Affine reference system

Example 2

A new affine reference system of $\mathbb{R}^3$ could consist of the following points:

$$P_0=(1,1,1),\ P_1=(2,1,1),\ P_2=(1,2,1),\ P_3=(2,1,2).$$

What are the affine coordinates of a point Q w.r.t. the new affine reference system, if we know that its coordinates w.r.t. the standard affine reference system were $(5,-1,0)$?

$$(5,-1,0)=Q=O+\varphi(O,Q)=P_0+\alpha_1(P_1-P_0)+\alpha_2(P_2-P_0)+\alpha_3(P_3-P_0)$$

$$(5,-1,0)=Q=(1,1,1)+\alpha_1[(2,1,1)-(1,1,1)]+\alpha_2[(1,2,1)-(1,1,1)]+\alpha_3[(2,1,2)-(1,1,1)]$$

$$(5,-1,0)^{\mathsf{T}}-(1,1,1)^{\mathsf{T}}=\alpha_1[(2,1,1)-(1,1,1)]^{\mathsf{T}}+\alpha_2[(1,2,1)-(1,1,1)]^{\mathsf{T}}+\alpha_3[(2,1,2)-(1,1,1)]^{\mathsf{T}}$$

$$P_0=O \Rightarrow \left[(P_1-P_0)^{\mathsf{T}}\quad (P_2-P_0)^{\mathsf{T}}\quad (P_3-P_0)^{\mathsf{T}}\right]\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=[Q-P_0]^{\mathsf{T}}$$

This is a determined linear system because the points are affinely independent

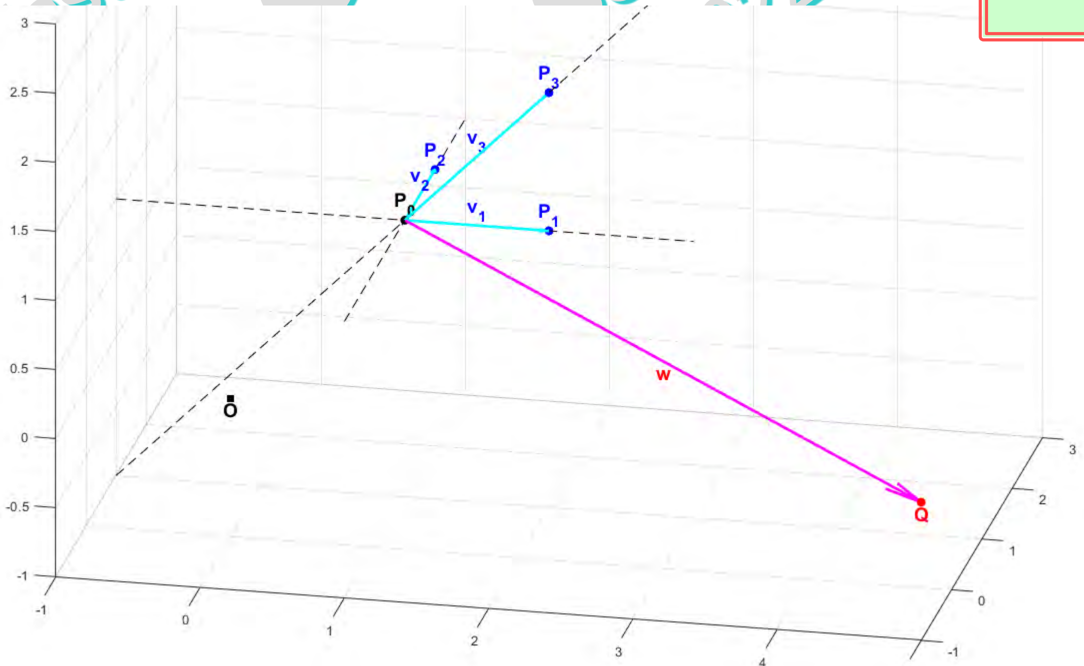
Example 2 (contd.)

```
P0=[1 1 1]; P1=[2 1 1]; P2=[1 2 1]; P3=[2 1 2];
v1=(P1-P0)'; v2=(P2-P0)'; v3=(P3-P0)';
A=[v1 v2 v3]; basis of the direction space
rank(A)
ans =
     3
Q=[5 -1 0]; w=(Q-P0)'; alpha=A\w
alpha =
     5
    -2
    -1
```

4 affinely independent points

check

```
A*alpha
ans =
     4
    -2
    -1
P0' + A*alpha
ans =
     5
    -1
     0
```



Affine reference system

Example 3

A new affine reference system of a plane π, considered as an affine subspace of $\mathbb{R}^3$, could consist of the following points:

$$P_0=(1,1,1),\ P_1=(2,1,1),\ P_2=(2,1,2).$$

What are the affine coordinates of a point Q (on the plane) w.r.t. the new affine reference system, if we know that its coordinates w.r.t. the standard affine reference system were $(4,1,3)$?

$$(4,1,3) = Q = O+\varphi(O,Q) = P_0 + \alpha_1(P_1 - P_0) + \alpha_2(P_2 - P_0)$$

$$(4,1,3) = Q = (1,1,1) + \alpha_1[(2,1,1)-(1,1,1)] + \alpha_2[(1,2,1)-(1,1,1)]$$

$$(4,1,3)^T - (1,1,1)^T = \alpha_1[(2,1,1)-(1,1,1)]^T + \alpha_2[(1,2,1)-(1,1,1)]^T$$

$$P_0 = O \Longrightarrow \left[(P_1-P_0)^T \quad (P_2-P_0)^T\right]\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = [Q-P_0]^T$$

This is a determined linear system because the points are affinely independent

Example 3 (contd.)

```
P0=[1 1 1]; P1=[2 1 1]; P2=[2 1 2];
v1=(P1-P0)'; v2=(P2-P0)'; A=[v1 v2];
rank(A)
ans =
     2
Q=[4 1 3]; w=(Q-P0)'; alpha=A\w
alpha =
     1
     2
```

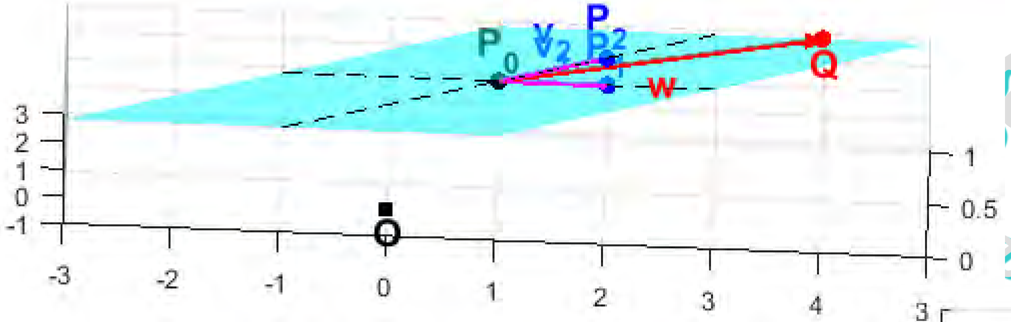
basis of the direction space

3 affinely independent points

affine coordinates of Q on the plane

check

```
A*alpha
ans =
     3
     0
     2
P0' + A*alpha
ans =
     4
     1
     3
```



in front of the plane