SIS Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST UNIVERSITÀ PARTHENOPE DIPARTIMENTO DI SCIENZE E TECNOLOGIE

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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➢ **Fundamental Theorem of Linear Algebra***

* https://www.engineering.iastate.edu/~julied/classes/CE570/Notes/strangpaper.pdf

Gilbert Strang: professor at MIT (Massachusetts Institute of Technology).
https://math.mit.edu/~gs/

His lectures on Linear Algebra are freely available at **MITOPENCOURSEWARE**
https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/

Gilbert Strang: MIT course 18.065
"Matrix Methods in Data Analysis, Signal Processing, and Machine Learning" (2018)
[https://ocw.mit.edu/courses/18-065-matrix-methods-in-data-analysis-signal-processing-and-machine-learning-spring-2018/
https://www.youtube.com/playlist?list=PLUl4u3cNGP63oMNUHXqIUcrkS2PivhN3k]

Fundamental Theorem of Linear Algebra

If A is a matrix of size $(m \times n)$ then

$$\mathcal{N}(A) = \mathcal{R}(A^{\mathsf{T}})^{\perp} \quad \text{and} \quad \mathcal{R}(A^{\mathsf{T}}) = \mathcal{N}(A)^{\perp}$$

$$\mathcal{N}(A^{\mathsf{T}}) = \mathcal{R}(A)^{\perp} \quad \text{and} \quad \mathcal{R}(A) = \mathcal{N}(A^{\mathsf{T}})^{\perp}$$

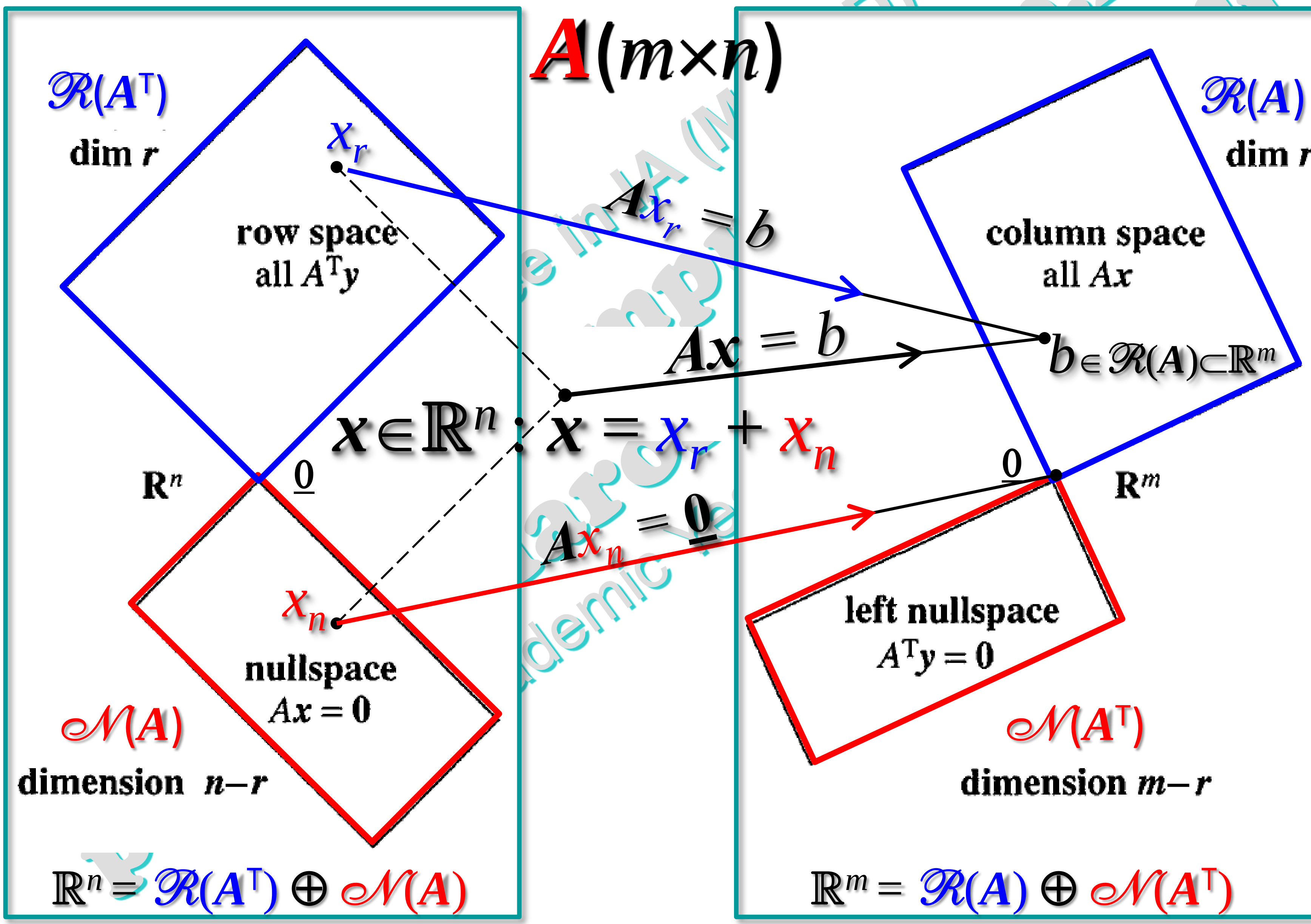
Its **proof** is based on the definition of $\mathcal{N}(A)$

$$\mathcal{N}(A) = \{ v \in \mathbb{R}^n : A\,v = \underline{0} \}$$

It contains all the vectors (v) which are orthogonal to the rows in A, so that it is orthogonal to $\mathcal{R}(A^{\mathsf{T}})$.

For the other two subspaces we consider A^{T} in place of A.

Graphical representation of Fundamental Theorem of Linear Algebra



Lab.: compute the 4 fund. subspaces of A where

$$A=\begin{pmatrix}1 & 4\\ 2 & 7\\ 3 & 5\end{pmatrix} \qquad A(3\times 2)$$

$\mathcal{R}(A), \mathcal{N}(A^\top) \subseteq \mathbb{R}^3$: $\mathcal{R}(A) = \mathcal{N}(A^\top)^\perp$

$\mathcal{R}(A^\top), \mathcal{N}(A) \subseteq \mathbb{R}^2$: $\mathcal{R}(A^\top) = \mathcal{N}(A)^\perp$

$\mathcal{N}(A^\top) = \mathcal{R}(A)^\perp \Leftrightarrow$

Grassmann Formula

$\dim[\mathcal{N}(A^\top)] = 3 - \dim[\mathcal{R}(A)] = 1$

MATLAB Symbolic Math Toolbox

```
A=sym([1 2 3; 4 7 5]');
disp(rank(A))
     2
RA=colspace(A)
RA =
[   1, 0]
[   0, 1]
[ -11, 7]
disp(rank(double([RA A])))
     2
NAT=null(A')
NAT =
 11
 -7
  1
disp(RA'*NAT)
 0
 0
```

→ the columns are independent

$\mathcal{R}(A) = \text{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\7\\5\end{pmatrix}\right\}$

→ two basis for the same subspace

$\mathcal{N}(A^\top) = \text{span}\left\{\begin{pmatrix}11\\-7\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\right\}$

$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \perp \mathcal{R}(A) \left\{\begin{array}{l}\left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\right\rangle = 0 \\ \left\langle \begin{pmatrix}4\\7\\5\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\right\rangle = 0\end{array}\right.$

$\mathcal{R}(A), \mathcal{N}(A^\top)$ are orthogonal

MATLAB

```
A=[1 2 3; 4 7 5]';
disp(rank(A))
     2
RAT=orth(A')
RAT =
     -0.34998   0.93676
     -0.93676  -0.34998
NA=null(A)
NA =
  2×0 empty double matrix
```

$\mathcal{R}(A^\top) = \mathbb{R}^2$

$\mathcal{N}(A) = \{\underline{0}\}$

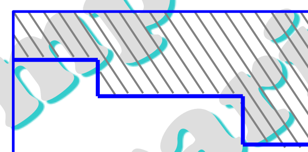
$\mathcal{N}(A)$ contains only the null vector

Fundamental Theorem and its applications

Gauss elimination

$$A=\begin{pmatrix}1 & 0 & 2 & 0\\ 0 & 2 & 2 & 0\\ 1 & -1 & 1 & 1\end{pmatrix}\xrightarrow{G\downarrow}\begin{pmatrix}1 & 0 & 2 & 0\\ 0 & 2 & 2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

3 pivots



"staircase" matrix S

$\dim[\mathscr{R}(A)] = 3 = \dim[\mathbb{R}^3]$

$$\mathscr{R}(A)=\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

we can compute $\mathscr{N}(A^T)$

without A^T, as $\mathscr{N}(A^T)=\mathscr{R}(A)^{\perp}=\{\underline{0}\}$ ... why?

Fundamental Theorem and its applications

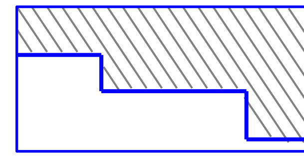
$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} \xrightarrow{G^{\downarrow}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

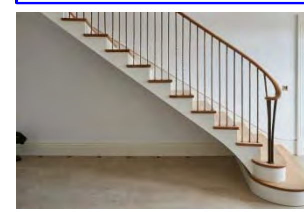
MATLAB Symbolic Math Toolbox

Gauss-Jordan elimination
(row-reduced echelon form → SC2_02c)

```
A=sym([1 0 2 0;0 2 2 0;1 -1 1 1]);
S=rref(A)
S =
[    1      0      2      0    ]
[    0      1      1      0    ]
[    0      0      0      1    ]
```

"staircase" matrix S





```
RAT=colspace(A')
RAT =
[ 1, 0, 0]
[ 0, 1, 0]
[ 2, 1, 0]
[ 0, 0, 1]
RST=colspace(S')
RST =
[ 1, 0, 0]
[ 0, 1, 0]
[ 2, 1, 0]
[ 0, 0, 1]
```

$$\mathscr{R}(A^\top) = \mathscr{R}(S^\top)$$

The row subspaces are the same

Fundamental Theorem and its applications

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

MATLAB Symbolic Math Toolbox

```
A=sym([1 0 2 0; 0 2 2 0; 1 -1 1 1]);
S=rref(A)
S =
[    1      0      0    ]
[    0      1      0    ]
[    0      0      1    ]
[    0      0      0    ]
```

"staircase" matrix S

row-reduced echelon form

```
RA=colspace(A)
RA =
[ 1, 0, 0]
[ 0, 1, 0]
[ 2, 1, 0]
[ 0, 0, 1]
RS=colspace(S)
RS =
[ 1, 0, 0]
[ 0, 1, 0]
[ 0, 0, 1]
[ 0, 0, 0]
```

```
rank(RA)
ans =
     3
rank([RA RS])
ans =
     4
```

$$\mathscr{R}(A) \neq \mathscr{R}(S)$$

The column subspaces differ

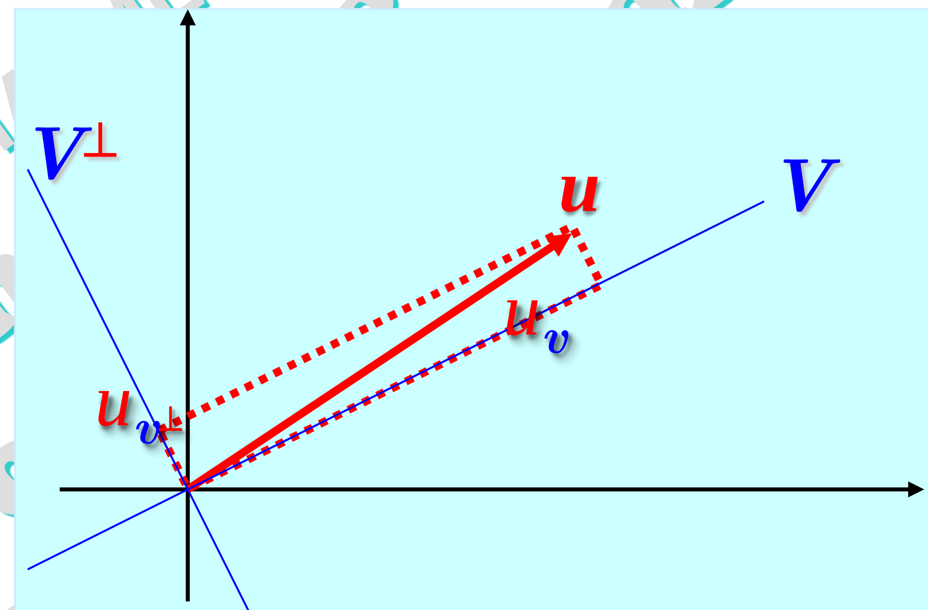
Example

Compute the components of $\boldsymbol{u}=(3,2)^T$, along the directions of $\boldsymbol{V}$ and $\boldsymbol{w}=\boldsymbol{V}^{\perp}$, where $\boldsymbol{V}=\text{span}\{\boldsymbol{v}\}$, $\boldsymbol{v}=(2,\ 1)^T$.

$$\boldsymbol{V}=\mathcal{R}(\boldsymbol{v})$$

$$\boldsymbol{w}=\boldsymbol{V}^{\perp}=\mathcal{R}(\boldsymbol{v})^{\perp}=\mathcal{N}(\boldsymbol{V}^T)$$

$$\boldsymbol{V}^{\perp}=\text{span}\{(-1,\ 2)^T\}$$



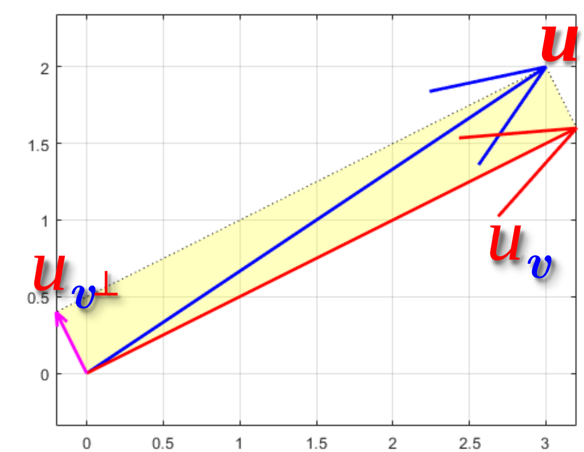
u_V ↓ $\quad$ $u_{V^{\perp}}$ ↓

$$u=\begin{pmatrix}3\\2\end{pmatrix}=\alpha\begin{pmatrix}2\\1\end{pmatrix}+\beta\begin{pmatrix}-1\\2\end{pmatrix}$$

linear system

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}1.6\\0.2\end{pmatrix}$$

```
u=[3 2]';   V=[2 1]';
W=null(V'); A=[V W];  c=A\u;
uV=c(1)*V;  uW=c(2)*W;
P=[zeros(2,1) uV u uW];
figure; h=patch(P(1,:),P(2,:),'y');
set(h,'LineStyle',':','FaceAlpha',0.25)
axis equal; grid on; hold on; box on
h=compass(complex(u(1),u(2)),'b');
h=compass(complex(uV(1),uV(2)),'r');
h=compass(complex(uW(1),uW(2)),'m');
```



Example

Compute, in $\mathbb{R}^3$, the angle between the planes:

$$\pi_1 = \text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right\}, \quad \pi_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

$$\pi_1 = \mathcal{R}(A), \quad A = \begin{pmatrix}1 & 2\\ 3 & 0\\ 3 & 1\end{pmatrix}$$

$$\pi_2 = \mathcal{R}(B), \quad B = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

$$\pi_1 \equiv 3x + 5y - 6z = 0$$

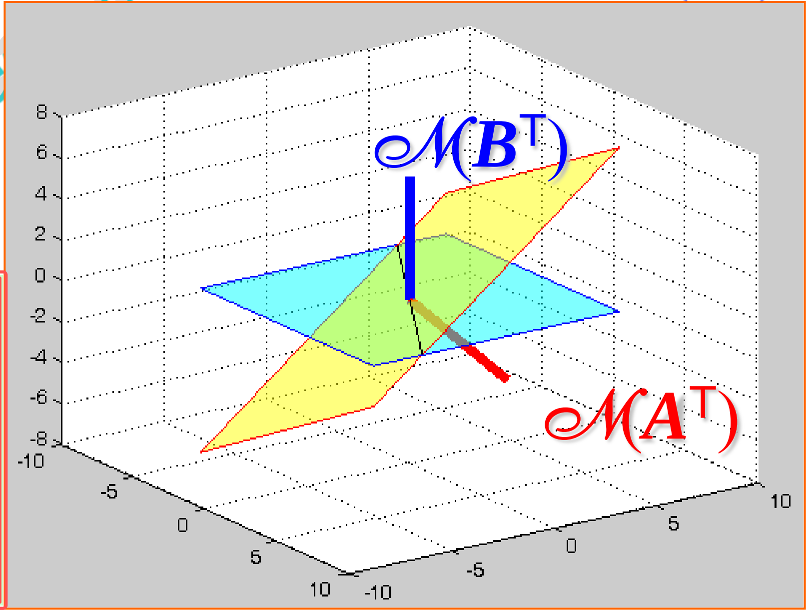
$$\pi_2 \equiv z = 0$$

$$\eta_1 = \begin{pmatrix}3\\5\\-6\end{pmatrix}, \quad \eta_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The coefficients of each cartesian equation form a basis for $\mathcal{R}(A)^\perp$ and $\mathcal{R}(B)^\perp$

$\mathcal{N}(A^\top) = \mathcal{R}(A)^\perp$ and $\mathcal{N}(B^\top) = \mathcal{R}(B)^\perp$

```
[x,y]=meshgrid([-6 6]); z1=(3*x+5*y)/6;
z2=zeros(size(x)); surf(x,y,z1);
set(h,'FaceAlpha',.5); hold on; h=surf(x,y,z2);
n1=[3 5 -6]'; n2=[0 0 1]'; % normals
cosTH=dot(n1,n2)/(norm(n1)*norm(n2));
TH=acos(abs(cosTH))*180/pi;
disp([TH  subspace(n1,n2)*180/pi])
       44.181       44.181
```



(prof. M. Rizzardi)