Exercises

SC2_04 – Particular Linear Subspaces and their properties.

1. Given in $\mathbf{R}^3$ the subspaces $V_1=\text{span}\{(1,2,0)^\top\}$ and $V_2=\text{span}\{(2,3,0)^\top\}$, find a basis and the dimension of the following subspaces, and check the *Grassmann Formula*:
   1.1 The sum subspace $\boldsymbol{W} = \boldsymbol{V}_1 + \boldsymbol{V}_2$.
   1.2 Some complementary subspaces of $\boldsymbol{W}$ in $\mathbf{R}^3$.
   1.3 The orthogonal complement of $\boldsymbol{W}$ in $\mathbf{R}^3$.

2. What is a formula that implies the inequality: $\text{rank}(\mathbf{A}+\mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$? Under which hypothesis does inequality become equality?

3. Find two different complementary subspaces and the orthogonal complement of the following linear subspaces W^* of $\mathbf{R}^n$ (if you are using the *Symbolic Math Toolbox* to solve the problem, you have to give a value to n: for example, n=5, or n=3 if you want to draw the subspaces):
   3.1 $W=\{\boldsymbol{x}\in\mathbf{R}^n : \boldsymbol{x}=(x_1,x_2,\ldots,x_j,\ldots,x_n)^\top \wedge x_j=0 \text{ for a certain } j\}$
   3.2 $W=\{\boldsymbol{x}\in\mathbf{R}^n : \boldsymbol{x}=(x_1,x_2,\ldots,x_n)^\top \wedge x_1=x_2=\ldots=x_n\}$
   3.3 $W=\{\boldsymbol{x}\in\mathbf{R}^n : \boldsymbol{x}=(x_1,x_2,\ldots,x_n)^\top \wedge \sum_{k=1,\ldots,n} x_k = 0\}$

   * already seen in `Exercises_02.pdf`.

4. Find and display the intersection between the following subspaces of $\mathbf{R}^3$:
   4.1 The planes xy and yz.
   4.2 The line $r=\text{span}\{(1,1,1)^\top\}$ and the plane $\pi=\text{span}\{(1,0,0)^\top, (0,1,1)^\top\}$.
   4.3 The plane orthogonal to $r_1=\text{span}\{(1,1,0)^\top\}$ and the plane orthogonal to $r_2=\text{span}\{(0,1,1)^\top\}$.

5. Find the intersection $W=U\cap V$, where U and V are the following subspaces:

   $U=\{\boldsymbol{x}\in\mathbf{R}^n : \boldsymbol{x}=(x_1,x_2,\ldots,x_j,\ldots,x_n)^\top \wedge x_j=0 \text{ for a certain } j\}$
   $V=\{\boldsymbol{x}\in\mathbf{R}^n : \boldsymbol{x}=(x_1,x_2,\ldots,x_n)^\top \wedge \sum_{k=1,\ldots,n} x_k = 0\}$

6. Write a function implementing the *Gram-Schmidt Orthonormalization Algorithm* (GSO) and apply it to vectors $(1,2,0)^\top$, $(8,1,-6)^\top$, $(0,0,1)^\top$, then compare its output to the result returned by the `orth()` function. If $V=\text{span}\{(1,2,0)^\top, (8,1,-6)^\top, (0,0,1)^\top\}$, is $\mathcal{R}(\mathbf{O})=V$? What happens if the vectors are $(1,2,0)^\top$, $(8,1,-6)^\top$, $(2,4,0)^\top$? If $V=\text{span}\{(1,2,0)^\top, (8,1,-6)^\top, (2,4,0)^\top\}$, is $\mathcal{R}(\mathbf{O})=V$ still true?

7. Find an orthonormal basis in the subspace containing the real algebraic polynomials of degree at most 2:

$$W = \text{span}\{p_1, p_2, p_3\}$$

   where

$$p_1(x) = 2 + x + 4x^2$$
$$p_2(x) = 1 - x + 3x^2$$
$$p_3(x) = 3 + 2x + 5x^2$$

and the scalar product is defined by: $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

8. Write a MATLAB function that accepts a sequence of vectors on input, then by means of GSO Algorithm checks their linear independence and, in this case, computes the two systems of orthogonal and orthonormal vectors, $\{\boldsymbol{v}_k\}$ and $\{\boldsymbol{u}_k\}$ respectively. What would change in the code if the input vectors were complex instead of real?

9. Write both the symbolic version of the code, suitable for function vectors too, and the numerical version of the code, only applicable to vectors of real components.