Exercises

SC2_03 – Dot products and norms.

1. Given in $\mathbf{R}^3$ the following bilinear forms with α as a parameter, establish which value of α makes the form to be a scalar product:

   1.1 $F(\boldsymbol{x},\boldsymbol{y}) = \alpha x_1y_1 + x_1y_3 + \alpha x_2y_2 + x_2y_3 + x_3y_1 + x_3y_2 + \alpha x_3y_3$

   1.2 $F(\boldsymbol{x},\boldsymbol{y}) = (2-\alpha)x_1y_3 + \alpha x_2y_2 + (2-\alpha)x_3y_1 + 5x_3y_3$

2. Verify that the following expressions are scalar products:

   2.1 $F(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} = \langle \boldsymbol{x},\boldsymbol{y}\rangle = \sum_{k=1}^{n} x_k y_k, \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ (*standard scalar product* in $\mathbb{R}^n$)

   2.2 $F(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{y}^{\mathsf{H}}\boldsymbol{x} = \langle \boldsymbol{x},\boldsymbol{y}\rangle = \sum_{k=1}^{n} x_k \overline{y}_k, \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{C}^n$ (*standard scalar product* in $\mathbb{C}^n$ *)

   * The MATLAB **`dot()`** function computes the scalar product between complex vectors as:

   $$\texttt{dot}(\boldsymbol{x},\boldsymbol{y}) = F(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^{\mathsf{H}}\boldsymbol{y} = \sum_{k=1}^{n} \overline{x}_k y_k, \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{C}^n$$

   In this case the ***linearity property*** must be applied to the second argument

   $$F(\boldsymbol{z}, a\boldsymbol{x} + b\boldsymbol{y}) = \boldsymbol{z}^{\mathsf{H}}(a\boldsymbol{x} + b\boldsymbol{y}) = aF(\boldsymbol{z},\boldsymbol{x}) + bF(\boldsymbol{z},\boldsymbol{y}), \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{C}^n, \ a,b \in \mathbb{C}$$

   or, if applied to the first argument, the ***linearity property*** should be written as follows:

   $$F(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}) = (a\boldsymbol{x} + b\boldsymbol{y})^{\mathsf{H}}\boldsymbol{z} = \overline{a}F(\boldsymbol{x},\boldsymbol{z}) + \overline{b}F(\boldsymbol{y},\boldsymbol{z}), \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{C}^n, \ a,b \in \mathbb{C}$$

3. Find the value of α such that the vectors $\boldsymbol{u}=(-3, 2)^{\mathsf{T}}$ and $\boldsymbol{v}=(2, \alpha)^{\mathsf{T}}$ be orthogonal.

4. Given the two vectors $\boldsymbol{r}=(2, -1)^{\mathsf{T}}$ and $\boldsymbol{s}=(-1, 2)^{\mathsf{T}}$, find a vector $\boldsymbol{x}=(x_1, x_2)^{\mathsf{T}}$ such that $\langle \boldsymbol{x},\boldsymbol{r}\rangle=1$ and $\langle \boldsymbol{x},\boldsymbol{s}\rangle=0$, where angle brackets denote the *standard scalar product* in $\mathbf{R}^2$.

5. If $\boldsymbol{A}$ is a matrix of size $(m\times n)$, prove that the *Null Space* of $\boldsymbol{A}$, $\mathcal{N}(\boldsymbol{A})$, is orthogonal to the *Row Space* of $\boldsymbol{A}$, $\mathcal{R}(\boldsymbol{A}^{\mathsf{T}})$.

6. Vector $\boldsymbol{p}=(150, 225, 375)^{\mathsf{T}}$ represents the price of certain models of bicycles, and vector $\boldsymbol{n}=(10, 7, 9)^{\mathsf{T}}$ represents the number of bicycles sold for each model, respectively. Compute the scalar product of $\boldsymbol{n}$ and $\boldsymbol{p}$, and state its meaning.

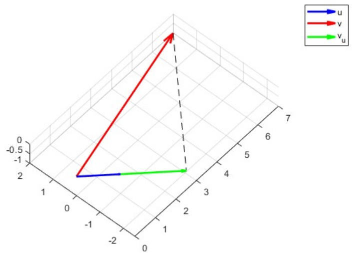
7. Check whether the following vectors are mutually orthogonal or not, and draw vectors:

   7.1 $\boldsymbol{u}=(2, 1)^{\mathsf{T}}$ e $\boldsymbol{v}=(-1, 2)^{\mathsf{T}}$.

   7.2 $\boldsymbol{u}=(3, -1, -2)^{\mathsf{T}}$ e $\boldsymbol{v}=(-2, -3, 1)^{\mathsf{T}}$.

   7.3 $\boldsymbol{u}=(1, -1, 0)^{\mathsf{T}}$ e $\boldsymbol{v}=(7, 2, -1)^{\mathsf{T}}$.

   Compute and draw the component of the vector $\boldsymbol{v}$ along $\boldsymbol{u}$. For example, in 7.3 the figure might look as follows:



8. Find and draw all vectors $\boldsymbol{v}$ orthogonal to vector $\boldsymbol{u}$, where:
   8.1 $\boldsymbol{u}=(3, 4)^{\mathsf{T}}$.
   8.2 $\boldsymbol{u}=(1, -1, -1)^{\mathsf{T}}$.

   Is this set a linear subspace?

9. What is the set of vectors orthogonal simultaneously to $(1, 1, 1)^{\mathsf{T}}$ and $(1, 2, 3)^{\mathsf{T}}$. Fit the answer in the subject matter of *Fundamental Subspaces of a matrix*?

10. Compute the scalar product between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ starting from:
    10.1 $\|\boldsymbol{u}\|=5$, $\|\boldsymbol{v}\|=3$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ equals 30°.
    10.2 $\|\boldsymbol{u}\|=20$, $\|\boldsymbol{v}\|=15$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ equals 225°.

11. For a random real matrix A, of size (2×2), estimate its condition number $\kappa(A)$ by means of MATLAB *Symbolic Math Toolbox*, and compare the result with **cond(*A*)**. $\kappa(A)$ is defined by:

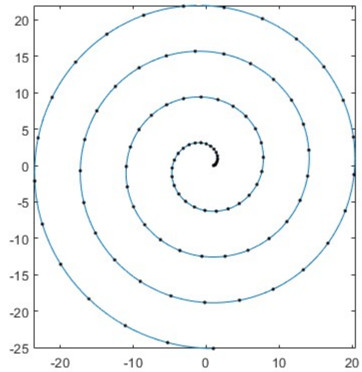
$$\kappa(A)=\frac{\max\limits_{v\neq 0}\dfrac{\|A\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2}}{\min\limits_{v\neq 0}\dfrac{\|A\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2}}$$

12. Given a triangle with vertices A, B and C, draw it and compute the angle in A in degrees:
    12.1 A(2,–7), B(1,1) e C(6,3).
    12.2 A(1,1,8), B(4,–3,–4) e C(–3,1,5).

13. Given a triangle with vertices A, B and C, draw it and identify “efficiently” whether it is an acute, obtuse or right triangle:
    13.1 A(2,3,0), B(3,1,–2) e C(–1,4,5).
    13.2 A(5,1,0), B(7,1,1) e C(6,3,2).

14. Given N samples on a regular curve Γ in the plane, choose a random sample point (P_0) and compute (“numerically”) the tangent line and the normal line to the curve at P_0. The parametric equation of the curve can be only used to produce samples and p to check the goodness of the results in their graphical display. For example, the curve *Involute of a Circle* has the following parametric equations:

$$x = \cos(\theta) + \theta\sin(\theta)$$
$$y = \sin(\theta) - \theta\cos(\theta)$$



[Other curves at: http://mathshistory.st-andrews.ac.uk/Curves/Curves.html]

Compute the angle between the tangent line at P_0 and the x-axis.
What happens if P_0 differs from all samples?