Course of
"Automatic Control Systems"
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Real Bode diagrams – Fourier analysis

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Real Bode diagrams

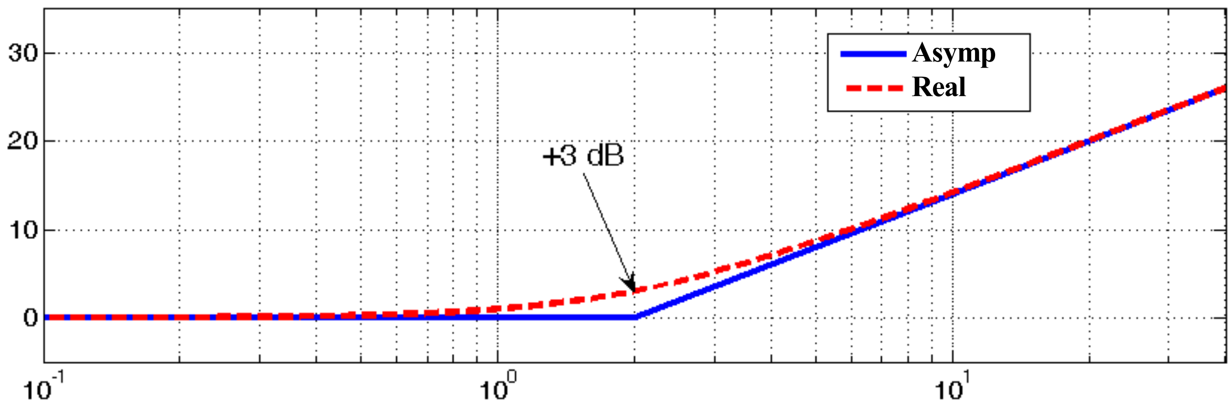
- ***In the real Bode diagrams the magnitude and phase of a transfer function $W(s)$ with $s = j\omega$ are drawn accurately*** also the in two decades around the break points of the binomial and trinomial terms.

- The real Bode diagrams are usually traced applying some ***corrections to the asymptotic Bode diagrams***

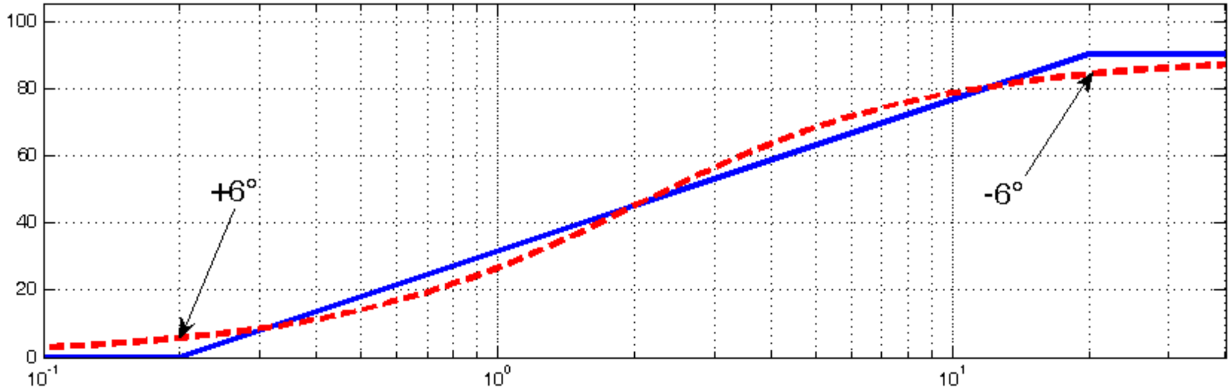
- The real Bode diagrams ***can be drawn in MATLAB using the command 'bode'***

Zero of multiplicity one $W(s) = (1 + s\tau)$

Magnitude [dB]



Phase [deg]



This result can be easily generalized to a generic binomial term

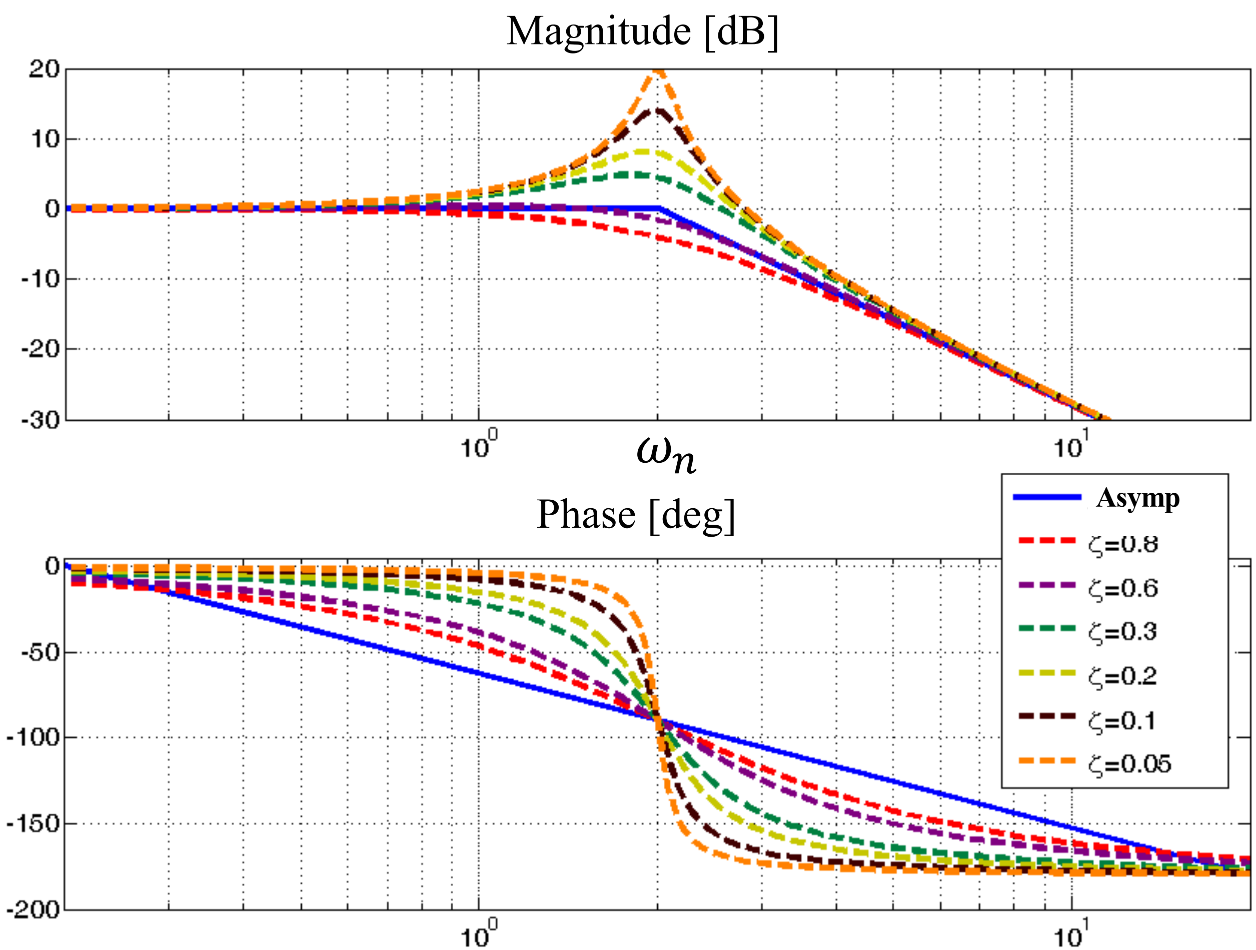
Complex conjugate poles of multiplicity one $W(s) = \left(1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\right)^{-1}$

Peak module

$$M_p = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

Peak frequency

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$



This result can be easily generalized to a generic trinomial term

Bode magnitude table

- Monomial terms of multiplicity 1. The slope is constant in $\omega \in [0 \ \ \infty[$

Zero in the origin	**+20 dB/decade**
Pole in the origin	**-20 dB/decade**

- Binomial and trinomial terms of multiplicity 1. The slope changes on the break point

	Indipendent from the sign of the real part
Real Zero	**+20 dB/decade**
Real Pole	**-20 dB/decade**
Comp. Conjug. zeros	**+40 dB/decade**
Comp. Conjug. poles	**-40 dB/decade**

- When the term has a multiplicity greater than one, the slopes should be multiplied by the multiplicity.

Bode phase table

- Constant and monomial terms of multiplicity 1. The slope is constant in $\omega \in [0 \ \infty[$

K < 0	**-180° per ω∈[0,∞)**
Zero in the origin	**+90° per ω∈[0,∞)**
Pole in the origin	**-90° per ω∈[0,∞)**

- Binomial and trinomial terms of multiplicity 1. The slope changes one decade before and after the breaking point.

	Negative real part	Positive real part
Real Zero	**+90° +45→ -45 °/decade**	**-90° -45→ +45 °/decade**
Real Pole	**-90° -45→ +45 °/decade**	**+90° +45→ -45 °/decade**
Comp. Conjug. zeros	**+180° +90→ -90 °/decade**	**-180° -90→ +90 °/decade**
Comp. Conjug. poles	**-180° -90→ +90 °/decade**	**+180° +90→ -90 °/decade**

- When the term has a multiplicity greater than one, the phase variation should be multiplied by the multiplicity.

Examples

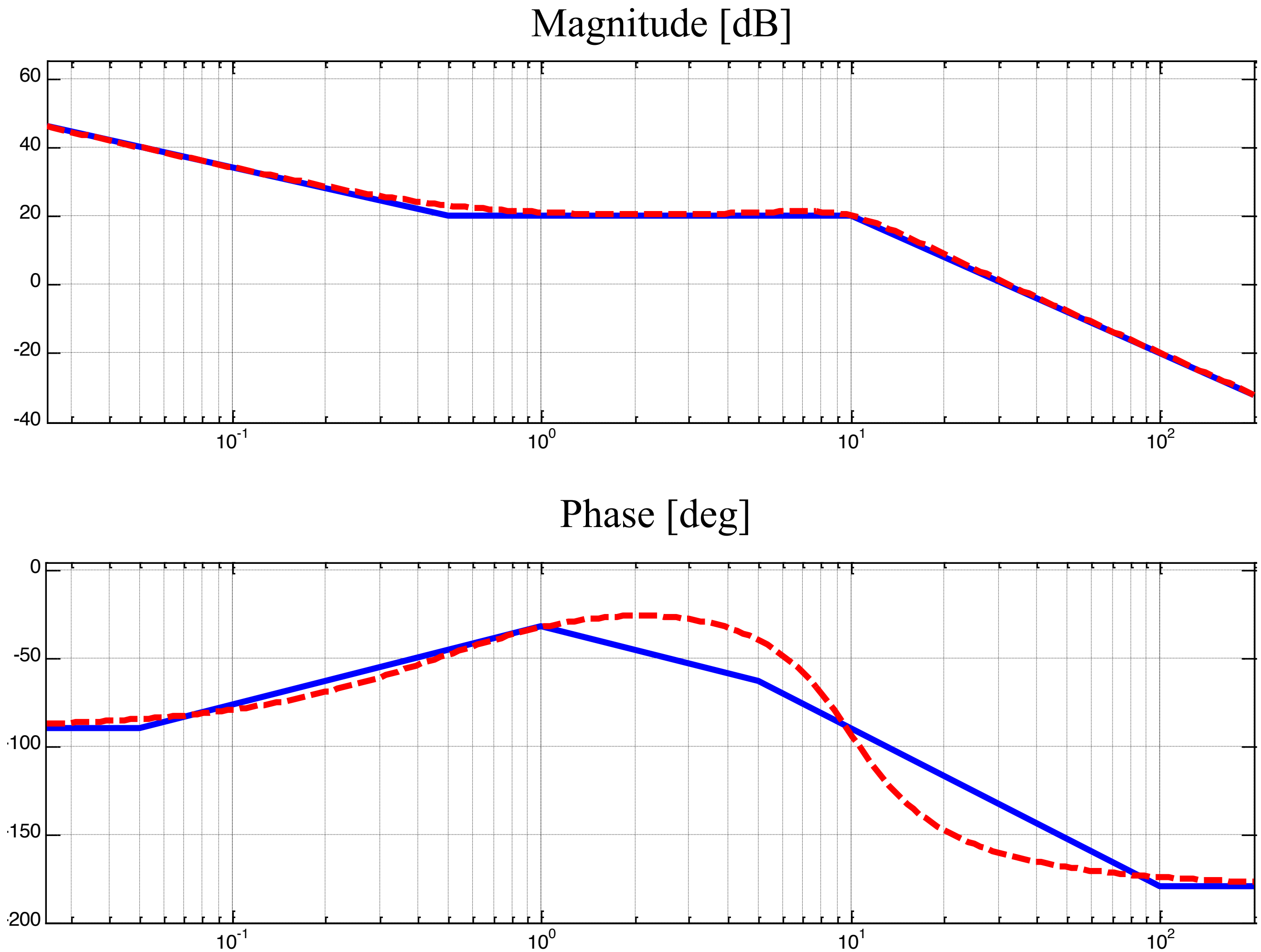
- Trace the real Bode diagrams of the functions

$$W(s) = \frac{1000(s+0.5)}{s(s^2+10s+100)}$$

$$W(s) = \frac{s(s-2)}{(s^2+5s+25)}$$

Example 1

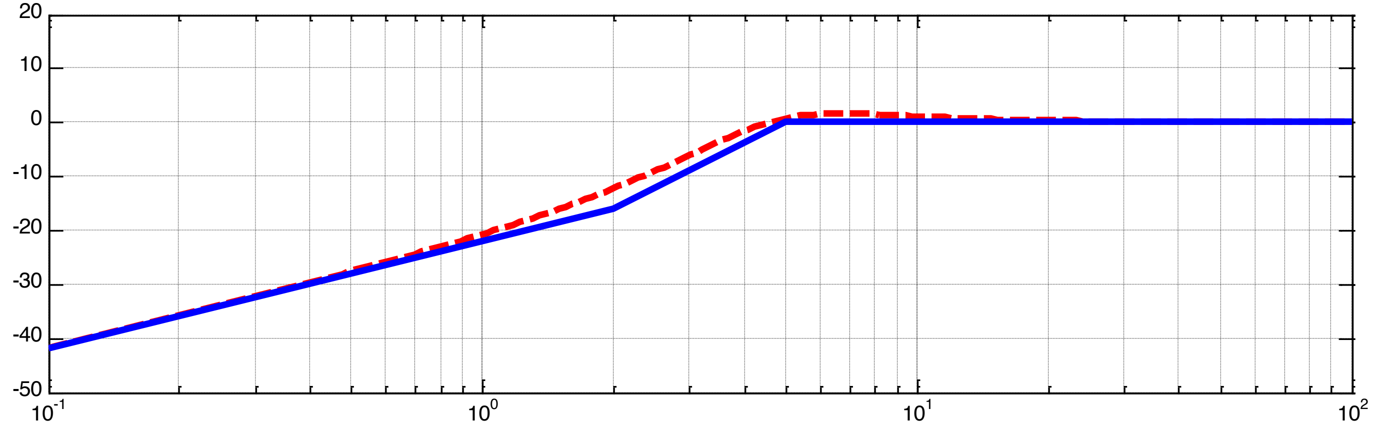
$$W(s) = \frac{1000(s+0.5)}{s(s^2+10s+100)}$$



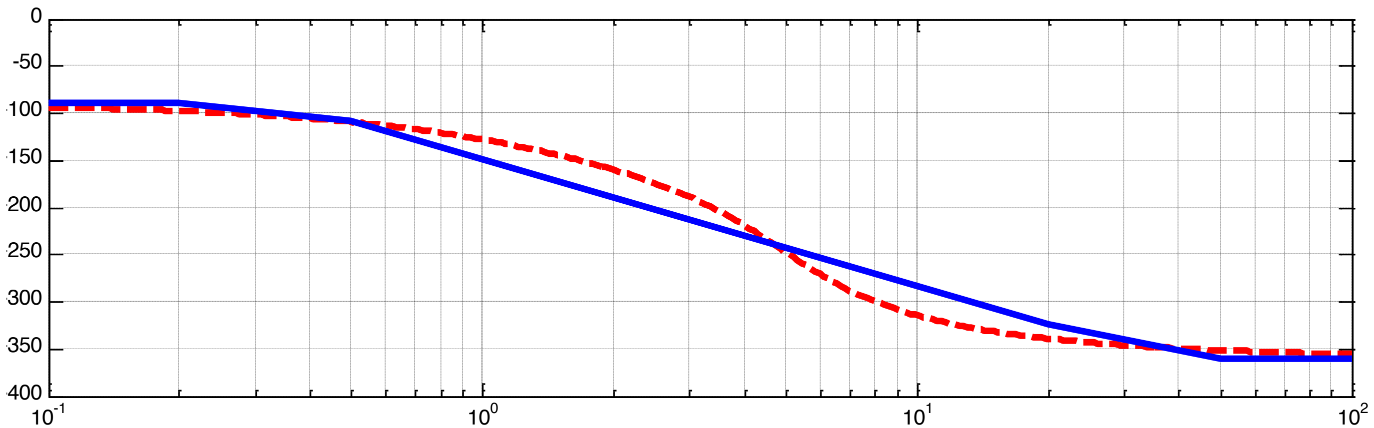
Exmple 2

$$W(s) = \frac{s(s-2)}{(s^2 + 5s + 25)}$$

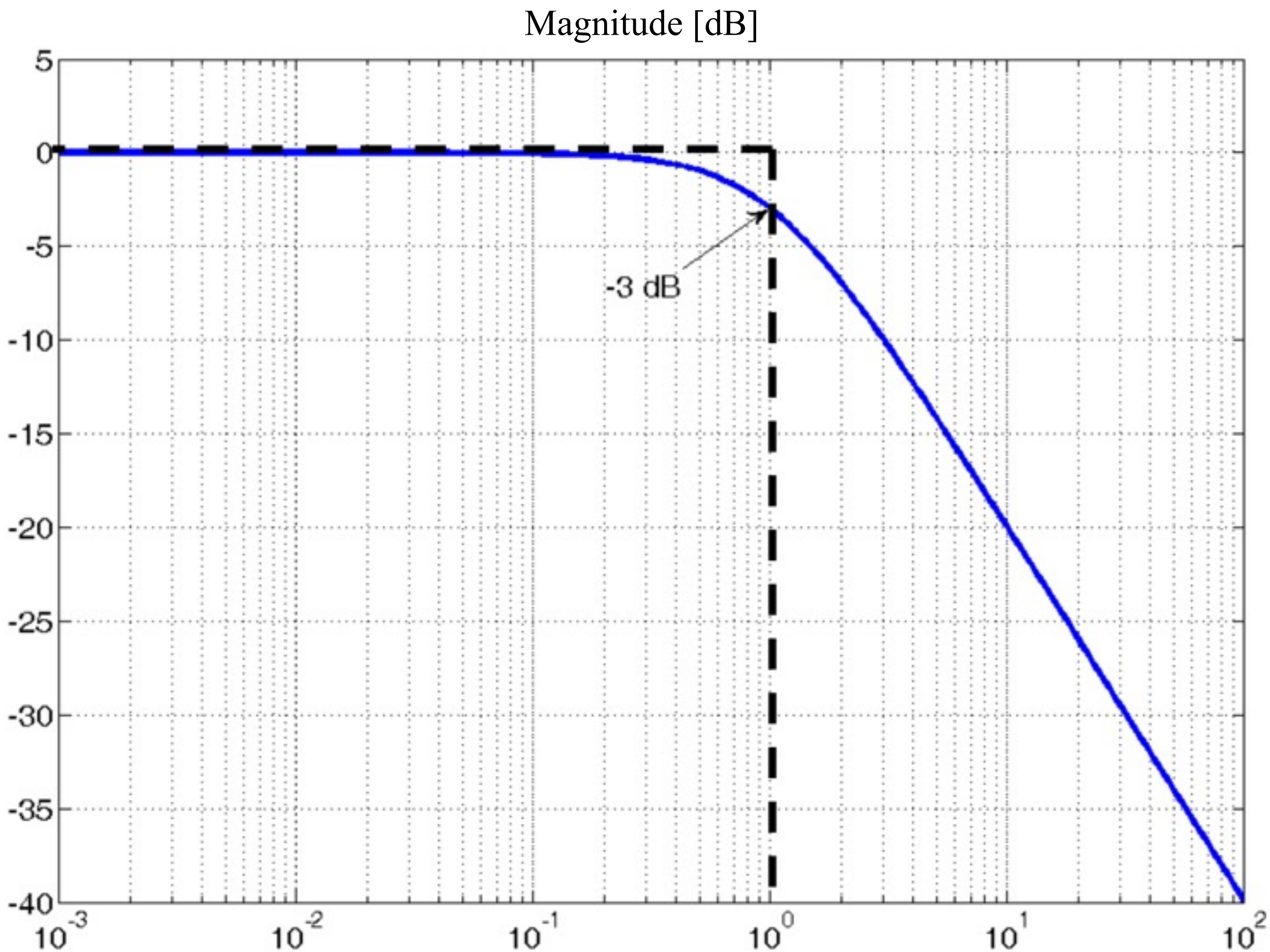
Magnitude [dB]



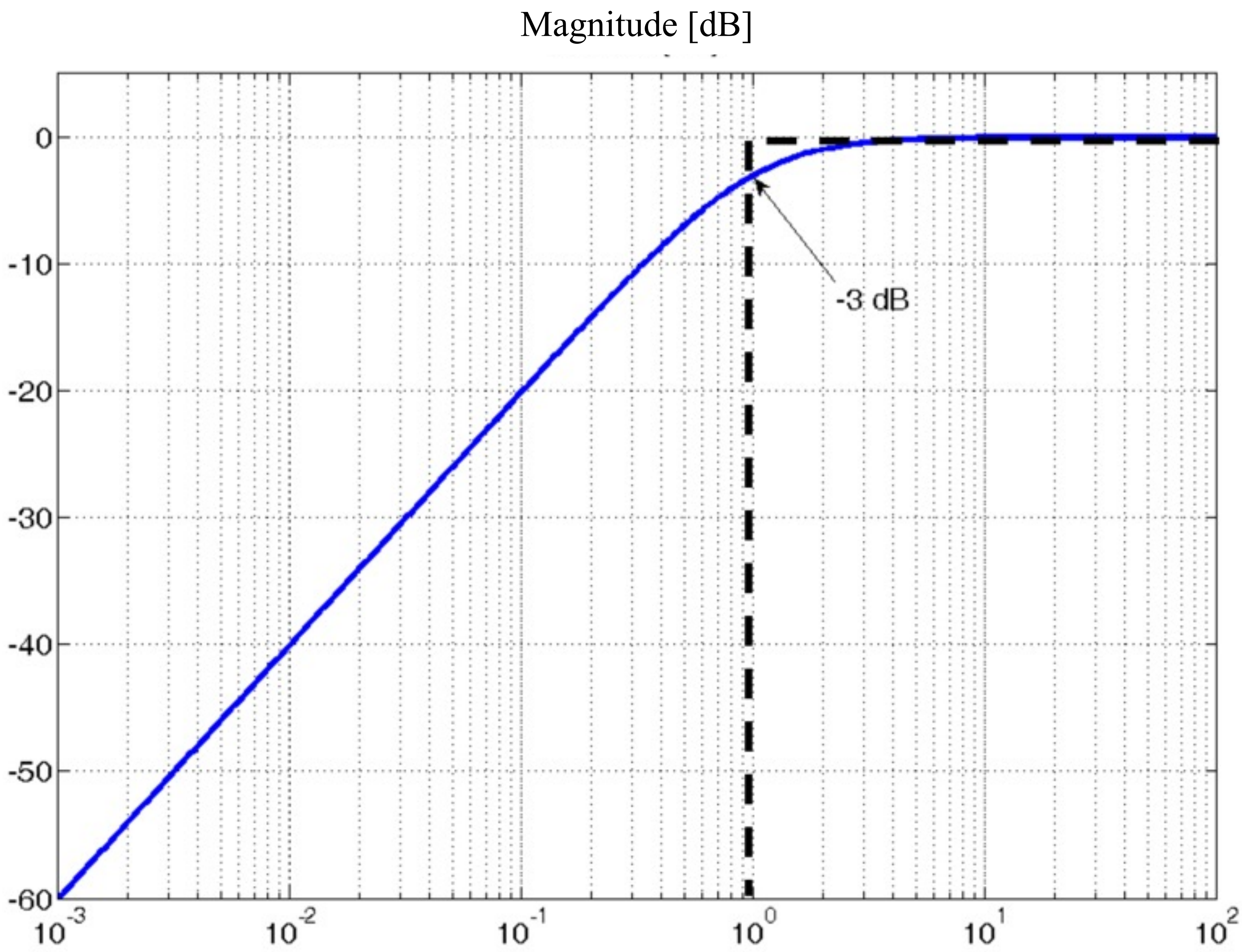
Phase [deg]



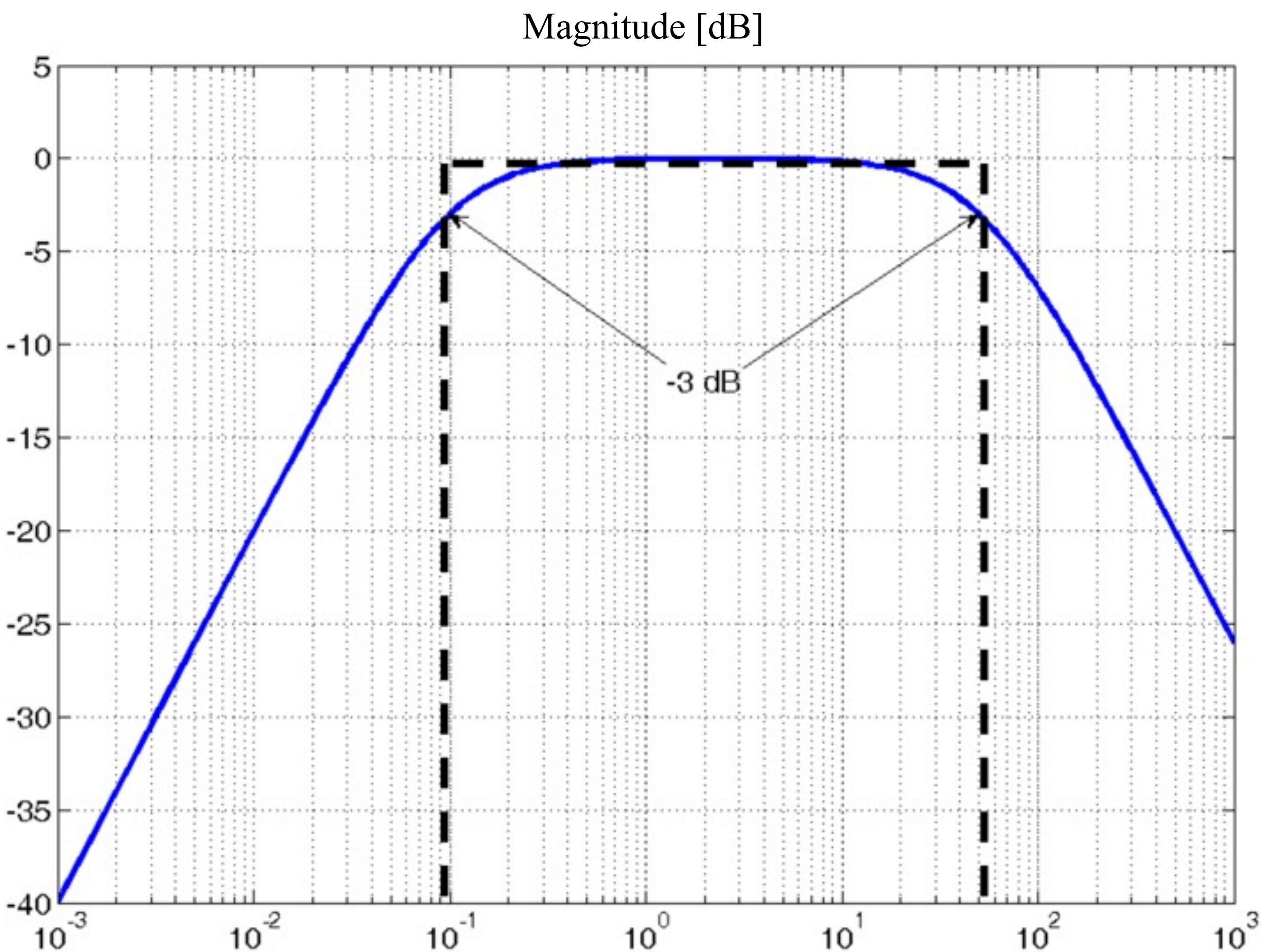
Low-pass filter



High-pass filter



Band-pass filter



Fourier analysis

- **Any periodic function *f*(*t*) with period *T*,**

$$f(t) = f(t+T),$$

can be written as

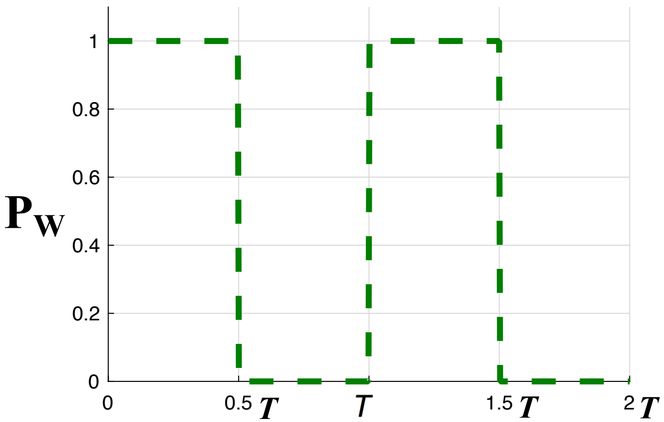
$$f(t) = F_0 + \sum_{n=1}^{\infty}\left[F_{cn}\cos(n\omega_0 t) + F_{sn}\sin(n\omega_0 t)\right]$$

where $\omega_0 = \frac{2\pi}{T}$ **,**

$$F_0 = \frac{1}{T}\int_T f(t)dt \qquad F_{cn} = \frac{2}{T}\int_T f(t)\cos(n\omega_0 t)dt \qquad F_{sn} = \frac{2}{T}\int_T f(t)\sin(n\omega_0 t)dt.$$

F_0 is the average value of f over a single period.
The component with ω_0 is the fundemental armonic or 1st harmonic, that with $n\omega_0$ is n-th harmonic.

Example: square wave



$$P_{\rm w}(t) = \begin{cases} 1 & \text{if} \quad 0 < t \le T/2 \\ 0 & \text{if} \quad T/2 < t \le T \end{cases}$$

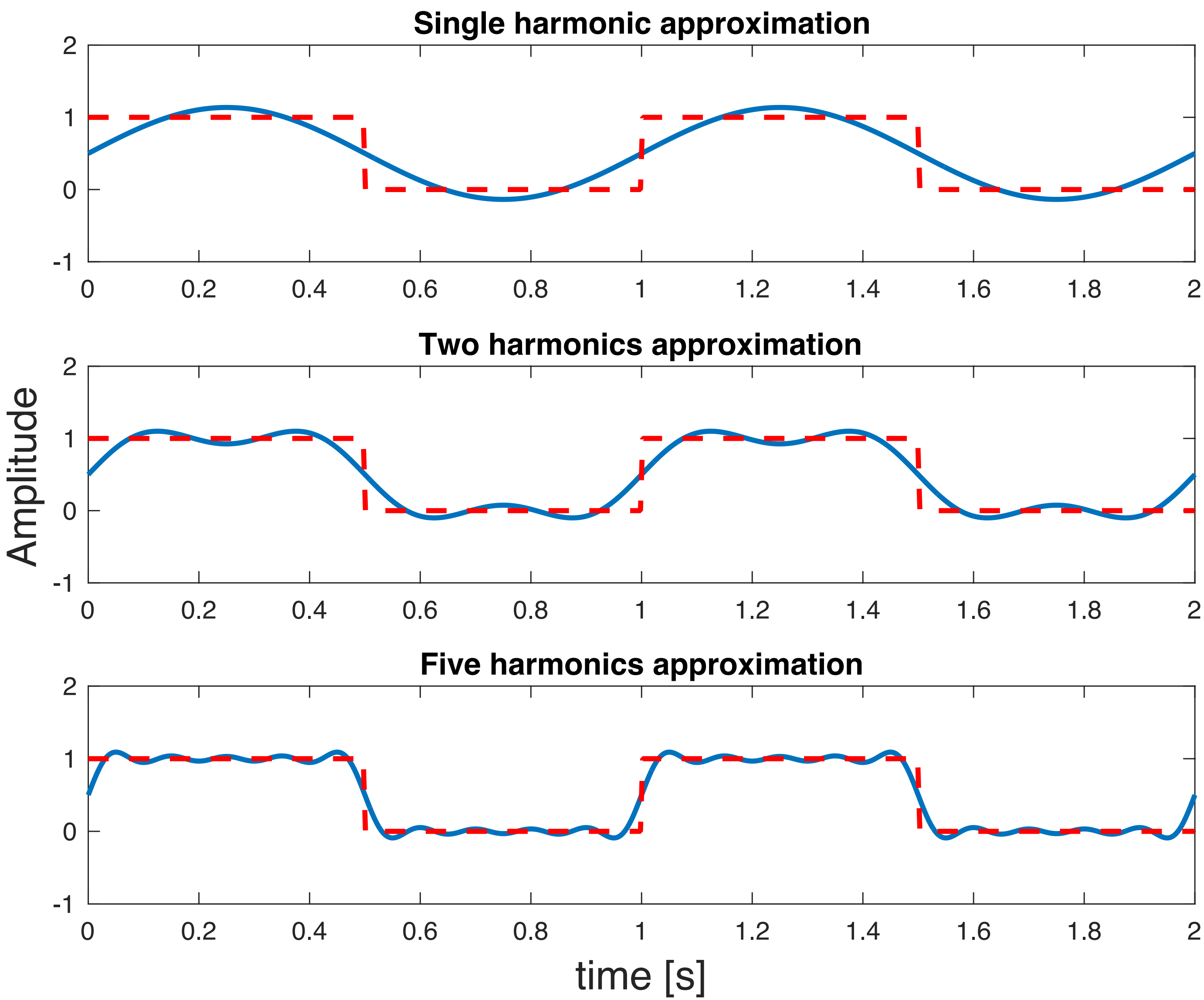
Using Fourier analysis:

$$F_0 = \frac{1}{2}\,, \quad F_{\rm cn} = 0 \quad \forall n \in \mathbb{N}\,, \quad F_{\rm sn} = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Therefore, the square wave can be written

$$P_{\rm w}(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t) + \cdots$$

Example: approximation of a square wave



$\omega_0 = 2\pi$ rad/s

$\omega_0, 3\omega_0$

$\omega_0, 3\omega_0,$
$5\omega_0, 7\omega_0,$
$9\omega_0,$

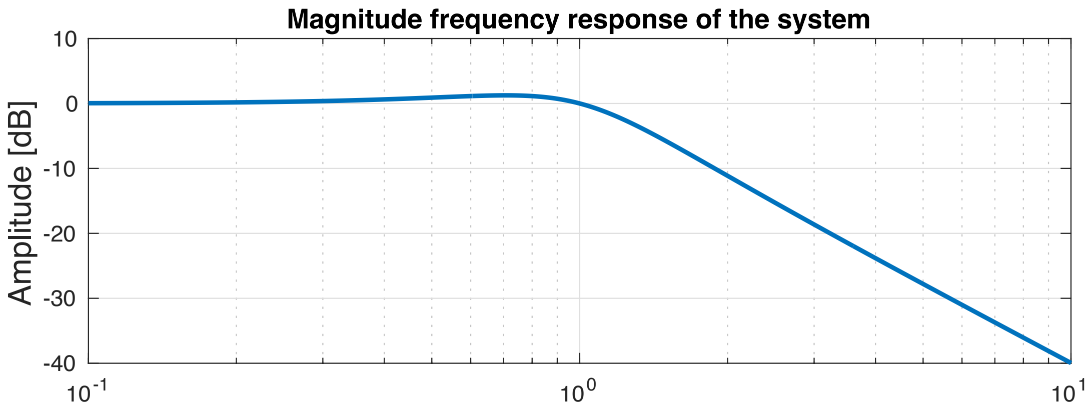
Example: steady state response to a square wave

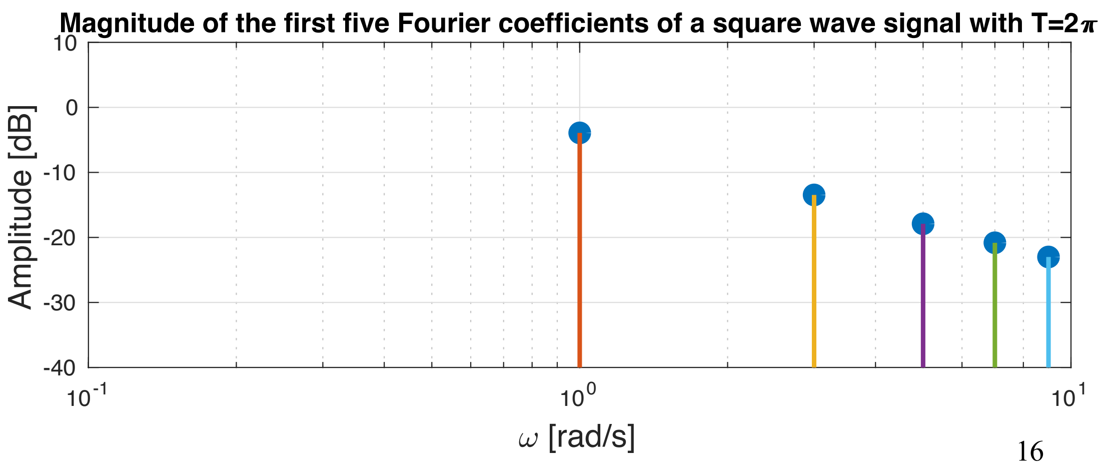
Let us consider the system with transfer function:

$$G(s) = \frac{1}{s^2 + s + 1}$$

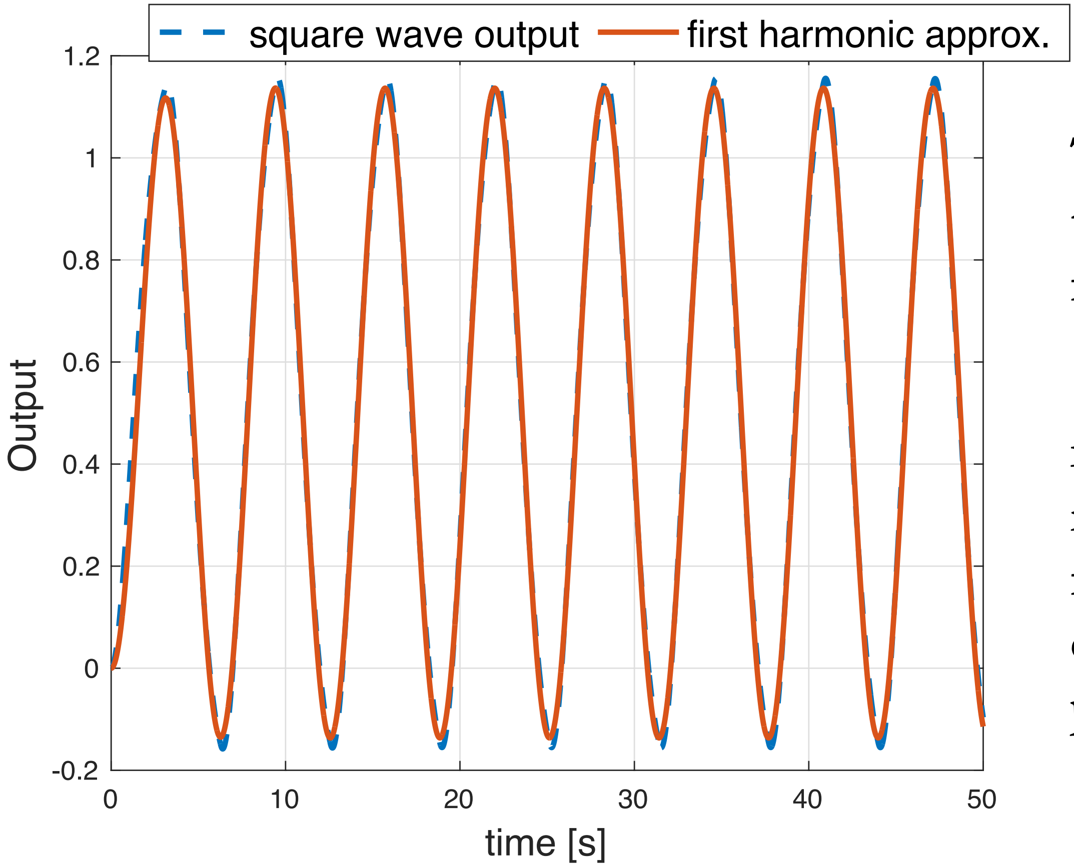
and assume we want to compute the steady-state response to the square wave with period T=2π.

- $u(t) = \frac{1}{2} + \frac{2}{\pi}\sin t + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \cdots$





Example: steady state response to a square wave



The stead state response of the system with transfer function

$$G(s) = \frac{1}{s^2 + s + 1}$$

is practically identical to the response assuming just the first two terms of the Fourier expansion (the average value plus the first harmonic)