L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

prof. Mariarosaria Rizzardi

Centro Direzionale di Napoli – Bldg. C4
room: n. 423 – North Side, 4th floor
phone: 081 547 6545
email: mariarosaria.rizzardi@uniparthenope.it

Contents

- **Dot products.**

Definition: dot product of real vectors

Let $\langle \boldsymbol{X}, \mathbb{R}, +, * \rangle$ be a Linear Space with real scalars.

The ***dot product*** (or ***scalar product*** or ***inner product***) of two real vectors is a map

angle brackets → $\langle \bullet, \bullet \rangle$

$$\langle \bullet, \bullet \rangle : \; (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{X} \longrightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{R}$$

Cartesian product of X with itself

satisfying the following properties:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in X$ — Linearity law
2. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$ — Commutative law
3. $\langle x, x \rangle > 0 \quad \forall x \in X \wedge x \neq \underline{0}$
4. $\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = \underline{0}$

Positive definiteness

Symmetric bilinear form

A dot product $\langle \bullet, \bullet \rangle$ between real vectors is a real symmetric bilinear form that is positive definite.

This means that it is:

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

2. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$

$\forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in X$

It is linear w.r.t. both of its arguments (x,y) ⟺ bilinear form

2. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$ — symmetric

3. $\langle x, x \rangle > 0 \quad \forall x \in X \wedge x \neq \underline{0}$

4. $\langle x, x \rangle = 0 \Leftrightarrow x = \underline{0}$

positive definite quadratic form

Example 1

Is the following bilinear form a **dot product**?

$$\langle x, y\rangle \overset{?}{=} 2x_1y_1 + 2x_1y_2 + x_2y_1 + x_2y_2$$

It can be written as: $\langle x, y\rangle = x^T A y$ where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

Verify by means of

MATLAB *Symbolic Math Toolbox*

```
syms x y [2 1] real
x, y
x =
x1
X2
y =
y1
y2
```

```
A=[2 2; 1 1];
x=sym('x',[2 1],'real');
y=sym('y',[2 1],'real');
expand(x'*A*y)
ans =
2*x1*y1 + 2*x1*y2 + x2*y1 + x2*y2
```

expand and simplify expressions

```
syms x y [2 1] real
f=2*x1*y1+2*x1*y2+x2*y1+x2*y2;
[c,t] = coeffs(f)
c =
[2, 2, 1, 1]
t =
[x1*y1, x1*y2, x2*y1, x2*y2]
n = sqrt(numel(c)); % num. elems.
A = (reshape(c,n,n))'
A =
[2, 2]
[1, 1]
```

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

Example 1 (contd.)

MATLAB *Symbolic Math Toolbox*

```
A=[2 2; 1 1];
syms x y z [2 1] real
syms a b real
F=@(x,y) x'*A*y
F =
  function_handle with value:
    @(x,y)x'*A*y
```

$$\langle x, y \rangle = x^T A y$$

Verify

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in X$ (bilinear)

```
expand( F(a*x+b*y,z) - ( a*F(x,z)+b*F(y,z) ) )
ans =
     0
```

```
simplify(F(a*x+b*y,z) == (a*F(x,z)+b*F(y,z)))
ans =
symtrue
```

OK!

2. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$ it does not hold! (symmetric)

```
expand( F(x,y) - F(y,x) )
ans =
x1*y2 - x2*y1
```

```
simplify(expand(F(x,y) == F(y,x)))
ans =
x1*y2 == x2*y1
```

NO!

It is <u>not</u> a dot product

Example 1 (contd.)

Also if this bilinear form is <u>not</u> a dot product, now we go to verify if it results positive definite anyway.

positive definite ⟺

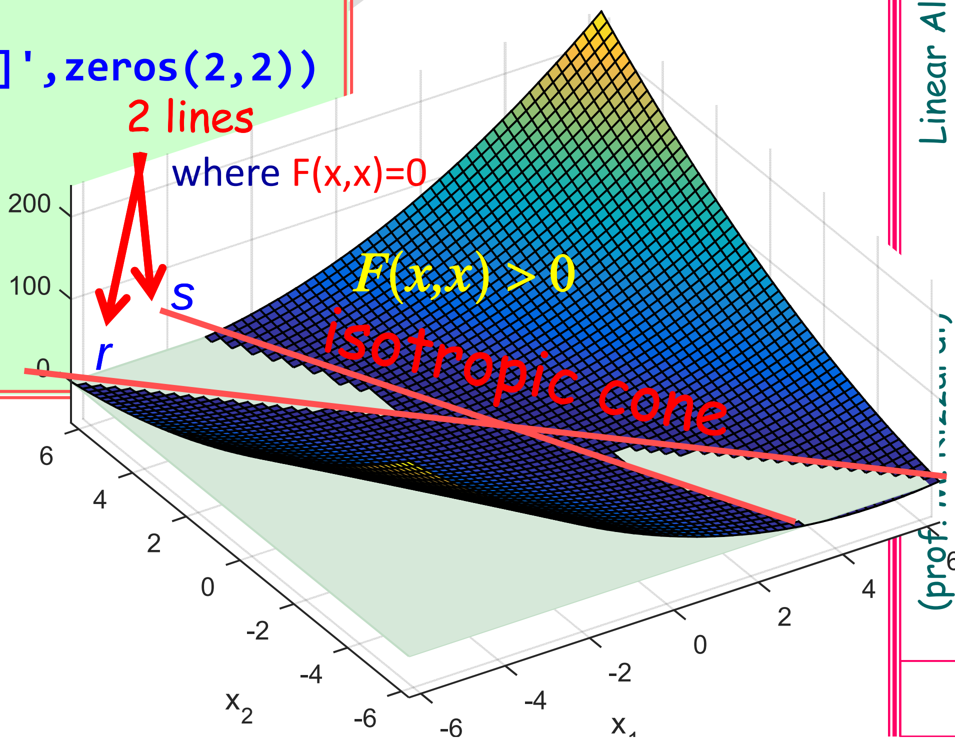
3. $\langle x,x\rangle = x^T Ax > 0 \quad \forall x \in X \wedge x \neq \underline{0}$

4. $\langle x,x\rangle = x^T Ax = 0 \;\Leftrightarrow\; x = \underline{0}$

only the null vector gives a bilinear form equal to 0

```
...S=simplify(F(x,x),50)
S =
(x1 + x2)*(2*x1 + x2)
fsurf(S); hold on; AX=axis;
surf([AX(1:2);AX(1:2)],[AX(3:4);AX(3:4)]',zeros(2,2))
X2=solve(S,x2)
X2 =
   -x1
 -2*x1
disp(factor(S))
[x1 + x2, 2*x1 + x2]
```

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \qquad F(x,y) = x^{\mathrm{T}}Ax$$



r : x2 = -x1
corresponds to $r = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

s : x2 = -2*x1
corresponds to $s = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$

isotropic cone of a quadratic form

r : x2 = -x1

corresponds to $r = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

s : x2 = -2*x1

corresponds to $s = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$

What is the connection between the lines r and s, and the matrix A?

```
A=[2 2; 1 1];
NA=null(A)
NA =
   -0.7071
    0.7071
NAT=null(A')
NAT =
   -0.4472
    0.8944
```

```
A=sym([2 2; 1 1]);
NA=null(A)
NA =
     -1
      1
NAT=null(A')
NAT =
   -1/2
      1
```

$$r = \mathcal{N}(A)$$

$$s = \mathcal{N}(A^T)$$

Besides the null vector, there are infinitely many vectors x such that

$$x^T A x = 0$$

The equation $x^T A x = 0$ is satisfied if:

- $Ax = 0 \quad \Leftrightarrow \quad x \in \mathcal{N}(A)$ [x in the Null Space of A]

or

- $x^T A = 0 \quad \Leftrightarrow \quad x \in \mathcal{N}(A^T)$ [x in the Left Null Space of A]

or

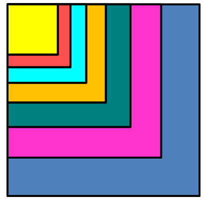
- $x \perp Ax \quad \Leftrightarrow \quad x$ and Ax are orthogonal (or perpendicular)

(prof. M. Rizzardi)

Positive definiteness

- A quadratic form is positive definite if, and only if, its associated matrix is positive definite.
- A symmetric matrix is positive definite if, and only if, all its leading principal minors* are strictly positive. it is not computationally convenient
- A symmetric matrix is positive definite if, and only if, all its eigenvalues are > 0.

* The **leading principal minors** are the determinants of all the leading principal submatrices of order k. The leading principal submatrix of order k, of a matrix (n×n) is obtained by deleting the last n−k rows and columns of the matrix.



$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

```
A=[2 2; 1 1];
disp(eig(A))
     3
     0
```

semidefinite positive

The matrix is not positive definite

Example 2

Is the following bilinear form a **dot product**?

$$\langle x, y \rangle \overset{?}{=} 2x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$

It can be written as: $\langle x, y \rangle = x^T A y$ where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

symmetric matrix

MATLAB ***Symbolic Math Toolbox***

```
A=[2 1; 1 -1];
syms x y [2 1] real
expand(x'*A*y)
ans =
2*x1*y1 + x1*y2 + x2*y1 - x2*y2
```

```
syms x y [2 1] real
f=2*x1*y1+x1*y2+x2*y1-x2*y2;
[c,t] = coeffs(f)
c =
[2, 1, 1, -1]
t =
[x1*y1, x1*y2, x2*y1, x2*y2]
n = sqrt(numel(c)); % num. of elem.
A = (reshape(c,n,n))'
A =
[2,  1]
[1, -1]
```

Example 2 (contd.)

```
A=[2 1; 1 -1];
syms x y z [2 1] real; syms a b real
F = @(x,y) x'*A*y
F =
    @(x,y)x'*A*y
```

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y, z \in X$ (bilinear)

```
expand( F(a*x+b*y,z) - (a*F(x,z)+b*F(y,z)) )
ans =
0
```

```
simplify(F(a*x+b*y,z) == (a*F(x,z)+b*F(y,z)))
ans =
symtrue
```

OK!

2. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$ (symmetric)

```
expand( F(x,y) - F(y,x) )
ans =
0
```

```
simplify(F(x,y) == F(y,x))
ans =
symtrue
```

OK!

3. 4. $\langle x, x \rangle > 0 \quad \forall x \in X \wedge x \neq \underline{0} \qquad \langle x, x \rangle = 0 \iff x = \underline{0}$ (positive definite)

```
disp(eig(A))
   -1.3028
    2.3028
```

←

NO!

Example 2 (contd.)

Also if this bilinear form is <u>not</u> a dot product, now we go to verify if it results positive definite anyway.

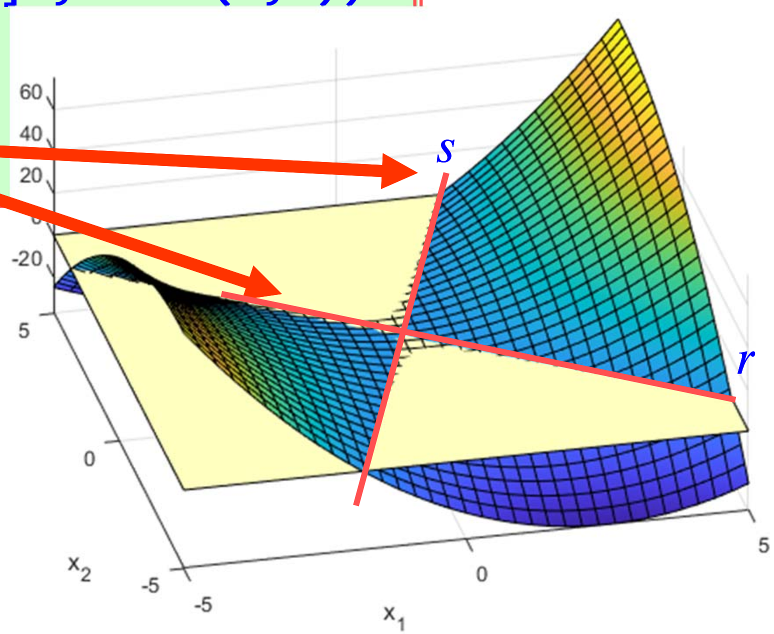
positive definite ⇔

3. $\langle x,x\rangle = x^T A x > 0 \quad \forall x \in X \wedge x \neq \underline{0}$

4. $\langle x,x\rangle = x^T A x = 0 \iff x = \underline{0}$

```
S=simplify(F(x,x),50)
S =
2*x1^2 + 2*x1*x2 - x2^2
fsurf(S); hold on; AX=axis; xlabel(...); ylabel(...)
surf([AX(1:2);AX(1:2)],[AX(3:4);AX(3:4)]',zeros(2,2))
X2=solve(S,x2)
X2 =
 x1 + 3^(1/2)*x1
 x1 - 3^(1/2)*x1
```

isotropic cone



r : x2 = x1(1-√3)
corresponds to $r = \text{span}\left\{\begin{pmatrix}1\\1-\sqrt{3}\end{pmatrix}\right\}$

s : x2 = x1(1+√3)
corresponds to $s = \text{span}\left\{\begin{pmatrix}1\\1+\sqrt{3}\end{pmatrix}\right\}$

isotropic cone of a quadratic form

r : x2 = x1(1-√3) corresponds to $r = \text{span}\left\{\begin{pmatrix} 1 \\ 1-\sqrt{3} \end{pmatrix}\right\}$

s : x2 = x1(1+√3) corresponds to $s = \text{span}\left\{\begin{pmatrix} 1 \\ 1+\sqrt{3} \end{pmatrix}\right\}$

What is the connection between the lines r and s, and the matrix A?

```
A=[2 1; 1 -1];
NA=null(A)
NA =
Empty matrix: 2-by-0
NAT=null(A')
NAT =
Empty matrix: 2-by-0
```

```
A=sym([2 1; 1 -1]);
NA=null(A)
NA =
Empty sym: 1-by-0
NAT=null(A')
NAT =
Empty sym: 1-by-0
```

$\mathcal{N}(A) = \{\underline{0}\}$

$\mathcal{N}(A^T) = \{\underline{0}\}$

a linear subspace cannot be empty: it contains the null vector at least!

No connection!

The symmetric matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ is not positive definite

Why?

```
A=[2 1; 1 -1];
disp(eig(A))
   -1.3028
    2.3028
```

The matrix is not positive definite: why?

The equation $x^{\mathsf{T}}Ax = 0$ is satisfied if:

- ~~$Ax = 0$~~ in this case $\mathcal{N}(A) = \{\underline{0}\}$
- ~~$x^{\mathsf{T}}A = 0$~~ in this case $\mathcal{N}(A^{\mathsf{T}}) = \{\underline{0}\}$
- $x \perp Ax \iff x$ and Ax are orthogonal

```
X2=solve(S,x2)
X2 =
 (1+3^(1/2))*x1
 (1-3^(1/2))*x1
v=subs(x,x2,X2(1))
v =
             x1
 (1+3^(1/2))*x1
simplify(v'*A*v)
ans =
0
```

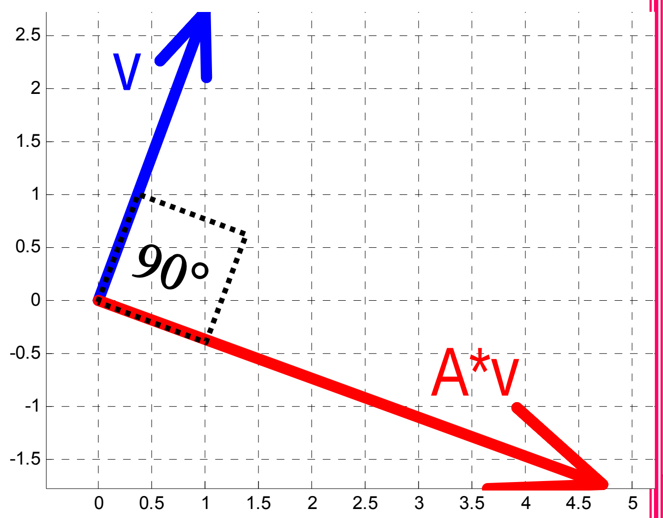
solve $2x_1^2 + 2x_1x_2 - x_2^2 = 0$ with respect to x_2

$$r = \text{span}\left\{\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}\right\}, \quad s = \text{span}\left\{\begin{pmatrix} 1 \\ +\sqrt{3} \end{pmatrix}\right\}$$

put v on s

compute $v^{\mathsf{T}}Av$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$



the vectors v and Av are orthogonal!

Example 3

Is the following bilinear form a **dot product**?

$$\langle x,y\rangle \overset{?}{=} x_1y_1+5x_2y_2+12x_3y_3-2x_1y_2-2x_2y_1+3x_1y_3+3x_3y_1-7x_2y_3-7x_3y_2$$

$$\langle x,y\rangle = x^T A y \quad \Longleftrightarrow \quad x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\ y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\ A=\begin{pmatrix}1&-2&3\\-2&5&-7\\3&-7&12\end{pmatrix}$$

symmetric matrix

```
A=[1 -2 3;-2 5 -7;3 -7 12];
syms a b real; syms x y z [3 1] real
F = @(x,y) x'*A*y;
```

```
expand( F(a*x+b*y,z) - (a*F(x,z)+b*F(y,z)) )
ans =
0
```

```
simplify(F(a*x+b*y,z)==(a*F(x,z)+b*F(y,z)))
ans =
symtrue
```

OK!

```
expand( F(x,y) - F(y,x) )
ans =
0
```

```
simplify(F(x,y)==(F(y,x)))
ans =
symtrue
```

OK!

```
all(eig(A) > 0)
ans =
  logical
   1
```

OK!

→ It is a dot product