Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- **Applications of dot products and vector norms:**
  - angles between subspaces;
  - component of a vector along another vector;
  - tangent line and tangent plane;
  - normal line.

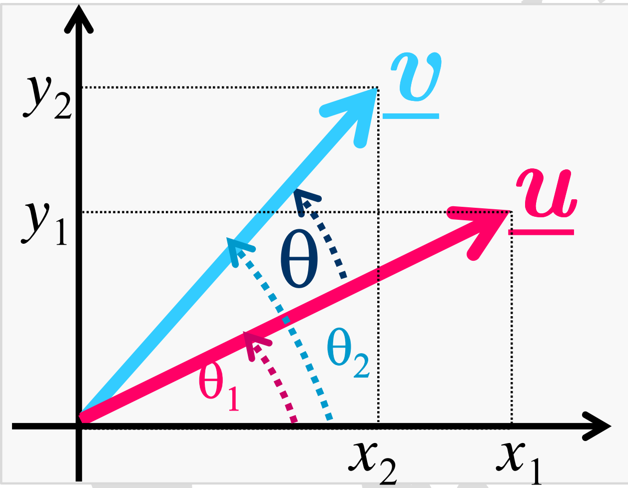
REMINDER (from the previous lecture)

standard scalar product in $\mathbb{R}^n$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{k=1}^{n} u_k v_k$$

the same as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \|\boldsymbol{u}\| \times \|\boldsymbol{v}\| \times \cos\theta$$



$$c = \cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

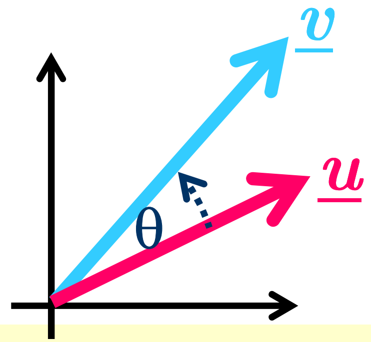
Angles between lines

X is a normed vector space

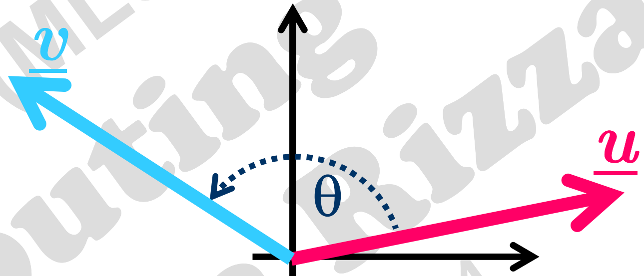
1) Angle between vectors

$u, v \in X$ $\quad c = \cos(\theta) = \dfrac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$ $\quad \theta$ = acos(□)

acute angle



$\cos(\theta) > 0 \iff 0 \le \theta < \pi/2$



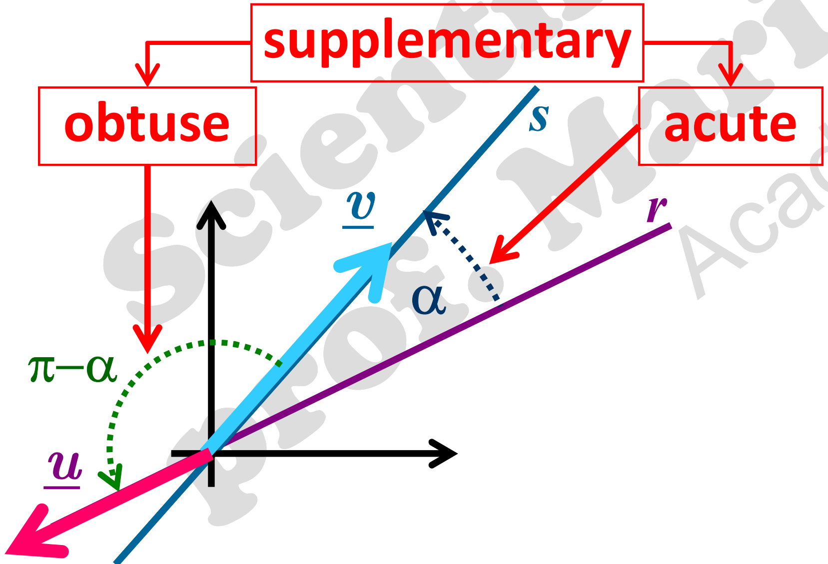
obtuse angle

$\cos(\theta) < 0 \iff \pi/2 < \theta \le \pi$

the formula θ = acos(c) always returns $\theta \in [0, \pi]$

2) Angle between lines

r = span{u}, $\quad s$ = span{v}



$u, v \in X$ $\quad c = \cos(\theta) = \dfrac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$ $\quad \theta$ = acos(c)

acute θ $\iff$ theta = acos(abs(c))

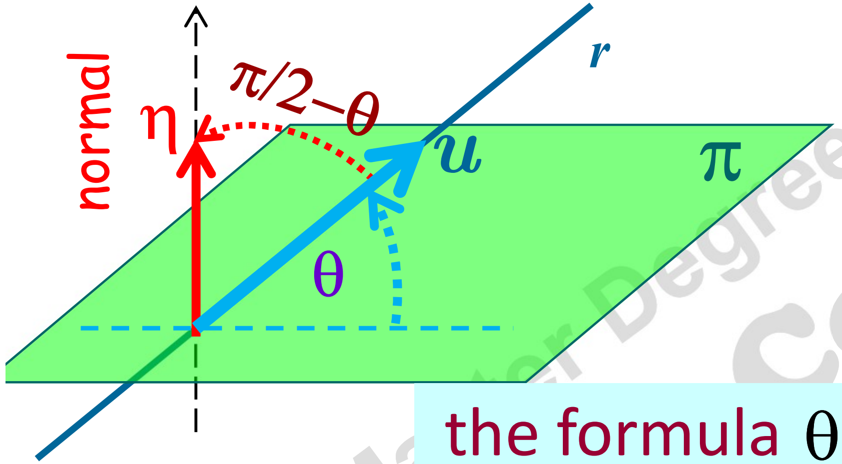
obtuse θ $\iff$ theta = acos(-abs(c))

Angles between subspaces

X is a normed vector space

3) Angle between a line r=span{u} and a plane π

as the complementary to the angle between the vector and the normal to the plane



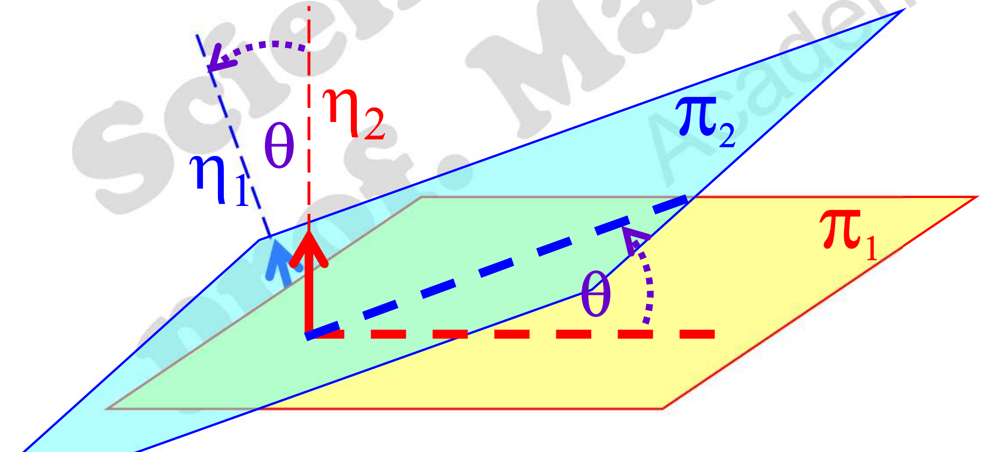
$$s = \sin\theta = \cos\widehat{u\eta} = \frac{\langle u, \eta\rangle}{\|u\|\cdot\|\eta\|} \qquad \theta = \text{asin}(s)$$

θ acute: $\theta>0$ ⇨ theta = asin(abs(s))

the formula θ = asin(s) always returns $\theta\in[-\pi/2, +\pi/2]$

4) Angle between planes

as the angle between the normals to the planes

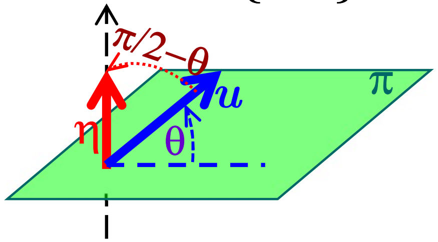


$$c = \cos\widehat{\pi_1\pi_2} = \cos\widehat{\eta_1\eta_2} = \frac{\langle \eta_1, \eta_2\rangle}{\|\eta_1\|\|\eta_2\|}$$

acute θ ⇨ theta = acos(abs(c))

Lab: angle between a line and a plane in $\mathbb{R}^3$

$$r = \text{span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \quad \pi = \text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \iff \pi = \mathcal{R}(A), \quad A = \begin{pmatrix}1 & 2\\3 & 0\\3 & 1\end{pmatrix}$$



$$\sin\theta = \cos\widehat{u\eta} = \frac{\langle u, \eta\rangle}{\|u\|\cdot\|\eta\|}$$

How can we find the normal η to the plane π ?

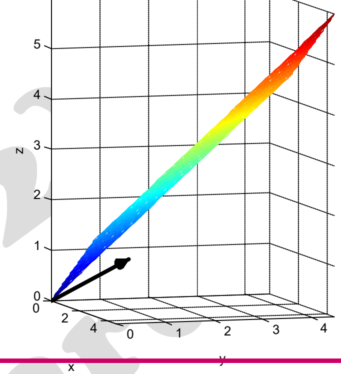
$$\pi \perp \eta \iff \begin{cases} \left\langle \begin{pmatrix}\eta_1\\\eta_2\\\eta_3\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix} \right\rangle = 0 \\ \left\langle \begin{pmatrix}\eta_1\\\eta_2\\\eta_3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle = 0 \end{cases} \iff \begin{pmatrix}1 & 3 & 3\\2 & 0 & 1\end{pmatrix}\begin{pmatrix}\eta_1\\\eta_2\\\eta_3\end{pmatrix} = 0$$

$$A^{\mathsf{T}}\eta = 0$$

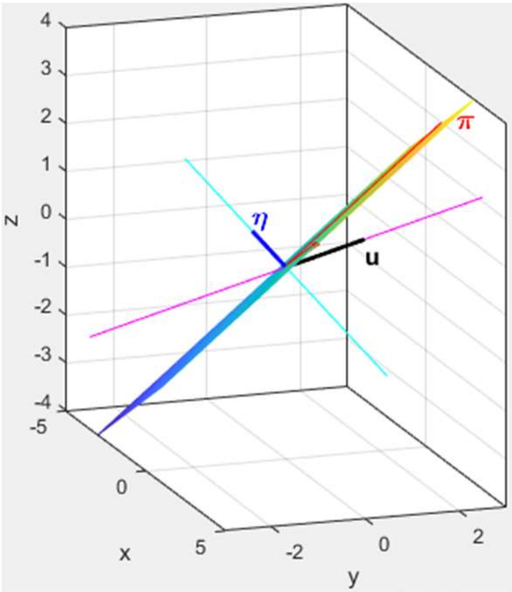
$$\pi = \mathcal{R}(A) \perp \mathcal{N}(A^{\mathsf{T}}) = \text{span}\{\eta\}$$

Lab (contd.)

$$r=\text{span}\left\{\overset{\boldsymbol{u}}{\begin{pmatrix}2\\1\\1\end{pmatrix}}\right\},\quad \pi=\text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \iff \pi=\mathcal{R}(A),\quad A=\begin{pmatrix}1&2\\3&0\\3&1\end{pmatrix}$$



How can we find the normal η to the plane π ? $\eta\in\mathcal{N}(A^T)$




```
A=sym([1 3 3;2 0 1]'); n=null(A')
n =
  -1/2
  -5/6
     1
[num,den]=numden(n); % extract numerators and denominators
comden=lcm(den);     % lcm: least common multiple
eta=comden*n; disp(eta)
-3
-5
 6
```

a vector that is normal to the plane

to avoid fractions $\eta=\text{span}\left\{\begin{pmatrix}-3\\-5\\6\end{pmatrix}\right\}$

$$\sin\theta=\cos\widehat{u\eta}=\frac{\langle u,\eta\rangle}{\|u\|\cdot\|\eta\|}$$

acute $\theta>0$ ⇒ theta = asin(abs(s))

The acute angle amplitude between r and π is:

```
u=[2 1 1]'; A=[1 3 3;2 0 1]'; n=null(A');
SINtheta=dot(u,n)/(norm(u)*norm(n));
degrees=asin(abs(SINtheta))*180/pi
degrees =
        14.1213
```

```
degrees=rad2deg(asin(abs(SINtheta)))
```

Cartesian equation

$$\pi \equiv 3x+5y-6z=0$$

as an orthogonality condition

$$P\in\pi \quad P=\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix}x\\y\\z\end{pmatrix}\perp\begin{pmatrix}-3\\-5\\6\end{pmatrix}$$

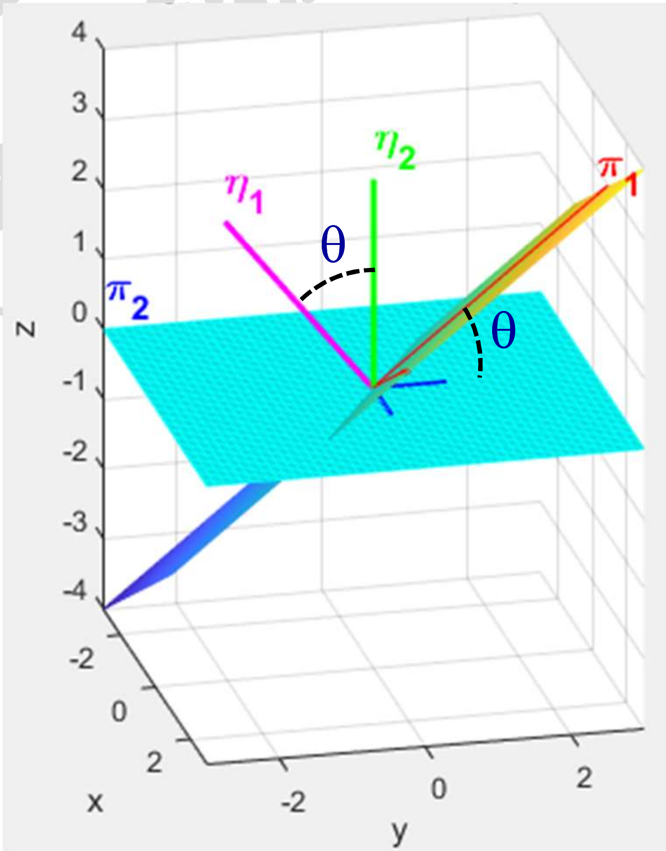
Lab: angle between two planes

$$\pi_1 = \text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right\},\quad \pi_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

A ***B***

```
[x,y]=meshgrid(linspace(-6,6,25));
z1=(3*x+5*y)/6;     % the plane π1
z2=zeros(size(x)); % the plane π2
mesh(x,y,z2); hold on; surf(x,y,z1); axis equal
A=[1 3 3;2 0 1]'; B=[1 0 0;0 1 0]';
n1=null(A'); % basis for the normal to ℛ(A)
n2=null(B'); % basis for the normal to ℛ(B)
theta=acos(abs(dot(n1,n2)/(norm(n1)*norm(n2))));
degrees=theta*180/pi
degrees =
        44.1814
```

```
degrees=rad2deg(theta)
```



Geometric interpretation of the Cartesian equation of a plane in $\mathbb{R}^3$: orthogonality condition between any vector on the plane $v=(x,y,z)^T$ and the normal η to the plane

$\pi_1 \equiv 3x + 5y - 6z = 0$ $\pi_2 \equiv z = 0$

The Cartesian equation coefficients are the components of a normal vector to the plane

$\eta_1 = (3,5,-6)^T$

$\eta_2 = (0,0,1)^T$

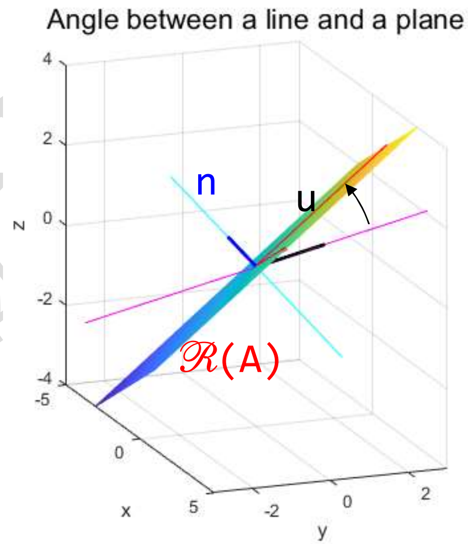
normals to planes

Lab: angle between subspaces in MATLAB

In MATLAB the function **subspace(A,B)** returns the angle (in radians) between the subspaces $\mathscr{R}(A)$ and $\mathscr{R}(B)$.

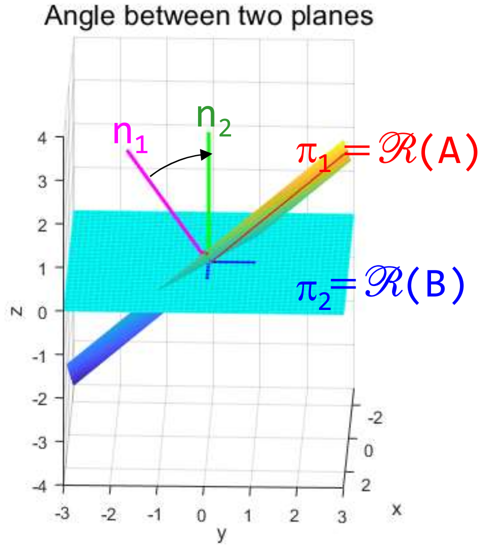
1) Angle between r and π $\quad r = \text{span}\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \quad \pi = \text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$

```
u=[2 1 1]';  A=[1 2;3 0;3 1];
n=null(A');
SINtheta=dot(u,n)/(norm(u)*norm(n));
degree=asin(abs(SINtheta))*180/pi
degree =
        14.121
theta=rad2deg( subspace(u,A) )
theta =
        14.121
```



2) Angle between π_1 and π_2 $\quad \pi_1 = \text{span}\left\{\begin{pmatrix}1\\3\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right\}, \quad \pi_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

```
A=[1 2;3 0;3 1];  B=[1 0;0 1;0 0];
nA=null(A');      nB=null(B');
COStheta=dot(nA,nB)/(norm(nA)*norm(nB));
degree=acos(abs(COStheta))*180/pi
degree =
        44.181
theta=rad2deg( subspace(A,B) )
theta =
        44.181
```



Lab: component of a vector along another vector

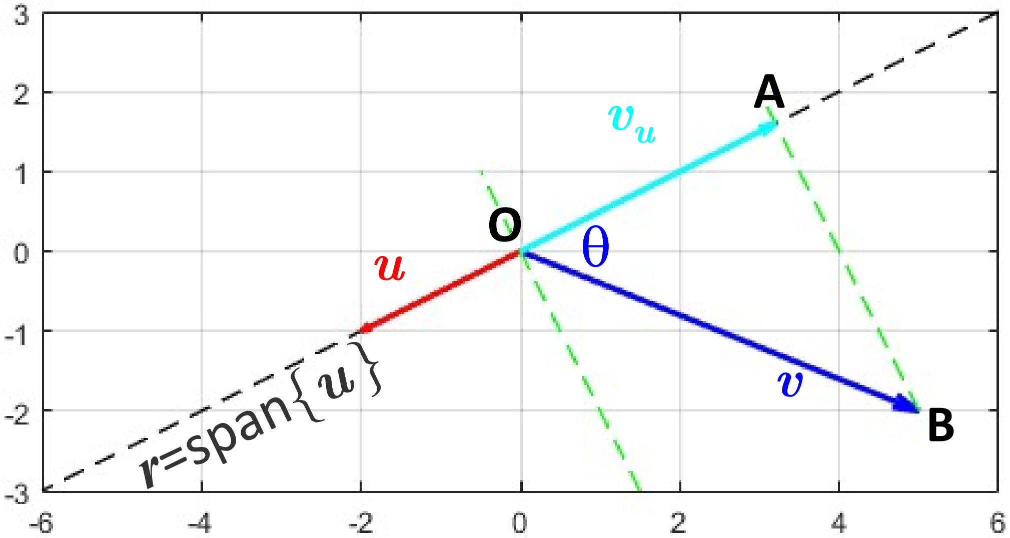
Find in $\mathbb{R}^2$ the component of a vector v along the line spanned by the vector u, where:

$$u=\begin{pmatrix}-2\\-1\end{pmatrix} \quad \text{e} \quad v=\begin{pmatrix}5\\-2\end{pmatrix}$$

The component of v along u is the vector v_u given by the orthogonal projection of the point **B** on the line r=span{u}.

In the right triangle **OAB**, the cathetus **OA** equals the hypotenuse **OB**=$\|v\|$ times the cosine of the angle θ which is adjacent to the cathetus:

$$\mathbf{OA}=\|v_u\|=\mathbf{OB}\cos(\theta)=\|v\|\cos(\theta)$$



$$\langle u,v\rangle=\|u\|\|v\|\cos\theta \iff \|v\|\cos\theta=\frac{\langle u,v\rangle}{\|u\|} \Rightarrow$$

$$v_u=\|v\|\cos\theta\frac{u}{\|u\|}=\frac{\langle u,v\rangle}{\|u\|}\frac{u}{\|u\|}=\left\langle\frac{u}{\|u\|},v\right\rangle\frac{u}{\|u\|}=\langle u,v\rangle\frac{u}{\|u\|^2}$$

versor of the line (or unit vector, i.e. its norm equals 1)

```
u=[-2;-1]; v=[5;-2];
u1=u/norm(u); % versor
u=dot(u1,v)*u1;
```

```
u=[-2;-1];  v=[5;-2];
vu=dot(u,v)/norm(u)^2*u;
```

Lab: tangent line to a regular curve in $\mathbb{R}^2$

$$\Gamma\begin{cases} x_1(\theta) = \rho\cos\theta \\ x_2(\theta) = \rho\sin\theta \end{cases} \qquad \theta \in [0, 2\pi]$$

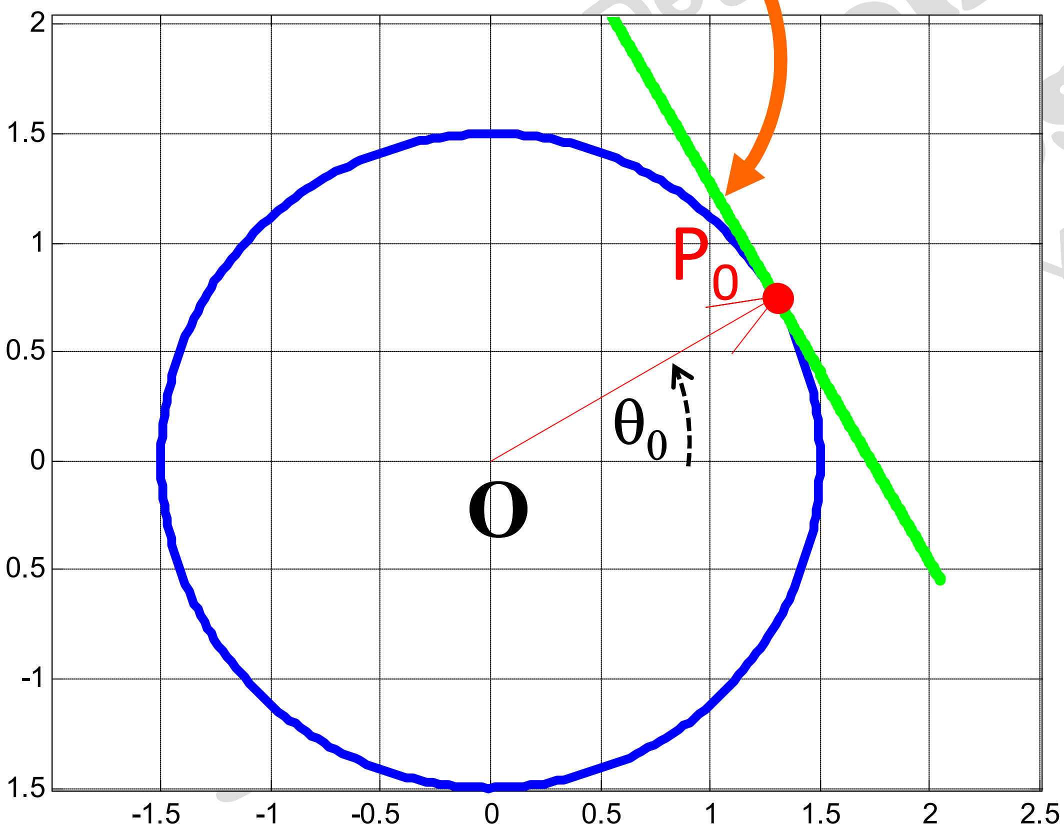
scalar parametric equations of a circle Γ centered at $\mathbf{O}$ and with radius ρ

regular curve

τ: tangent line to Γ at a point $P_0(x_1(\theta_0), x_2(\theta_0)) \in \Gamma$

$$\tau\begin{cases} \tau_1 = x_1(\theta_0) + \lambda x_1'(\theta_0) \\ \tau_2 = x_2(\theta_0) + \lambda x_2'(\theta_0) \end{cases} \qquad \lambda \in \mathbb{R}$$

scalar parametric equations of the tangent to Γ at P_0



MATLAB Symbolic Math Toolbox

```
syms q t real;  rho=1.5;
x1=rho*cos(t); x2=rho*sin(t);
x=[x1;x2]; % Γ curve
ezplot(x1,x2); hold on
xp=diff(x); % derivative w.r.t. t
t0=pi/6;
P0=subs(x,'t',t0);
xpP0=subs(xp,'t',t0); % direction
T=P0 + q*xpP0; % tangent line at P0
ezplot(T(1),T(2),[-1 1]); grid
```

Lab: normal line to a regular curve in $\mathbb{R}^2$

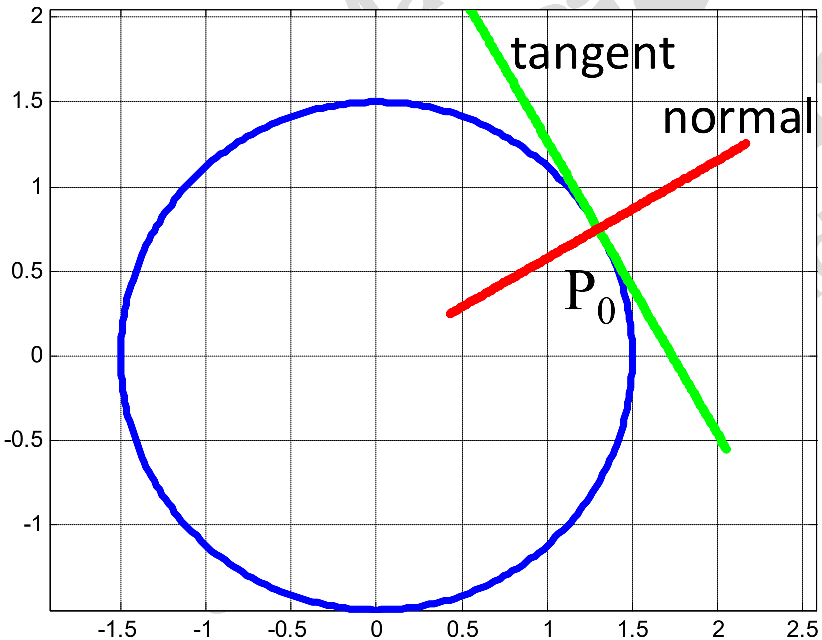
$$\Gamma: \quad x = \begin{pmatrix} x_1(\theta) \\ x_2(\theta) \end{pmatrix}, \quad \begin{cases} x_1(\theta) = \rho\cos\theta \\ x_2(\theta) = \rho\sin\theta \end{cases} \quad \theta \in [0, 2\pi]$$

The **direction vector** of the tangent at P_0 is $\quad x'(\theta_0) = \begin{pmatrix} x_1'(\theta_0) \\ x_2'(\theta_0) \end{pmatrix}$

The **normal vector** $\eta(\theta_0)$ to Γ at P_0 is such that $\langle \eta(\theta_0), x'(\theta_0) \rangle = 0$

$\Leftrightarrow$ $(x'(\theta_0))^T \eta(\theta_0) = 0$ $\Leftrightarrow$ $\eta(\theta_0) \in \mathcal{N}\left[(x'(\theta_0))^T\right]$

standard scalar product in $\mathbb{R}^2$



MATLAB Symbolic Math Toolbox

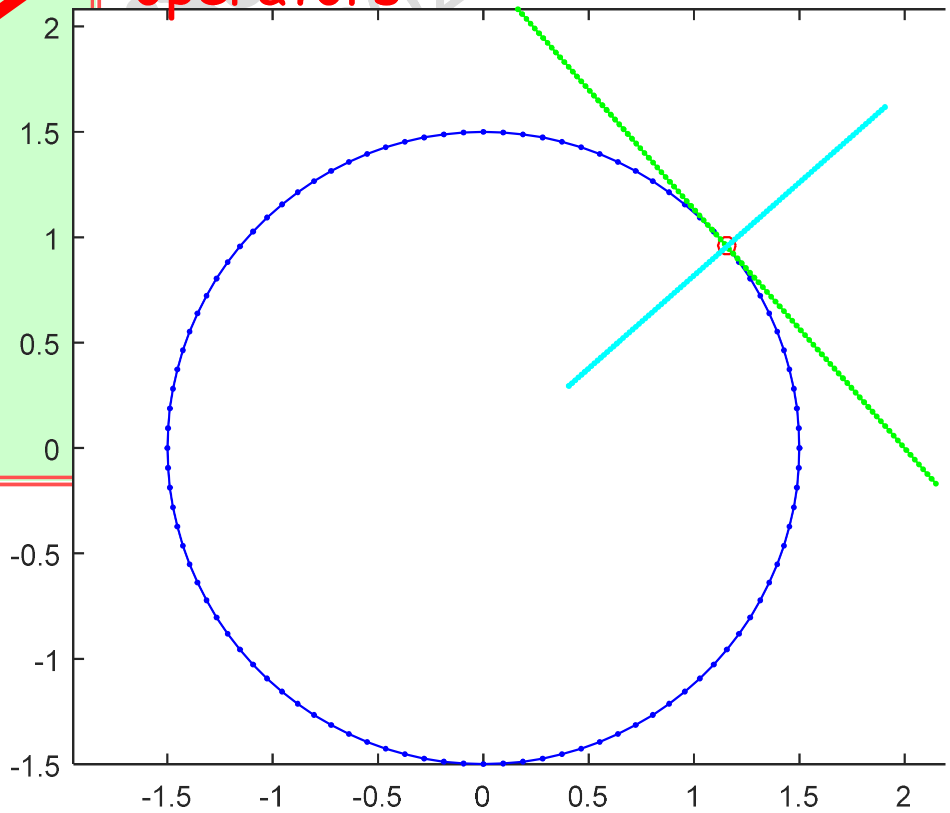
```
syms q t real ......
xp=diff(x); % derivative w.r.t. t
t0=pi/6;
P0=subs(x,'t',t0);
xpP0=subs(xp,'t',t0); % direction vector
N0 = null(xpP0.');
N=P0 + q*N0; % normal line at P0
ezplot(N(1),N(2),[-1 1])
```

Exercise: solve the same problem numerically

You must use only samples for $\Gamma(0,\rho)$ $\Gamma: \quad P_k = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}, \quad k = 1,\dots,n$

```
t=linspace(0,2*pi,101); rho=1.5;
x1=rho*cos(t); x2=rho*sin(t);
P=[x1;x2]; % sample points
j=12; t0=t(j); P0=P(:,j); % a particular point on Γ
plot(P(1,:),P(2,:),'.b-',P0(1),P0(2),'or');
axis equal
X1p=…; X2p=…;      % approximated derivative
q=linspace(-1,1);
T(1,:)=... ; T(2,:)=... ; % tangent
nj=null(...);
N(1,:) = ... ; N(2,:) = ... ; % normal
plot(T(1,:),T(2,:),'.g-',N(1,:),N(2,:),'.c-')
grid on
```

use difference quotient or other finite difference operators



What happens if P_0 is not a sample point?

Lab: display the tangent plane to a surface

$$S: \quad z = f(x,y) \quad \text{differentiable at } P_0$$

parametric equations

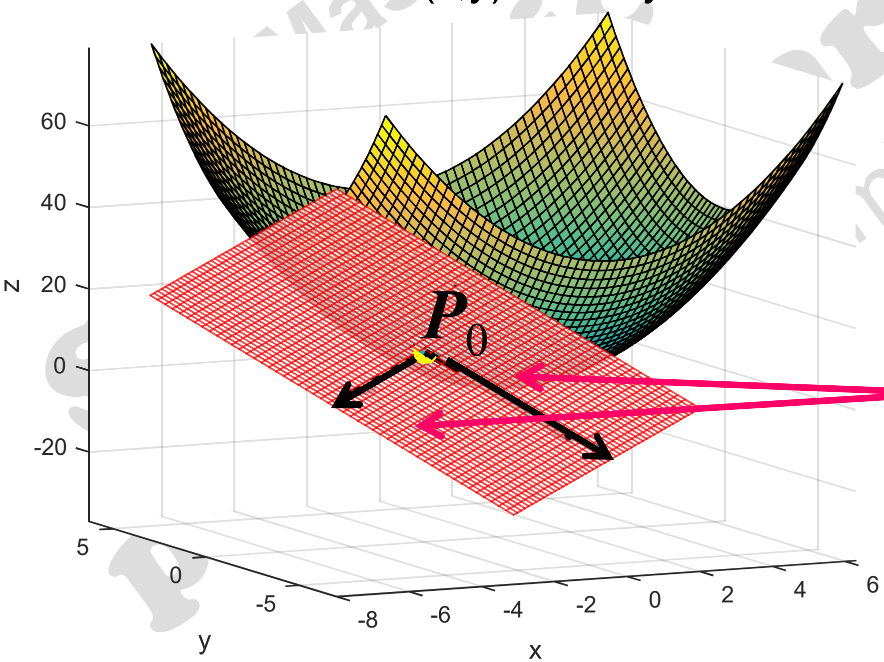
- surface $\quad S: \quad P = P(x,y) = \begin{pmatrix} X(x,y) \\ Y(x,y) \\ Z(x,y) \end{pmatrix} : \begin{cases} X = x \\ Y = y \\ Z = f(x,y) \end{cases}$
- tangent plane at P $\quad P = P(x,y) + \left[\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right]\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = P(x,y) + J\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$

Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial X}{\partial x}(x_0,y_0) & \frac{\partial X}{\partial y}(x_0,y_0) \\ \frac{\partial Y}{\partial x}(x_0,y_0) & \frac{\partial Y}{\partial y}(x_0,y_0) \\ \frac{\partial Z}{\partial x}(x_0,y_0) & \frac{\partial Z}{\partial y}(x_0,y_0) \end{pmatrix}$$

Jacobian matrix

surface f(x,y) = $x^2 + y^2$



MATLAB Symbolic Math Toolbox

```
syms x y real;   f=x^2+y^2;
X=-3;  Y=-2;   Z=subs(f,{'x','y'},{X,Y});
P=[x;y;f];   J=jacobian(P);
J0=subs(J,{x,y},{X,Y}); v=double(J0);
P0=[X Y Z]'; % point P0
ezsurf(f); hold on; plot3(P0(1),P0(2),P0(3))
syms a b real;  P=P0+J0*[a;b]; % tangent plane
ezmesh(P(1),P(2),P(3))
quiver3([X X],[Y Y],[Z Z],v(1,:),v(2,:),v(3,:),1)
```

J0: Jacobian matrix at P_0

```
J=[sym(eye(2));jacobian(f)];
J0=subs(J,{x,y},{X,Y}); v=double(J0);
```

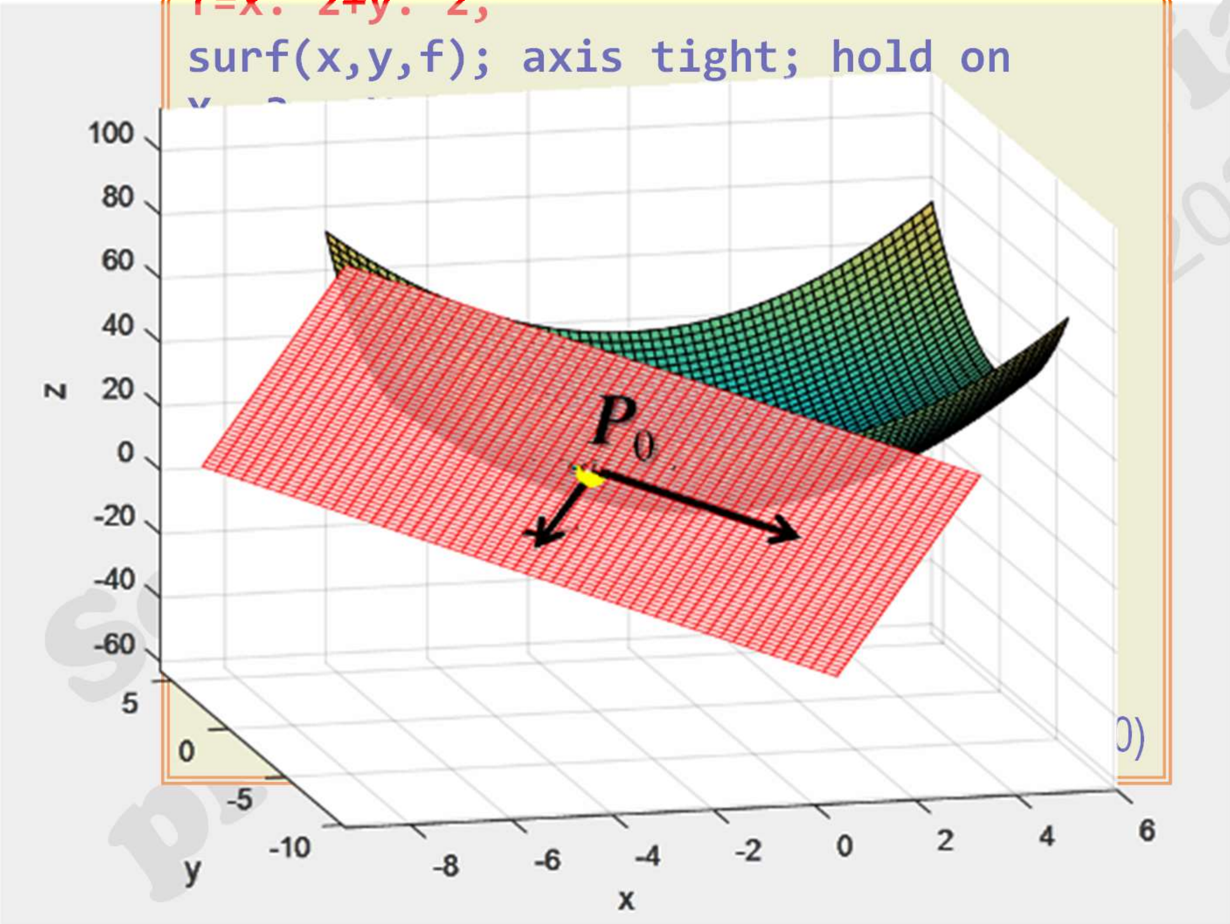
```
J=[sym(eye(2)); gradient(f).'];
J0=subs(J,{x,y},{X,Y}); v=double(J0);
```

Exercise: solve the same problem numerically

$z_{h,k} = f(x_h, y_k)$ only samples for $z = f(x, y)$ (differentiable at P_0)

Use MATLAB (num) gradient() in place of jacobian()

```
w=0.2;  [x,y]=meshgrid(-6:w:6);
f=x.^2+y.^2;
surf(x,y,f); axis tight; hold on
```




```
[fx,fy]=gradient(f,w);
```

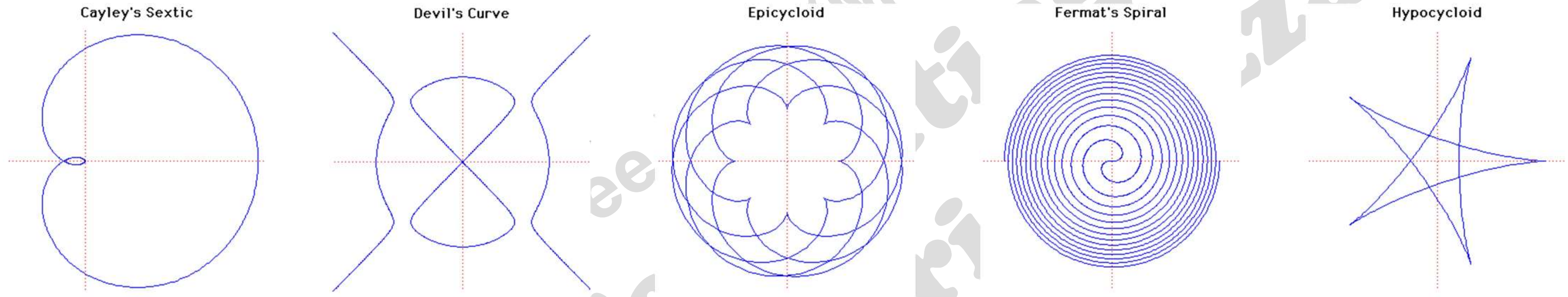
approximate the gradient of f() at grid points

What happens if P_0 is not a sample point?

Useful links for parametric eqs

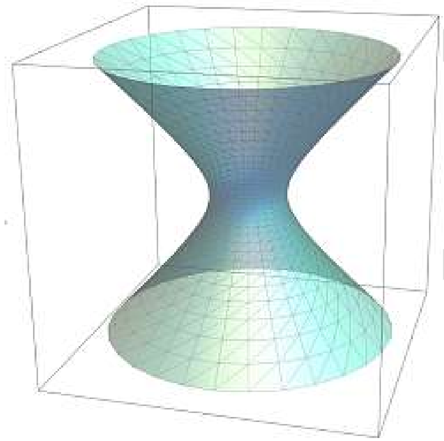
Famous Curves Index

http://mathshistory.st-andrews.ac.uk/Curves/Curves.html

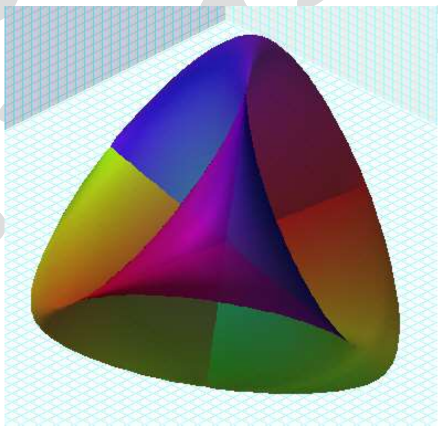


Gallery of Surfaces

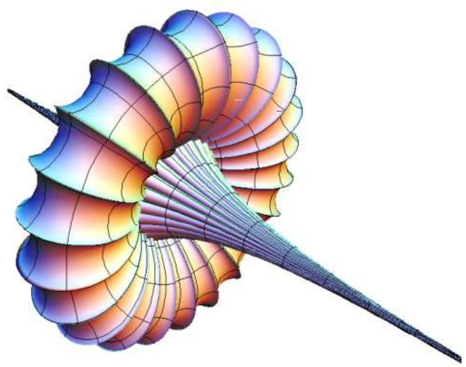
http://xahlee.info/surface/gallery.html



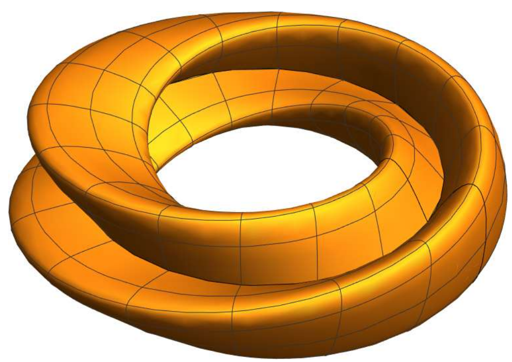
Hyperboloid of One Sheet



Steiner Surface



Breather Surface



Klein Bottle