L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- Dot product for complex vectors.
- Vector norms.
- Induced vector norms.
- Parallelogram Law.

Definition: dot product of complex vectors

Let $\langle X, \mathbb{C}, +, * \rangle$ be a Linear Space with complex scalars.

The ***dot product*** (or ***scalar product*** or ***inner product***) of two complex vectors is a map

$$\langle \bullet, \bullet \rangle : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{X} \longrightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{C}$$

angle brackets

satisfying the following properties:

Linearity law w.r.t. the first argument

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in X$

Conjugate simmetry

2. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$

complex conjugate

If $a + ib$ is a complex number, then its complex conjugate is $a - ib$.

3. $\langle x, x \rangle \in \mathbb{R}^{+} \quad \forall x \in X \wedge x \neq \underline{0}$

Positive definiteness

4. $\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = \underline{0}$

Hermitian form

The dot product $\langle \bullet, \bullet \rangle$ between complex vectors is a hermitian sesquilinear form that is **positive definite**; i.e. the following properties hold:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

2. $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$ $\quad \forall \alpha, \beta \in \mathbb{C}, \forall x, y, z \in X$

sesquilinear form

2. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$

hermitian (conjugate symmetric)

3. $\langle x, x \rangle \in \mathbb{R}^+ \quad \forall x \in X \wedge x \neq \underline{0}$

4. $\langle x, x \rangle = 0 \iff x = \underline{0}$

positive definite

Example

In the Linear space C[a,b] of complex-valued functions that are continuous on [a,b], the standard scalar product is defined by:

$$\langle x, y\rangle = \int_a^b x(t)\,\overline{y(t)}\,\mathrm{d}t$$

Proof:

1. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall \alpha, \beta \in \mathbb{C}, \forall x, y, z \in X$ **OK!**

$$\langle \alpha x + \beta y, z\rangle = \int_a^b [\alpha x(t) + \beta y(t)]\overline{z(t)}dt = \alpha\int_a^b x(t)\overline{z(t)}dt + \beta\int_a^b y(t)\overline{z(t)}dt = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

2. $\langle x, y\rangle = \overline{\langle y, x\rangle} \quad \forall x, y \in X$ **OK!**

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}dt = \overline{\int_a^b \overline{x(t)}y(t)dt} = \overline{\langle y, x\rangle}$$

3. $\langle x, x\rangle \in \mathbb{R}^+ \quad \forall x \in X \wedge x \neq \underline{0}$ **OK!**

$$\langle x, x\rangle = \int_a^b x(t)\overline{x(t)}dt = \int_a^b |x(t)|^2\,dt > 0$$

if the function does not vanish everywhere

Example: particular case of the previous

In the Linear space C[a,b] of real-valued functions that are continuous on [a,b], the standard scalar product is defined by:

$$\langle x, y\rangle = \int_a^b x(t)\, y(t)\, \mathrm{d}t$$

Proof:

1. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \quad \forall \alpha, \beta \in \mathbb{C}, \forall x, y, z \in X$ **OK!**

$$\langle \alpha x + \beta y, z\rangle = \int_a^b [\alpha x(t) + \beta y(t)] z(t) dt = \alpha \int_a^b x(t) z(t) dt + \beta \int_a^b y(t) z(t) dt = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

2. $\langle x, y\rangle = \langle y, x\rangle \quad \forall x, y \in X$ **OK!**

$$\langle x, y\rangle = \int_a^b x(t) y(t) dt = \int_a^b y(t) x(t) dt = \langle y, x\rangle$$

3. $\langle x, x\rangle \in \mathbb{R}^+ \quad \forall x \in X \wedge x \neq \underline{0}$ **OK!**

$$\langle x, x\rangle = \int_a^b x(t) x(t) dt = \int_a^b [x(t)]^2 dt > 0$$ if the function does not vanish everywhere

Test the real case by means of Symbolic Math Toolbox

```
syms w real
syms x(w) y(w) z(w)
assumeAlso([x(w) y(w) z(w)],'real')
assumeAlso([x(w) y(w) z(w)] ~= 0)
syms a b t real
assumptions ...
dp = @(f,g) int(f*g, -1,1); % dot product
```

declare x(w), y(w), z(w) as non-zero real symbolic functions

$$\langle f,g\rangle = \int_{-1}^{+1} f(t)\,g(t)\,\mathrm{d}t$$

Check if properties are satisfied:

1. (linearity law) $\langle \alpha x+\beta y,z\rangle = \alpha\langle x,z\rangle+\beta\langle y,z\rangle \quad \forall \alpha,\beta\in\mathbb{R}, \forall x,y,z\in X$

```
simplify(expand(dp(a*x+b*y,z) - (a*dp(x,z) + b*dp(y,z))))
ans =
     0
```

```
simplify(expand(dp(a*x+b*y,z) == (a*dp(x,z) + b*dp(y,z))))
ans =
symtrue
```

OK!

2. (symmetric law) $\langle x,y\rangle = \langle y,x\rangle \quad \forall x,y\in X$

```
simplify( dp(x,y) - dp(y,x) )
ans =
     0
```

```
simplify( dp(x(w),y(w)) == dp(y(w),x(w)) )
```

```
simplify( dp(x,y) == dp(y,x) )
ans =
symtrue
```

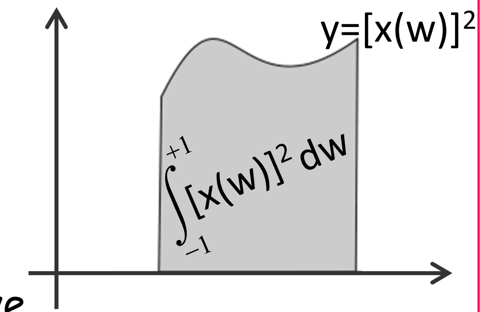
OK!

3. (positive definiteness) $\langle x,x\rangle > 0 \quad \forall x\in X \wedge x\neq \underline{0}$

```
disp(dp(x,x))
ans =
int(x(w)^2, w, -1, 1)
```

```
disp(dp(x(w),x(w)))
```

OK!



it represents the area under the positive curve, so that it is positive

Exercise

Verify if the following are dot products by means of Symbolic Math Toolbox

standard scalar product of real vectors

- $\langle x, y \rangle = y^T x \quad \forall x, y \in \mathbb{R}^n \iff \langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$

standard scalar product of complex vectors

- $\langle x, y \rangle = y^H x \quad \forall x, y \in \mathbb{C}^n \iff \langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$

where x and y are column vectors, $\overline{y}_k$ is the complex conjugate of y_k and y^H denotes the transposed and conjugate vector of y. You must choose a value for n: for example, $n=5$

Recall: Vector norm

Let $\langle X, \mathbb{R}, +, * \rangle$ be a Vector Space with real scalars.

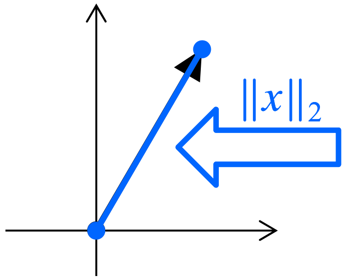
The ***vector norm*** of a vector is defined by the map

$$\|\cdot\| \; : \; x \in X \longrightarrow \|x\| \in \mathbb{R}$$

satisfying the following properties:

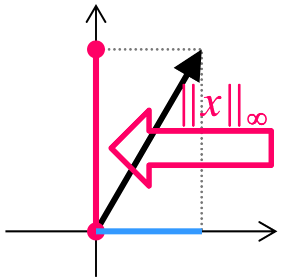
1. $\|x\| \geq 0 \qquad \forall x \in X$
2. $\|x\| = 0 \Leftrightarrow x = \underline{0}$
3. $\|\alpha x\| = |\alpha| \, \|x\|, \qquad \forall x \in X, \forall \alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\| \qquad$ (*triangle inequality*)

Examples of vector norms in $\mathbb{R}^n$



euclidean norm
(or **2-norm**)

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

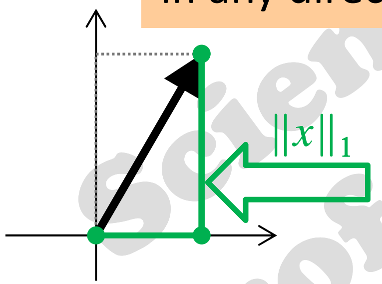


uniform norm (or **∞-norm**
or **maximum norm** or
Chebyshev norm)

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$$

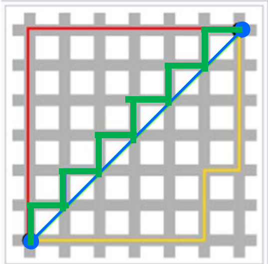
It is also named as chessboard distance, since it corresponds to the minimum number of moves needed by a King to go from a square on a chessboard to another, provided that the squares have side length one. The King can move 1 square in any direction (horiz., vert. and diag.) each time.

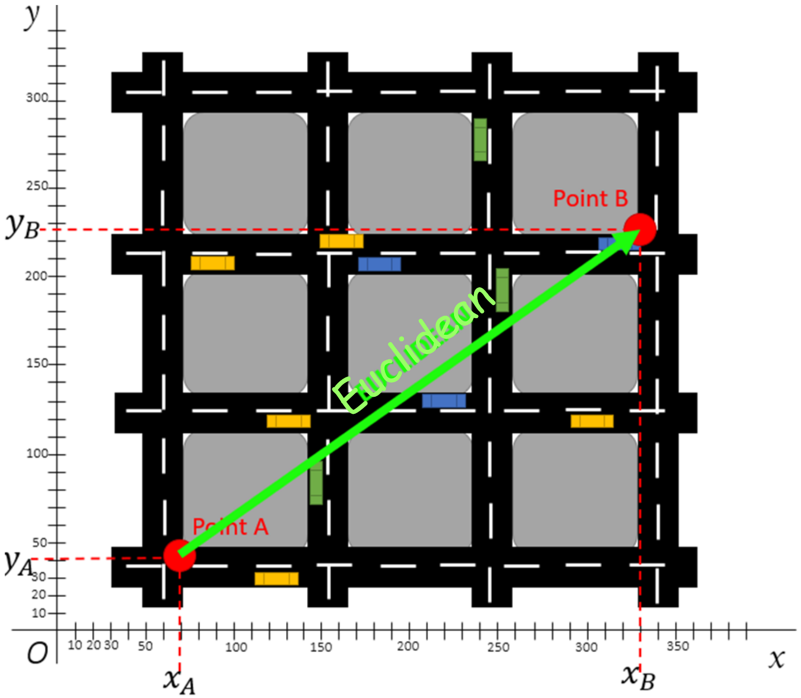
	a	b	c	d	e	f	g	h	
8	5	4	3	2	2	2	2	2	8
7	5	4	3	2	1	1	1	2	7
6	5	4	3	2	1	♔	1	2	6
5	5	4	3	2	1	1	1	2	5
4	5	4	3	2	2	2	2	2	4
3	5	4	3	3	3	3	3	3	3
2	5	4	4	4	4	4	4	4	2
1	5	5	5	5	5	5	5	5	1
	a	b	c	d	e	f	g	h	

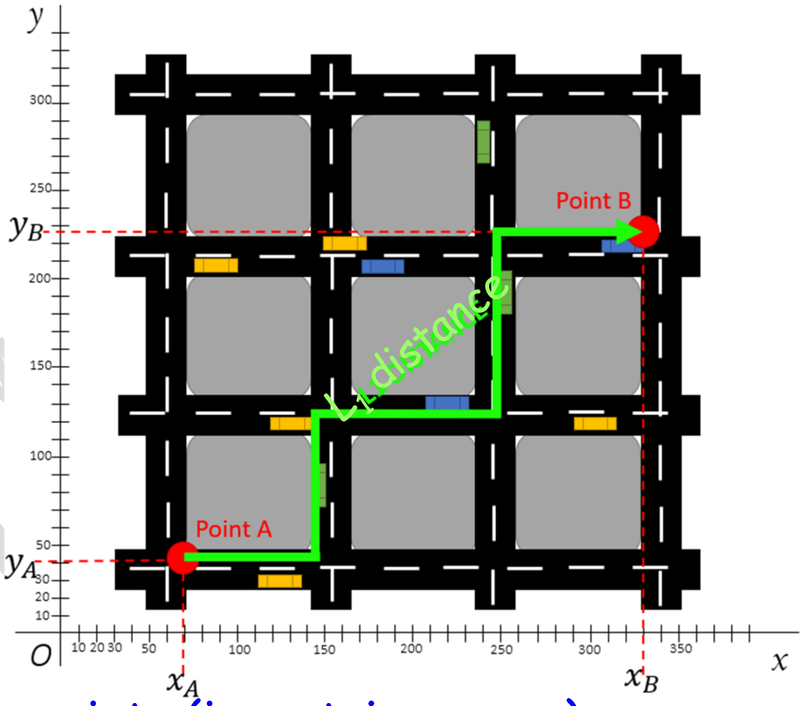


Taxicab norm (or **1-norm**
(or **Manhattan norm**)

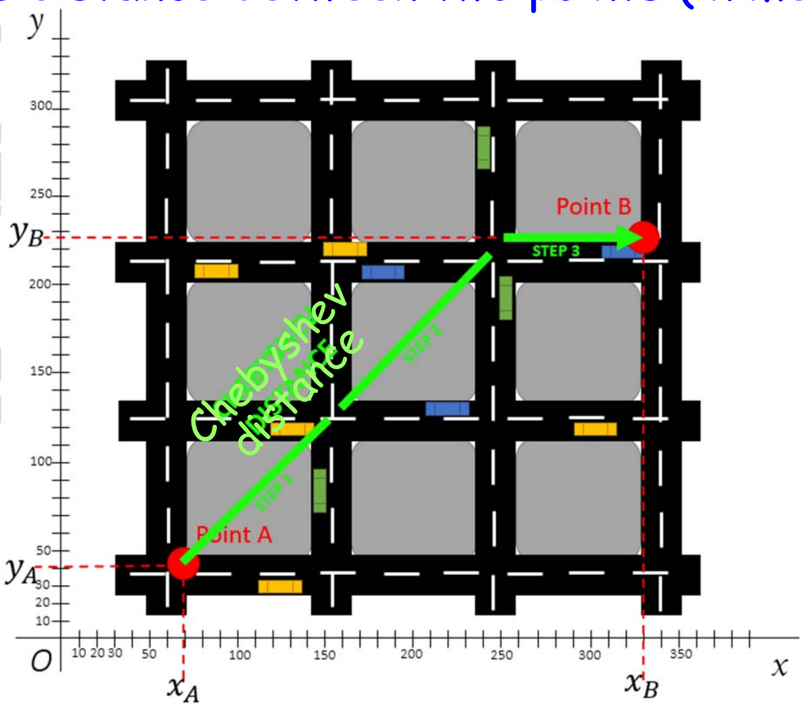
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$







Norms as distance between two points (in metric spaces)



norms in MATLAB

```
x=[1 2 -3 -4 5];
disp([norm(x); norm(x,2); norm(x,inf); norm(x,1)])
    7.4162
    7.4162
    5.0000
   15.0000

disp(sqrt(sum(x.^2)))
    7.4162

disp(max(abs(x)))
    5

disp(sum(abs(x)))
    15
```

2-norm (default)

∞-norm

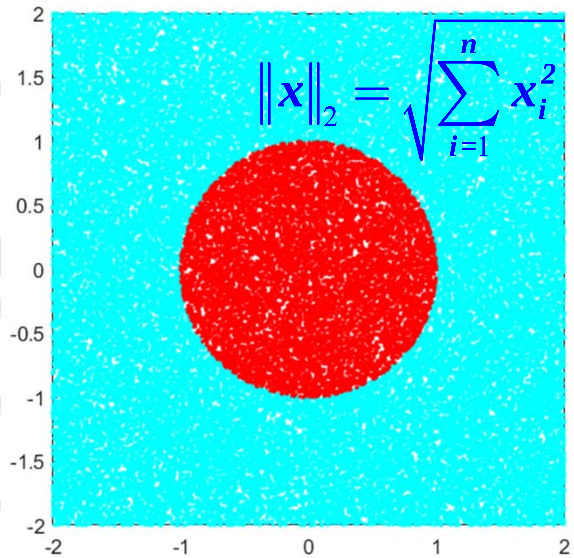
1-norm

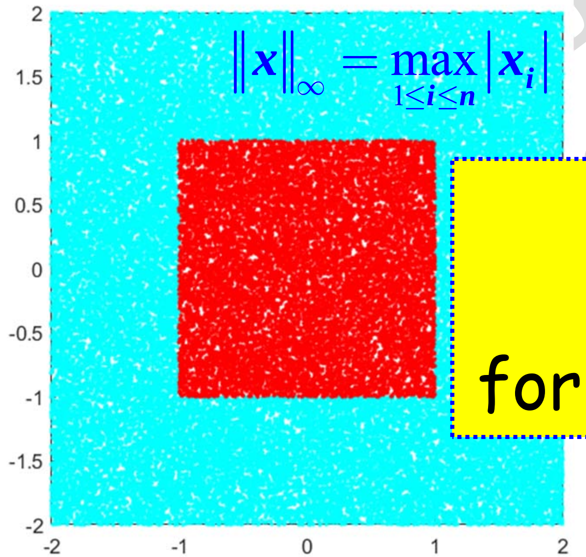
Neighborhoods in metric spaces

$$I_{\rho=1}(Q)=\{P : d(Q,P) = \|P-Q\| \le \rho=1\}$$

$\|x\|_2$

```
x1=-2+4*rand(1,50000);
x2=-2+4*rand(1,50000); x=[x1;x2];
j=find(sqrt(sum(x.^2)) <= 1);
plot(x1,x2,'.c',x1(j),x2(j),'.r')
axis equal
```



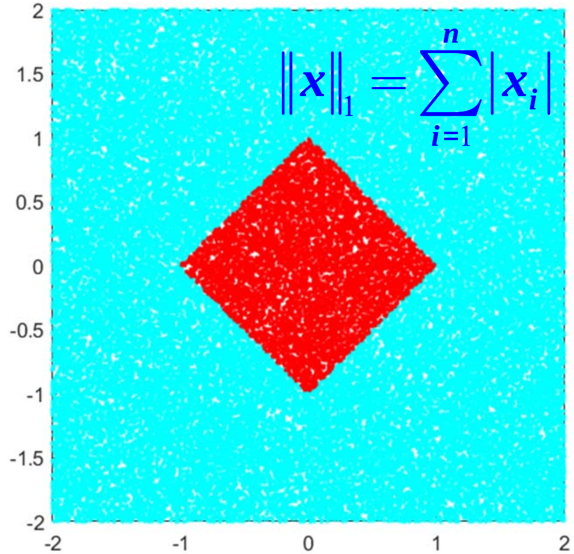


$\|x\|_1$

```
...
j=find(sum(abs(x))<=1);
...
```

May we use **norm(x)** for these examples?

see: **vecnorm(x,p)**




```
...
j=find(max(abs(x))<=1);
...
```

$\|x\|_\infty$

Induced vector norm

By definition of the ***standard dot product*** in $\mathbb{R}^n$

$$\forall u,v \in \mathbb{R}^n \qquad \langle u,v \rangle = \sum_{k=1}^{n} u_k \cdot v_k$$

or in $\mathbb{C}^n$

$$\forall u,v \in \mathbb{C}^n \qquad \langle u,v \rangle = \sum_{k=1}^{n} u_k \cdot \overline{v}_k$$

the ***Euclidean norm*** can be written as

$$\|x\|_2^2 = \sum_{k=1}^{n} |x_k|^2 = \langle x, x \rangle$$

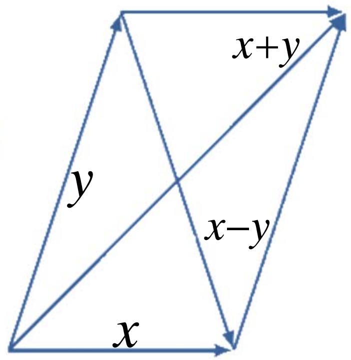
Then the 2-norm is said to be "**induced**" by the standard dot product.

Parallelogram Law

In a Linear Space X, with a dot product $\langle\cdot,\cdot\rangle$ that induces a norm $||x||^2 = \langle x,x\rangle$, the Parallelogram Law holds:

$$\forall x,y\in X \qquad ||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

Indeed, **property**: "a norm is induced by an inner product iff the Parallelogram Law holds".



Its name is due to the fact that it relates the squared sides to the squared diagonals of a parallelogram.

Proof: $\forall x,y\in X$ (real case)

By properties of scalar *products*:

$$||x+y||^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle = ||x||^2 + ||y||^2 + 2\langle x, y\rangle$$

$$||x-y||^2 = \langle x-y, x-y\rangle = \langle x, x\rangle + \langle y, y\rangle - 2\langle x, y\rangle = ||x||^2 + ||y||^2 - 2\langle x, y\rangle$$

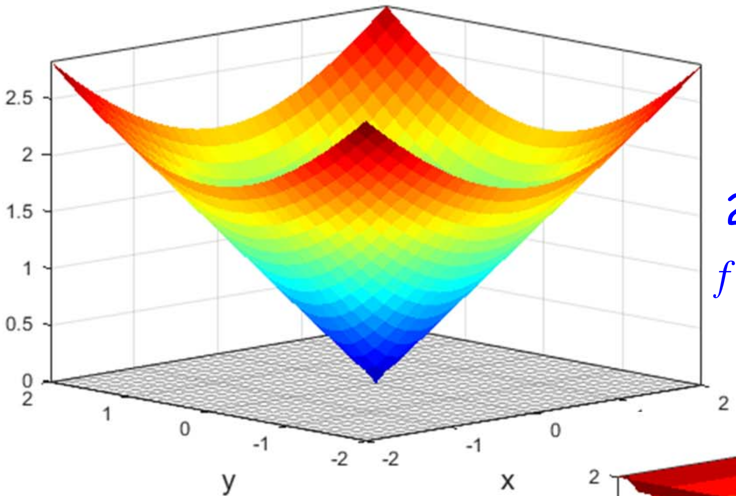
and adding the two equations side-by-side, we get:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

Norms are always convex functions*, whatever $\|x\|$ is.

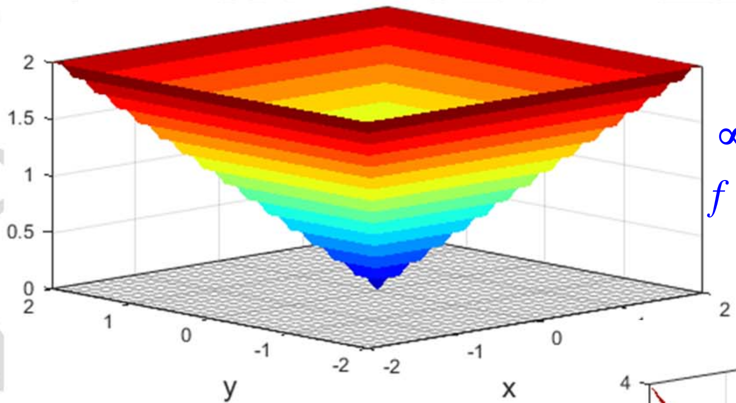
Examples in $\mathbb{R}^2$

Convex functions are very important for many optimization problems used in applications



2-norm

$$f(x_1,x_2)=\|x\|_2=\sqrt{\sum_{i=1}^{2} x_i^2}$$

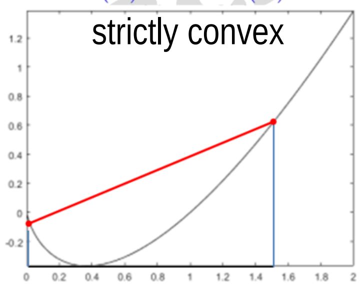


∞-norm

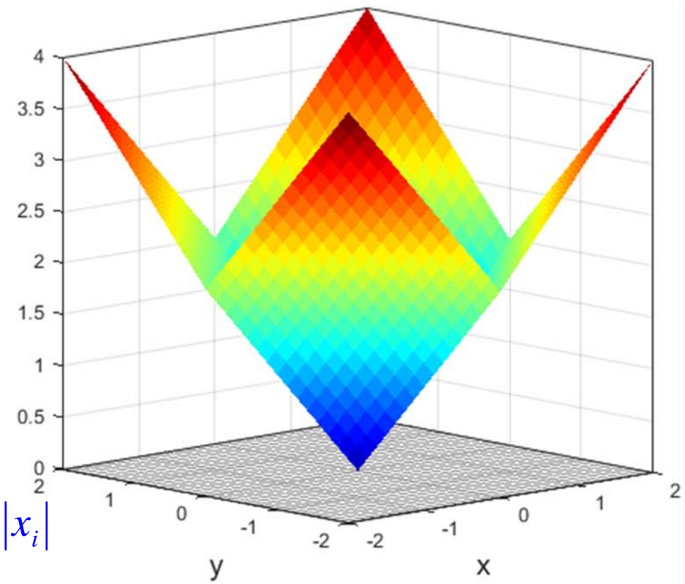
$$f(x_1,x_2)=\|x\|_\infty=\max_{1\le i\le 2}|x_i|$$

* A function $f : X\subset\mathbb{R}^n \to \mathbb{R}$ (X convex set), is **convex** if
$\forall x,y\in X,\ \forall\lambda\in[0,1]\quad f[\lambda x + (1-\lambda)y] \le \lambda f(x) + (1-\lambda)f(y)$
f strictly convex: $<$ for $x\neq y$ and $\lambda\in]0,1[$

$f(x)=x\log(x)$



From a geometrical point of view, the chord between (x,f(x)) and (y,f(y)) lies above the graph of f between the two points.



1-norm

$$f(x_1,x_2)=\|x\|_1=\sum_{i=1}^{2}|x_i|$$