Scuola Interdipartimentale delle Scienze, dell'Ingegneria e della Salute

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- “Sum” and “direct sum” of two linear subspaces.
- “Complementary” subspace and “orthogonal complement” subspace of a given subspace.
- Grassmann Formula and other properties of linear subspaces.

Definitions (1 of 2)

Let S be a Vector Space and U, V, W are subspaces of S.

1. The subspace $U = V + W$ is said to be the **sum of V and W** if:

$$V + W = \{ s \in S \ : \ s = v + w, \ \forall v \in V \wedge \forall w \in W \}$$

2. The subspace $U = V \oplus W$ is said to be the **direct sum of V and W** if:

$$V \oplus W = V + W \quad \wedge \quad V \cap W = \{ \underline{0} \}$$

They are subspaces of S

Definitions (2 of 2)

Let S be a Vector Space and U, V, W are subspaces of S.

3 V is a complementary subspace of W in S if:

$$V \quad : \quad V \oplus W = S$$

4 $V = W^{\perp}$ is the orthogonal complement of W in S if:

$$W^{\perp} = \left\{ s \in S \; : \; s \perp w, \forall w \in W \right\}$$

Example 1

In $\mathbb{R}^3$

if $V = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\-1\end{pmatrix}\right\}, \quad W = \text{span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

planes

we set

$$A = \begin{pmatrix}1 & 0 & 2 & 0\\0 & 2 & 2 & 0\\1 & -1 & 1 & 1\end{pmatrix}$$

the Sum Subspace of V and W is:

$$V+W = \mathcal{R}(A)$$

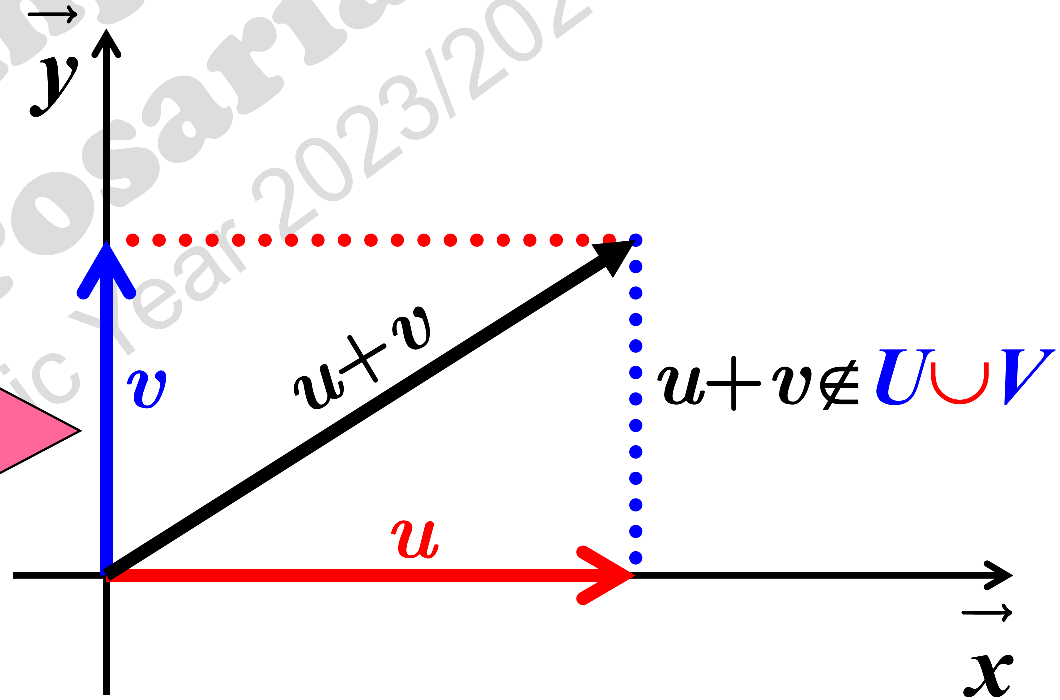
Column Space of A

$U \cup V$ **is not a subspace** of S because, in general,

$$\forall u \in U, \quad \forall v \in V \qquad u+v \notin U \cup V$$

Example

in $\mathbb{R}^2$ $\quad U = \vec{x}, \quad V = \vec{y}$



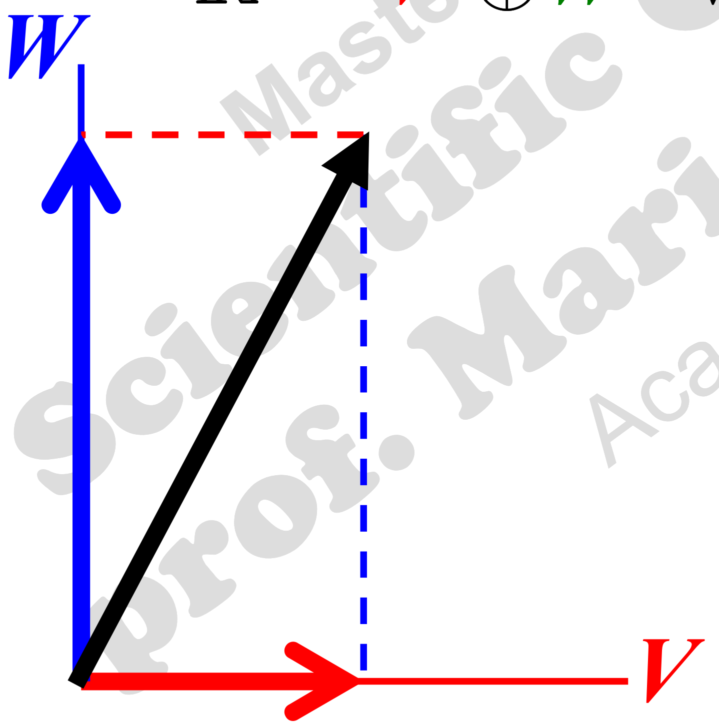
For each subspace in S, there are infinitely many complementary subspaces in the same space S.

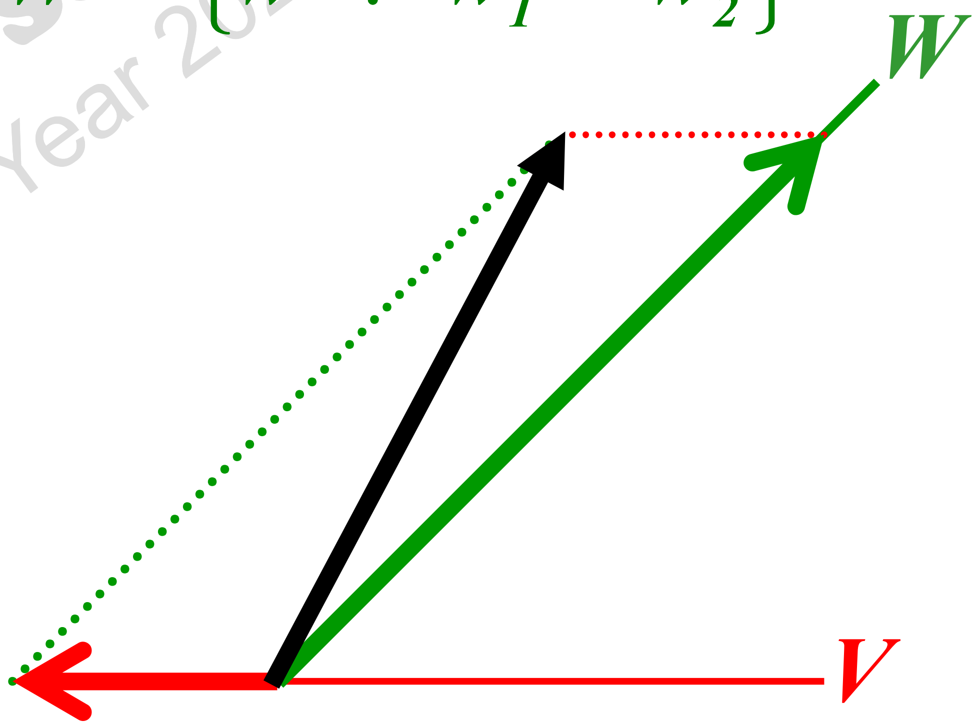
Example 2

If $V = x$-axis, then

$$\mathbb{R}^2 = V \oplus W \quad \text{where} \quad W = y\text{-axis}$$

$$\mathbb{R}^2 = V \oplus W \quad \text{where} \quad W = \{w : w_1 = w_2\}$$



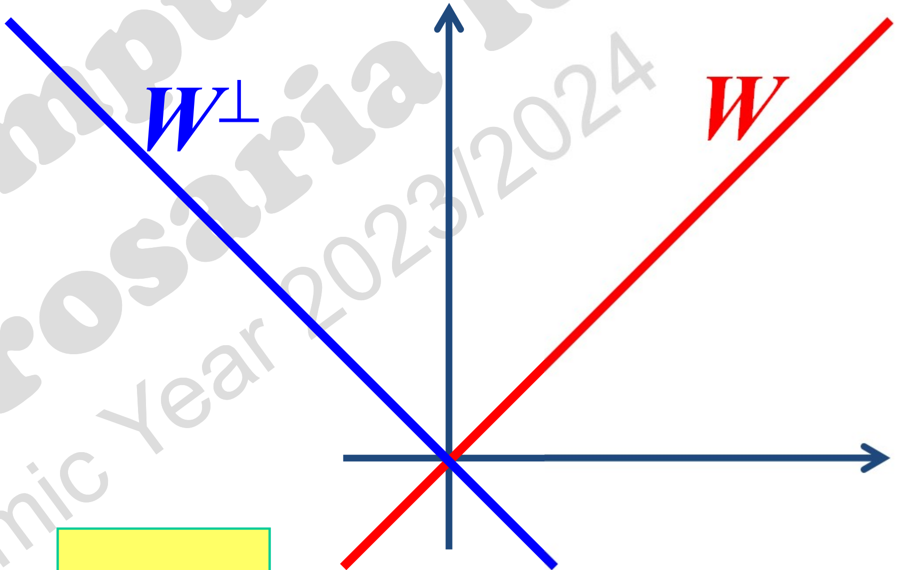


Example 3

If $W \subset \mathbb{R}^2 : W = \{w \in \mathbb{R}^2 \ : \ w_1 = w_2\}$ then its orthogonal complement $V = W^{\perp}$ is

$$V = W^{\perp} = \{v \ : \ v_1 = -v_2\}$$

$$W = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$



$$V = W^{\perp} = \left\{v \ : \ v \perp \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

$$\left\langle v, \begin{pmatrix}1\\1\end{pmatrix}\right\rangle = 0 \iff v_1 + v_2 = 0$$

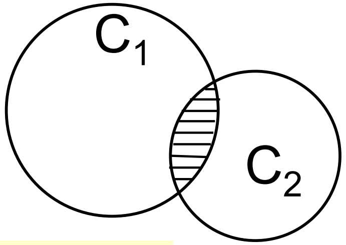
Properties (1/2)

V, W are subspaces of a vector space S:

1 $V+W$ and $V \cap W$ are linear subspaces.

2 **Grassmann formula**

to remember: similar to the area of $C_1 \cup C_2$



$$\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$$

⇓

3 $$\dim(V \oplus W) = \dim V + \dim W$$

4 $S = V \oplus W \;\Rightarrow\; \forall s \in S \; \exists! v \in V, \; \exists! w \in W : s = v + w$

5 $S = V \oplus W \;\Rightarrow\;$ a basis of S is given by the union of a basis of V and a basis of W.

Properties (2/2)

6 The zero-vector is orthogonal to all the vectors in the space, and it is the only vector orthogonal to itself.

7 The orthogonal complement is a subspace.

8 The orthogonal complement of W is a complementary subspace of W, that is $W \oplus W^{\perp} = S$.

9 The orthogonal complement of W is unique.

10 $(W^{\perp})^{\perp} = W$.

Intersection of two subspaces

1. If S is a Linear Space and U, V are subspaces of S, then $U \cap V$ is a subspace of S.

Proof

We will apply to $U \cap V$ the Theorem that gives a necessary and sufficient condition for a subspace of a Linear Space:

$U \cap V$ must contain the linear combinations of all its vectors.

Thesis: $\forall x, y \in U \cap V \Rightarrow \alpha x + \beta y \in U \cap V$

The result immediately follows, because U and V are both subspaces.

$$\forall x, y \in U \Rightarrow \alpha x + \beta y \in U$$

$$\forall x, y \in V \Rightarrow \alpha x + \beta y \in V$$