L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- **MATLAB laboratories on:**
  - intersection of subspaces;
  - application of Grassmann Formula;
  - derivation of the cartesian equation of a plane from its bases.

Laboratory: Compute in $\mathbb{R}^3$ $V \cap W$, where we know their bases:

$$V = \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\-1\end{pmatrix} \right\}, \quad W = \text{span}\left\{ \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$\forall x \in V \cap W \Leftrightarrow x \in V$ and $x \in W$, that is

$$x = \alpha \begin{pmatrix}1\\0\\1\end{pmatrix} + \beta \begin{pmatrix}0\\2\\-1\end{pmatrix} = a \begin{pmatrix}2\\2\\1\end{pmatrix} + b \begin{pmatrix}0\\0\\1\end{pmatrix} = x$$

$x \in V$ $\qquad$ $x \in W$

$$\underbrace{\begin{pmatrix}1 & 0 & 2 & 0\\0 & 2 & 2 & 0\\1 & -1 & 1 & 1\end{pmatrix}}_{A} \begin{pmatrix}\alpha\\\beta\\-a\\-b\end{pmatrix} = 0$$

$\begin{pmatrix}\alpha\\\beta\\-a\\-b\end{pmatrix} \in \mathcal{N}(A)$

unknowns are (α, β) or (a, b)

Lab (contd)

To compute a basis for $V \cap W$, at first we find $\mathcal{N}(A)$ where

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} \Rightarrow \mathcal{N}(A) = span\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftarrow \begin{pmatrix} \alpha \\ \beta \\ -a \\ -b \end{pmatrix}$$

then we choose the values of α and β (or of a and b), and at last, we substitute them into the formula giving x

$$x = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} -1 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$$

```
B1=sym([1 0;0 2;1 -1]);
B2=sym([2 0;2 0;1  1]);
A=[B1 B2];
N=null(A);
x=B1*N(1:2)
x =
-2
-2
-1
```

(α,β)

```
x=-B2*N(3:4)
x =
-2
-2
-1
```

$(-a,-b)$

the numerical resolution is the same, without **sym**

we can equally choose (a,b) or (α,β) as unknows

$$V \cap W = \textbf{span}\left\{ \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \right\}$$

Lab: intersection between planes

Compute in $\mathbb{R}^3$ $\pi_1 \cap \pi_2$, where we know their cartesian equations:

$$\pi_1 \equiv -2x+y+2z=0 \qquad \pi_2 \equiv -x+y=0$$

$$\pi_1 \equiv \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 0 \qquad \pi_2 \equiv \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

orthogonality conditions

```
syms x y z real
Eq1=-2*x+y+2*z == 0;
Eq2=-x+y == 0;
S=solve(Eq1,Eq2, ...
'ReturnConditions',true);
B=[S.x;S.y;z]/z
B =
2
2
1
```

the simplest way: solve the system

```
n1=sym([-2 1 2]');
n2=sym([-1 1 0]');
A=[n1 n2]; step 1
N=null(A') step 2
N =
2
2
1
```

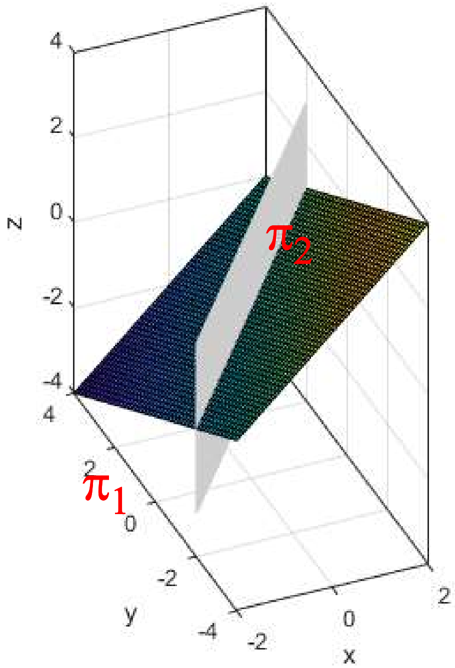
$\pi_1^\perp$ $\pi_2^\perp$

normals n1, n2

```
n1=sym([-2 1 2]'); % normal line to π1
n2=sym([-1 1 0]'); % normal line to π2
A1=null(n1'); % basis of π1
A2=null(n2'); % basis of π2
A=[A1 A2];     N=null(A);
A1*N(1:2)
ans =
-2
-2
-1
-A2*N(3:4)
ans =
-2
-2
-1
```

α β a b

bases for $\pi_1 \cap \pi_2$



Lab: the "simplest way" for the intersection between planes

$\pi_1 \equiv -2x+y+2z=0$ $\quad \pi_1 \equiv \left\langle \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \begin{pmatrix} -2\\ 1\\ 2 \end{pmatrix} \right\rangle = 0$ $\quad \pi_2 \equiv -x+y=0$ $\quad \pi_2 \equiv \left\langle \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \begin{pmatrix} -1\\ 1\\ 0 \end{pmatrix} \right\rangle = 0$

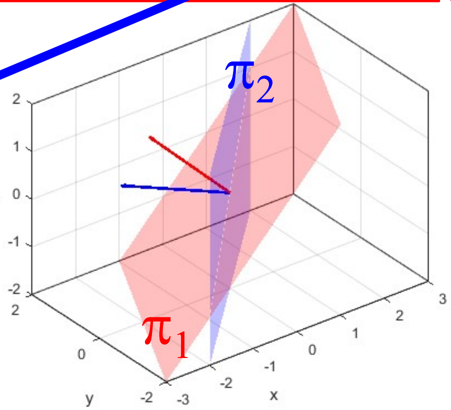
normals

1) Compute bases for both normals

```
n1=sym([-2;1;2]);
n2=sym([-1;1;0]);
```

the numerical solution is the same, without `sym`

$$\pi_1^\perp:\ \eta_1 = \text{span}\left\{\begin{pmatrix} -2\\ 1\\ 2 \end{pmatrix}\right\}, \quad \pi_2^\perp:\ \eta_2 = \text{span}\left\{\begin{pmatrix} -1\\ 1\\ 0 \end{pmatrix}\right\}$$

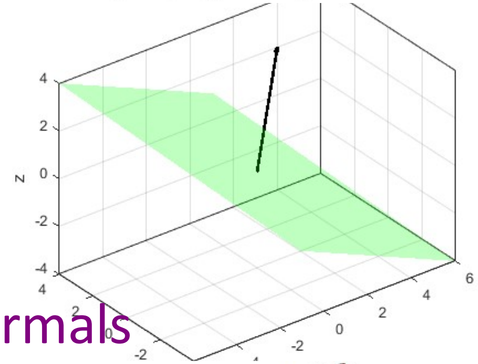


If required, from normals, compute the bases for both planes

```
syms a b real
p1=null(n1')*[a;b];
p2=null(n2')*[a;b];
```

parametric equations

$$\pi_1 = \text{span}\left\{\begin{pmatrix} 1\\ 2\\ 0 \end{pmatrix}, \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}\right\}, \quad \pi_2 = \text{span}\left\{\begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix}, \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}\right\}$$



If required, compute a basis of the plane spanned by the two normals

```
p3=[n1 n2]*[a;b];
```

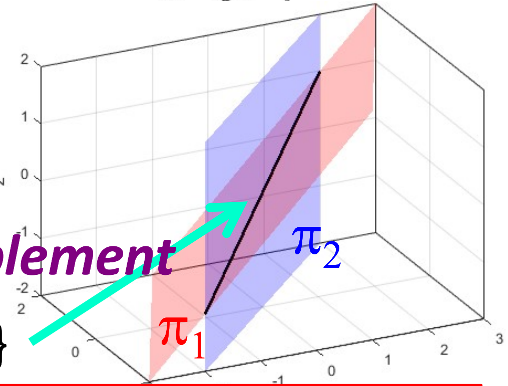
parametric equations

$$\pi_3 = \text{span}\left\{\begin{pmatrix} -2\\ 1\\ 2 \end{pmatrix}, \begin{pmatrix} -1\\ 1\\ 0 \end{pmatrix}\right\}$$

2) Compute the normal to that plane as the *Orthogonal Complement* of this plane

```
n3=null([n1 n2]');
```

$\eta_3 = \pi_1 \cap \pi_2 = \text{span}\{n_3\}$



Why is the normal to the plane spanned by the two normals n_1 and n_2 a basis for $\pi_1 \cap \pi_2$?

Lab.: are π and r complementary subspaces?

$\pi, r \subset \mathbb{R}^3$

$$\pi = \text{span}\left\{\begin{pmatrix}-5\\3\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \qquad r = \text{span}\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$$

```
V=[-5 3 0; 2 0 1]';
disp(rank(V))
2
u=[1 -2 1]';
A=[V u];
disp(rank(A))
3
```

$\dim(\mathbf{V}) = 2$

$\dim(\mathbf{u}) = 1$

$\dim(\mathbf{V}+\mathbf{u}) = 3 = \dim(\mathbb{R}^3)$
$= \dim(\mathbf{V})+\dim(\mathbf{u})$

⇓ Grassmann Formula

$\dim(\mathbf{V}\cap\mathbf{u}) = 0$ ⇒ **Yes!**

Question: is r the orthogonal complement of π?

Lab.

compute $V^{\perp}$ and its dimension where $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$.

```
A=[1 0 1;0 2 -1]';
disp(rank(A))
     2
Vperp=null(A')
Vperp =
         -0.6667
          0.3333
          0.6667
disp(Vperp'*A)
     0     0
RA=orth(A); disp(Vperp'*RA)
  1.0e-015 *
    0.1110         0
disp(rank([A RA]))
     2
```

$V \oplus V^{\perp} = \mathbb{R}^3$

$\dim(V) = 2$

$\dim(V^{\perp}) = 3 - 2 = 1$

verify orthogonality

compute $\mathscr{R}(\mathbf{A})$ and verify that it is orthogonal to $\mathscr{N}(\mathbf{A}^T)$

RA is equal to the Column Space of $\mathbf{A}$

Lab.: compute the cartesian equation of V

The plane π is given by a basis $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}.$

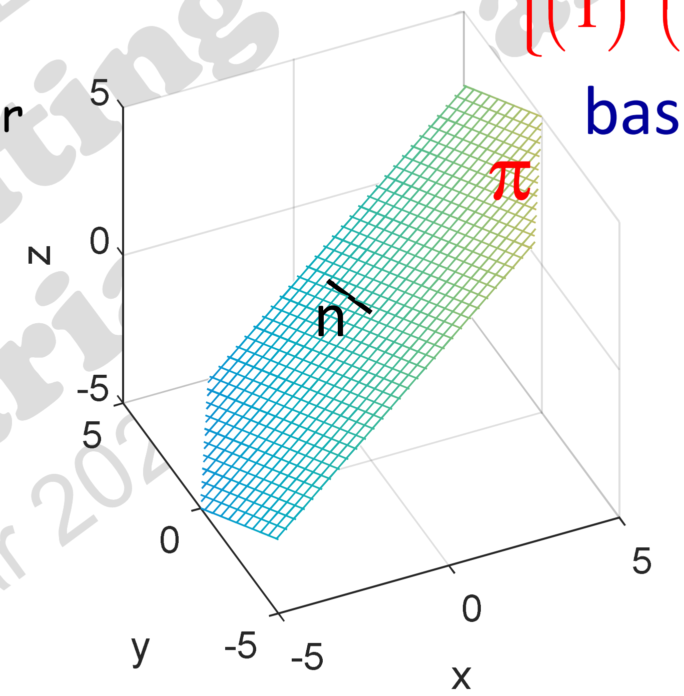
basis

```
A=sym([1 0 1;0 2 -1]');
n=null(A')
n =
   -1
   1/2
    1
```

compute a normal vector

```
[num,den]=numden(n);
mult=lcm(den);
n=mult*n
n =
   -2
    1
    2
```

without denominators




```
syms x y z real
Eqn=[char([x y z]*n) ' = 0']
Eqn =
    'y/2 - x + z = 0'
```

by means of a char array

char(): converts a symbolic expression into characters

```
syms x y z real
Eqn=string(char([x y z]*n)) + " = 0"
Eqn =
    "y - 2*x + 2*z = 0"
```

by means of a string

cartesian equation

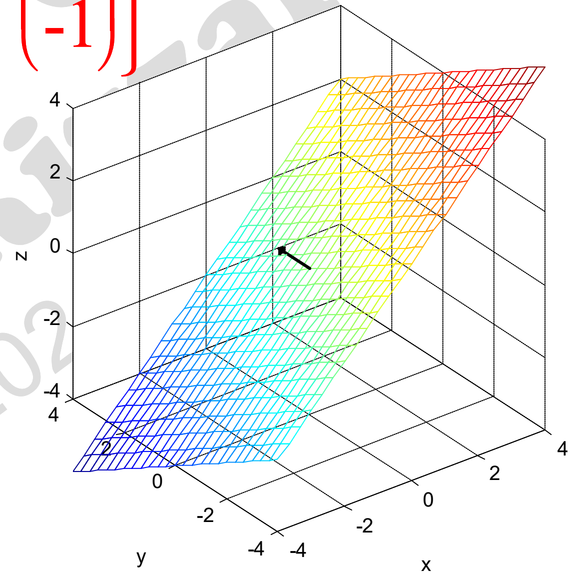
string(): converts a character array into a string

Lab.: display the plane with the observer in front of it

The plane π is given by a basis $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$.

```
A=[1 0 1;0 2 -1]';  n=null(A');
[x,y]=meshgrid(linspace(-4,4,25));
z=-(n(1)*x+n(2)*y)/n(3); mesh(x,y,z);
hidden off; hold on; axis equal
quiver3(0,0,0,n(1),n(2),n(3),1)
view([n(1),n(2),n(3)])
```

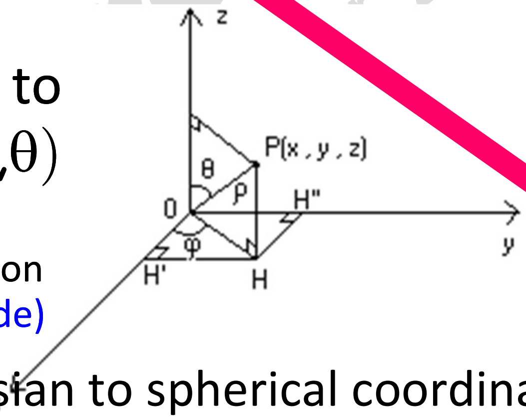
observer coordinates



cart2sph: from cartesian to spherical coordinates (ρ,φ,θ)

φ: azimuth

θ (colatitude) = $\pi/2$ – elevation (latitude)



cart2sph: from cartesian to spherical coordinates

or

```
[AZ,EL,rho]=cart2sph(n(1),n(2),n(3));
view(90+AZ*180/pi, EL*180/pi)
```

azimuth elevation